NOTES ON SEQUENTIAL Oligopoly

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A Sequential Move Oligopoly Game

Two firms, labelled 1 = 1, 2, play a sequential move Cournot (quantity-setting) game. In each period \( t, t = 0, 1, \ldots \), firm 1 moves first and picks an output level \( q_{1t} \). Than having seen \( q_{1t} \), firm 2 moves and picks an output level \( q_{2t} \). We call \( q_{it} \) the action of firm \( i \) at time \( t \). We assume \( q_{it} \) is a member of the action space \( A_i \) for all \( t \). The period \( t \) payoff to firm \( i \) when actions \( q_{1t} \) and \( q_{2t} \) are taken is

\[
\pi_{it}(q_{1t}, q_{2t}) = P(q_{1t} + q_{2t})q_{it}
\]

where \( P(\cdot) \) is the industry demand function. We assume \( P(q) \) is differentiable, monotonically decreasing in \( q \) on a finite interval \([0, m]\) and that \( P(q) \to 0 \) as \( q \) increases to \( m \) and \( P(q) \) equals zero for all \( q \geq m \).

A strategy \( \sigma_i \) for firm \( i \) is a sequence of functions \( \sigma_{i1}, \sigma_{i2}, \ldots \), one for each period \( t \). The function for period \( t \) determines player \( i \)'s actions as a function of the actions taken by both players times before time \( t \). Let the history faced by player 1 at time \( t \) be denoted \( h_{1t} \) where

\[
h_{1t} = (q_{11}, q_{21}, \ldots, q_{1t-1}, q_{2t-1}).
\]

That is, \( h_{1t} \) records the output levels of both players in all periods before \( t \). Let \( H_{1t} = \{h_{1t}|q_{is} \in A_i \text{ for all } 1 \leq s < t\} \). In period \( t \) player 2 moves after having observed the current output \( q_{1t} \) produced by player 1. Thus the history facing player 2 at time \( t \) is \( h_{2t} = (h_{1t}; q_{1t}) \). Let

\[
H_{2t} = \{h_{2t}|q_{1s} \in A_1 \text{ for } 1 \leq s \leq t \text{ and } q_{2s} \in A_2 \text{ for } 1 \leq s < t\}.
\]

A strategy \( \sigma_i \) for player \( i \) is a sequence of functions \( \{\sigma_{it}\}_{t=1}^{\infty} \) where \( \sigma_{it} : H_{it} \to A_i \). Let the strategy space of player \( i \) be

\[
\mathcal{S}_i = \{\sigma_i = (\sigma_{it})_{t=1}^{\infty}|\sigma_{it} : H_{it} \to A_i\}.
\]
Let $\sigma = (\sigma_1, \sigma_2)$ and $S = S_1 \times S_2$. We need to define payoffs over strategies. We first define payoffs over outcomes. An outcome path $q^0$ is a collection of actions for both players, one each $t$. That is $q^0 = \{q_{1t}, q_{2t}\}_{t=1}^\infty$. The payoff to firm $i$ under outcome path $q^0$ is

$$V_i(q^0) = \sum_{t=1}^\infty \delta^t \pi_i(q_{1t}, q_{2t}).$$

Likewise, the payoff to firm $i$ from $t$ onwards under the outcome path $q^t$ from $t$ onwards is

$$V_{it}(q^t) = \sum_{s=t}^\infty \delta^{s-t} \pi_i(q_{1s}, q_{2s})$$

where $q^t = (q_{1t}, q_{2t}, q_{1t+1}, q_{2t+1}, \ldots)$.

Given any history $h_{1t}$ a strategy vector $\sigma_1(\cdot|h_{1t})$, $\sigma_2(\cdot|h_{2t})$ generates an outcome path from $t$ onward, which is inductively defined as,

$$(1.4) \quad q_{1t} = \sigma_1(h_{1t}|h_{1t})$$

$$(1.4) \quad q_{2t} = \sigma_2(h_{2t}|h_{1t}) \text{ where } h_{2t} = (h_{1t}; q_{1t})$$

$$(1.4) \quad q_{1t+1} = \sigma_1(h_{1t+1}|h_{1t}) \text{ where } h_{1t+1} = (h_{1t}; q_{1t}; q_{2t})$$

$$(1.4) \quad q_{2t+1} = \sigma_2(h_{2t+1}|h_{1t}) \text{ where } h_{2t+1} = (h_{1t}; q_{1t}; q_{2t}; q_{1t+1}), \text{ and soon.}$$

Payoffs over strategies $\sigma_1(\cdot|h_{1t})$, $\sigma_2(\cdot|h_{2t})$ are given by

$$V_{it}(\sigma_1(\cdot|h_{1t}), \sigma_2(\cdot|h_{2t})) = \sum_{s=t}^\infty \delta^{s-t} \pi_i(q_{1s}, q_{2s})$$

where $q^t = (q_{1t}, q_{2t}, q_{1t+1}, q_{2t+1}, \ldots)$ is defined by (1.4).

Let $S_i(h_{1t})$ denote the set of strategies for player $i$ from $t$ onward, given history $h_{1t}$. That is

$$(1.6) \quad S_i(h_{1t}) = \{\sigma_i(\cdot|h_{1t}) = \{\sigma_{1s}(\cdot|h_{1t})\}_{s=t}^\infty \mid \sigma_{1s}(\cdot|h_{1t}); H_{1t}; A_1\}$$
where \( H^{s}_{1t} = \{ h^{s}_{1t} = (q_{1t}, q_{2t}, \ldots, q_{1s-1}, q_{2s-1}) | q_{ir} \in A_i \text{ all } t \leq r < s \} \). Let \( S_{2}(h_{2t}) \) be defined in an analogous fashion. We then have

**Definition.** \( \sigma = (\sigma_1, \sigma_2) \in S \) is a subgame perfect Nash equilibrium if for each \( t = 1, 2, \ldots \). The following conditions hold: for each \( h_{1t} \in H_{1t} \).

\[
V_{1t}(\sigma_1(\cdot|h_{1t}), \sigma_2(\cdot|h_{2t})) > V_{1t}\left(\sigma'_1(\cdot|h_{1t}), \sigma_2(\cdot|h_{2t})\right)
\]

for all \( \sigma'_1(\cdot|h_{1t}) \in S_{1}(h_{1t}) \) where \( h_{2t}' = (h_{1t}', \sigma'_1(h_{1t}|h_{1t})) \) and for each \( h_{2t} \in H_{2t} \).

\[
V_{2t}(\sigma_1(\cdot|h_{1t}), \sigma_2(\cdot|h_{2t})) > V_{2t}(\sigma_1(\cdot|h_{1t}), \sigma'_2(\cdot|h_{2t}))
\]

for all \( \sigma'_2(\cdot|h_{2t}) \in S_{2}(h_{2t}) \).

Notice that in each period \( t \) player 1 is a "Stackelberg leader" in the sense that player 1 when considering a deviation to some \( \sigma'_1(\cdot|h_{1t}) \) takes account of the fact that the action he adopts at \( t \), say \( \sigma'_1(h_{1t}|h_{1t}) \), will affect the action taken by player 2 by affecting the history that player 2 confronts when it is his turn to move.