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Modeling the Evolution of Age and Cohort Effects in Social Research: Technical Appendix*

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ABSTRACT

This document contains detailed algebra for the proofs of propositions 1 and 2.

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Proof of Proposition 1

Normalization 1. *$r_k \leq r_{k'}$ for all $k < k'$.*

Normalization 2. *$w_{k,0} = 1$ for all k.*

Normalization 3. *If $K > 1$, then $e_{k,s} = 0$ for $s < k - A$.*

Proposition 1. *Let $K \in \{1, 2, 3\}$ be known, and let the parameter space $\boldsymbol{\Theta}$ consist of all vectors $\left[\mu, \left\{e_{k,1-A}, \ldots, e_{k,T}, \{w_{k,a}\}_{a=0}^{A}, r_k\right\}_{k=1}^{K}\right]$ that satisfy normalizations 1 to 3. Suppose further that $A \geq K$, that $T \geq A + K$, and that if $K = 1$, then $T \geq 4$; if $K = 2$, then $T \geq 12$; and if $K = 3$, then $T \geq 32$. Then there exists a set $X_K \subset \boldsymbol{\Theta}$ such that X_K is of measure zero and all $\boldsymbol{\theta} \in \boldsymbol{\Theta} \setminus X_K$ are identified.*

Proof. We prove the result separately for $K = 1$, $K = 2$, and $K = 3$. In each case, we need to show that there exists a set $X_K \subset \boldsymbol{\Theta}$ such that X_K is of measure zero and such that, unless $\boldsymbol{\theta} = \left[\mu, \left\{\{e_{k,t}\}_{t=1-A}^{T}, \{w_{k,a}\}_{a=1}^{A}, r_k\right\}_{k=1}^{K}\right]$ is in X_K, the equality

$$\mu + \sum_{k=1}^{K}\sum_{a'=0}^{a} r_k^{a-a'} w_{k,a'} e_{k,t-a+a'} = \tilde{\mu} + \sum_{k=1}^{K}\sum_{a'=0}^{a} \tilde{r}_k^{a-a'} \tilde{w}_{k,a'} \tilde{e}_{k,t-a+a'} \quad \forall\, a, t \tag{1}$$

implies $\left[\mu, \left\{\{e_{k,t}\}_{t=1-A}^{T}, \{w_{k,a}\}_{a=1}^{A}, r_k\right\}_{k=1}^{K}\right] = \left[\tilde{\mu}, \left\{\{\tilde{e}_{k,t}\}_{t=1-A}^{T}, \{\tilde{w}_{k,a}\}_{a=1}^{A}, \tilde{r}_k\right\}_{k=1}^{K}\right] \equiv \tilde{\boldsymbol{\theta}}$. The strategy will be to construct a set X_K of measure zero in the parameter space and show that (1) implies $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$ when $\boldsymbol{\theta} \in \boldsymbol{\Theta} \setminus X_K$.

Case 1: $K = 1$.

Define the following sets:

$$X_{1,1} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon r_1 = 0\}$$

$$X_{1,2} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon r_1 + w_{1,1} = 1\}$$

$$X_{1,3} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon e_{1,t} = 0 \;\forall\, t \geq A + 1\}$$

$$X_{1,4} = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \operatorname{rank} \begin{bmatrix} 1 & e_{1,1} & e_{1,2} \\ 1 & e_{1,2} & e_{1,3} \\ \vdots & \vdots & \vdots \\ 1 & e_{1,T-1} & e_{1,T} \end{bmatrix} < 3 \right\}$$

Let $X_1 = \cup_{j=1}^{4} X_{1,j}$. Since X_1 is a finite union of sets of measure zero, X_1 is a set of measure zero.

Assume $\boldsymbol{\theta} \in \boldsymbol{\Theta} \setminus X_1$. Specializing (1) to $K = 1$, $a = 0$ and $a = 1$ (by hypothesis, $A \geq 1$) and using normalization 2, we have

$$\mu + e_{1,t} = \tilde{\mu} + \tilde{e}_{1,t}, \quad t = 1, \ldots, T, \tag{2a}$$

$$\mu + r_1 e_{1,t-1} + w_{1,1} e_{1,t} = \tilde{\mu} + \tilde{r}_1 \tilde{e}_{1,t-1} + \tilde{w}_{1,1} \tilde{e}_{1,t}, \quad t = 1, \ldots, T. \tag{2b}$$

Substituting (2a) into (2b) and collecting terms gives

$$0 = (\mu - \tilde{\mu})(1 - \tilde{r}_1 - \tilde{w}_{1,1}) - (\tilde{r}_1 - r_1) e_{1,t-1} - (\tilde{w}_{1,1} - w_{1,1}) e_{1,t}, \quad t = 2, \ldots, T. \tag{3}$$

By hypothesis, $T \geq 4$, so (3) contains at least three equations. Since (given $\boldsymbol{\theta} \notin X_{1,4}$) $e_{1,t-1}$ and $e_{1,t}$ are not collinear with a constant, (3) can hold only if $(\mu - \tilde{\mu})(1 - \tilde{r}_1 - \tilde{w}_{1,1}) = 0$

and the coefficients on $e_{1,t-1}$ and $e_{1,t}$ are both zero. Hence $\tilde{r}_1 = r_1$, $\tilde{w}_{1,1} = w_{1,1}$, and, since $1 - r_1 - w_{1,1} \neq 0$ for $\boldsymbol{\theta} \notin X_{1,2}$, we must have $\tilde{\mu} = \mu$. It follows from (2a) that $\tilde{e}_{1,t} = e_{1,t}$ for $t = 1, \ldots, T$.

It remains to show that $\tilde{w}_{1,a} = w_{1,a}$ for $a > 1$ and $\tilde{e}_{1,t} = e_{1,t}$ for $t \leq 0$. If $\tilde{w}_{1,a'} = w_{1,a'}$ for all $a' < a$, then, using the foregoing results, (1) at a reduces to

$$(w_{1,a} - \tilde{w}_{1,a})e_{1,t} = 0, \quad t = a+1, \ldots, T. \tag{4}$$

Since $T \geq A + 1$, (4) contains at least one equation, and since $\boldsymbol{\theta} \notin 1, 3$, $e_{1,t}$ cannot be zero in all of these equations. Therefore, the only solution is $\tilde{w}_{1,a} = w_{1,a}$. By induction, this gives $\tilde{w}_{1,a} = w_{1,a}$ for all $a > 1$. Now fix any $s \leq 0$. If $\tilde{e}_{1,\tau} = e_{1,\tau}$ for $\tau = s+1, \ldots, T$, then, using the foregoing results, (1) at $a = 1 - s$ and $t = 1$ reduces to

$$r_1^{1-s}(e_{1,s} - \tilde{e}_{1,s}) = 0. \tag{5}$$

Since $\boldsymbol{\theta} \notin X_{1,1}$, $r_1 \neq 0$ and hence $\tilde{e}_{1,s} = e_{1,s}$. Therefore, by induction, $\tilde{e}_{1,t} = e_{1,t}$ for $t = 0, -1, \ldots, 1 - A$. We have now identified all parameters for the case $K = 1$.

Case 2: $K = 2$.

Define the following sets:

$$X_{2,1} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \exists k \text{ s.t. } r_k = 0\}$$

$$X_{2,2} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon r_1 = r_2\}$$

$$X_{2,3} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon w_{1,1} = w_{2,1} \text{ or } w_{2,1} = w_{2,2}\}$$

$$X_{2,4} = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \text{rank} \begin{bmatrix} 1 & \begin{pmatrix} e_{k,j} \\ \vdots \\ e_{k,T-4+j} \end{pmatrix}_{\substack{k \in \{1,2\}, \\ j \in \{1,2,3,4\}}} \\ \vdots & \\ 1 & \end{bmatrix} < 9 \right\}$$

$$X_{2,5} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon (w_{1,1} - w_{2,1})(r_1 w_{2,1} - r_2 w_{1,1}) + (w_{1,2} - w_{2,2})(r_2 - r_1) = 0\}$$

$$X_{2,6} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon w_{1,1} - w_{2,1} + r_1 - r_2 + (r_2^2 + r_2 w_{2,1} + w_{2,2})(1 - r_1 - w_{1,1}) \\ - (w_{1,2} + r_1 w_{1,1} + r_1^2)(1 - r_2 - w_{2,1}) = 0\}$$

$$X_{2,7} = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \text{rank} \begin{bmatrix} e_{1,A+1} & e_{2,A+1} \\ e_{1,A+2} & e_{2,A+2} \end{bmatrix} < 2 \right\}$$

Let $X_2 = \cup_{j=1}^{7} X_{2,j}$. Since X_2 is a finite union of sets of measure zero, X_2 is a set of measure zero.

Assume $\boldsymbol{\theta} \in \boldsymbol{\Theta} \backslash X_2$. Specializing (1) to $K = 2$, $a = 0$, $a = 1$, and $a = 2$ (by hypothesis, $A \geq 2$) and using normalization 2, we have

$$\mu + e_{1,t} + e_{2,t} = \tilde{\mu} + \tilde{e}_{1,t} + \tilde{e}_{2,t}, \quad t = 1, \ldots, T, \tag{6a}$$

$$\mu + r_1 e_{1,t-1} + w_{1,1} e_{1,t} + r_2 e_{2,t-1} + w_{2,1} e_{2,t}$$

$$= \tilde{\mu} + \tilde{r}_1 \tilde{e}_{1,t-1} + \tilde{w}_{1,1} \tilde{e}_{1,t} + \tilde{r}_2 \tilde{e}_{2,t-1} + \tilde{w}_{2,1} \tilde{e}_{2,t}, \quad t = 2, \ldots, T, \quad (6b)$$

$$\mu + r_1^2 e_{1,t-2} + r_1 w_{1,1} e_{1,t-1} + w_{1,2} e_{1,t} + r_2^2 e_{2,t-2} + r_2 w_{2,1} e_{2,t-1} + w_{2,2} e_{2,t}$$

$$= \tilde{\mu} + \tilde{r}_1^2 \tilde{e}_{1,t-2} + \tilde{r}_1 \tilde{w}_{1,1} \tilde{e}_{1,t-1} + \tilde{w}_{1,2} \tilde{e}_{1,t} + \tilde{r}_2^2 \tilde{e}_{2,t-2} + \tilde{r}_2 \tilde{w}_{2,1} \tilde{e}_{2,t-1} + \tilde{w}_{2,2} \tilde{e}_{2,t},$$

$$t = 3, \ldots, T. \quad (6c)$$

Solving (6a) for $\tilde{e}_{2,t}$, substituting into (6b) and (6c), and collecting terms yields

$$(\mu - \tilde{\mu})(1 - \tilde{r}_2 - \tilde{w}_{2,1}) + (r_1 - \tilde{r}_2) e_{1,t-1} + (w_{1,1} - \tilde{w}_{2,1}) e_{1,t} + (r_2 - \tilde{r}_2) e_{2,t-1}$$

$$+ (w_{2,1} - \tilde{w}_{2,1}) e_{2,t} = (\tilde{r}_1 - \tilde{r}_2) \tilde{e}_{1,t-1} + (\tilde{w}_{1,1} - \tilde{w}_{2,1}) \tilde{e}_{1,t}, \quad t = 2, \ldots, T, \quad (7a)$$

$$(\mu - \tilde{\mu})(1 - \tilde{r}_2^2 - \tilde{r}_2 \tilde{w}_{2,1} - \tilde{w}_{2,2}) + (r_1^2 - \tilde{r}_2^2) e_{1,t-2} + (r_1 w_{1,1} - \tilde{r}_2 \tilde{w}_{2,1}) e_{1,t-1} + (w_{1,2} - \tilde{w}_{2,2}) e_{1,t}$$

$$+ (r_2^2 - \tilde{r}_2^2) e_{2,t-2} + (r_2 w_{2,1} - \tilde{r}_2 \tilde{w}_{2,1}) e_{2,t-1} + (w_{2,2} - \tilde{w}_{2,2}) e_{2,t}$$

$$= (\tilde{r}_1^2 - \tilde{r}_2^2) \tilde{e}_{1,t-2} + (\tilde{r}_1 \tilde{w}_{1,1} - \tilde{r}_2 \tilde{w}_{2,1}) \tilde{e}_{1,t-1} + (\tilde{w}_{1,2} - \tilde{w}_{2,2}) \tilde{e}_{1,t},$$

$$t = 3, \ldots, T. \quad (7b)$$

Rewrite (7) as

$$\sum_{\ell=0}^{1} \beta_{1,\ell}\tilde{e}_{1,t-\ell} = \delta_1 + \sum_{k=1}^{2}\sum_{\ell=0}^{1} \alpha_{1,k,\ell} e_{k,t-\ell}, \quad t = 2, \ldots, T, \tag{8a}$$

$$0 = \delta_2 + \sum_{k=1}^{2}\sum_{\ell=0}^{2} \alpha_{2,k,\ell} e_{k,t-\ell} + \sum_{\ell=0}^{2} \beta_{2,\ell}\tilde{e}_{1,t-\ell}, \quad t = 3, \ldots, T, \tag{8b}$$

where

$$\begin{aligned}
\delta_1 &= (\mu - \tilde{\mu})(1 - \tilde{r}_2 - \tilde{w}_{2,1}), \\
\delta_2 &= (\mu - \tilde{\mu})(1 - \tilde{r}_2^2 - \tilde{r}_2\tilde{w}_{2,1} - \tilde{w}_{2,2}), \\
\alpha_{1,k,1} &= r_k - \tilde{r}_2, \quad \alpha_{1,k,0} = w_{k,1} - \tilde{w}_{2,1}, \\
\alpha_{2,k,2} &= r_k^2 - \tilde{r}_2^2, \quad \alpha_{2,k,1} = r_k w_{k,1} - \tilde{r}_2\tilde{w}_{2,1}, \quad \alpha_{2,k,0} = w_{k,2} - \tilde{w}_{2,2}, \\
\beta_{1,1} &= \tilde{r}_1 - \tilde{r}_2, \quad \beta_{1,0} = \tilde{w}_{1,1} - \tilde{w}_{2,1}, \\
\beta_{2,2} &= \tilde{r}_2^2 - \tilde{r}_1^2, \quad \beta_{2,1} = \tilde{r}_2\tilde{w}_{2,1} - \tilde{r}_1\tilde{w}_{1,1}, \quad \beta_{2,0} = \tilde{w}_{2,2} - \tilde{w}_{1,2}.
\end{aligned} \tag{9}$$

Add $\beta_{1,1}$ times (8b) at $t-1$ to $\beta_{1,0}$ times (8b) at t, then use (8a) to eliminate $\tilde{e}_{1,t}$:

$$\begin{aligned}
0 = \sum_{j=0}^{1} \beta_{1,j} &\left[\delta_2 + \sum_{k=1}^{2}\sum_{\ell=0}^{2} \alpha_{2,k,\ell} e_{k,t-\ell-j}\right] \\
&+ \sum_{\ell=0}^{2} \beta_{2,\ell} \left[\delta_1 + \sum_{k=1}^{2}\sum_{j=0}^{1} \alpha_{1,k,j} e_{k,t-\ell-j}\right], \quad t = 4, \ldots, T.
\end{aligned} \tag{10}$$

Collecting terms:

$$0 = \sum_{\ell=0}^{1} \beta_{1,\ell}\delta_2 + \sum_{\ell=0}^{2} \beta_{2,\ell}\delta_1 + \sum_{k=1}^{2}\sum_{\ell=0}^{2}\sum_{j=0}^{1}(\beta_{1,j}\alpha_{2,k,\ell} + \beta_{2,\ell}\alpha_{1,k,j})e_{k,t-\ell-j},$$

$$t = 4, \ldots, T. \quad (11)$$

Combining all coefficients on $e_{k,s}$ for $s = t-3, \ldots, t$:

$$\begin{aligned} 0 = &\sum_{\ell=0}^{1} \beta_{1,\ell}\delta_2 + \sum_{\ell=0}^{2} \beta_{2,\ell}\delta_1 \\ &+ \sum_{k=1}^{2}\Bigg[(\beta_{1,1}\alpha_{2,k,2} + \beta_{2,2}\alpha_{1,k,1})e_{k,t-3} \\ &\qquad + (\beta_{1,0}\alpha_{2,k,2} + \beta_{2,2}\alpha_{1,k,0} + \beta_{1,1}\alpha_{2,k,1} + \beta_{2,1}\alpha_{1,k,1})e_{k,t-2} \\ &\qquad + (\beta_{1,0}\alpha_{2,k,1} + \beta_{2,1}\alpha_{1,k,0} + \beta_{1,1}\alpha_{2,k,0} + \beta_{2,0}\alpha_{1,k,1})e_{k,t-1} \\ &\qquad + (\beta_{1,0}\alpha_{2,k,0} + \beta_{2,0}\alpha_{1,k,0})e_{k,t}\Bigg], \\ &t = 4, \ldots, T. \end{aligned} \quad (12)$$

Since $T \geq 12$, (12) contains at least nine equations. But since $\boldsymbol{\theta} \notin X_{2,4}$, the constant and the coefficients on $e_{1,t-3}, e_{1,t-2}, e_{1,t-1}, e_{1,t}, e_{2,t-3}, e_{2,t-2}, e_{2,t-1}, e_{2,t}$ must all equal zero. In particular,

$$0 = \sum_{\ell=0}^{1} \beta_{1,\ell}\delta_2 + \sum_{\ell=0}^{2} \beta_{2,\ell}\delta_1, \tag{13a}$$

$$0 = \beta_{1,1}\alpha_{2,k,2} + \beta_{2,2}\alpha_{1,k,1}, \quad k \in \{1,2\}, \tag{13b}$$

$$0 = \beta_{1,0}\alpha_{2,k,2} + \beta_{2,2}\alpha_{1,k,0} + \beta_{1,1}\alpha_{2,k,1} + \beta_{2,1}\alpha_{1,k,1}, \quad k \in \{1,2\}, \tag{13c}$$

$$0 = \beta_{1,0}\alpha_{2,k,1} + \beta_{2,1}\alpha_{1,k,0} + \beta_{1,1}\alpha_{2,k,0} + \beta_{2,0}\alpha_{1,k,1}, \quad k \in \{1,2\}. \tag{13d}$$

Substituting (9) into (13b)–(13d) and collecting terms, we have

$$0 = (\tilde{r}_1 - \tilde{r}_2)(r_k - \tilde{r}_2)(r_k - \tilde{r}_1), \quad k \in \{1,2\}, \tag{14a}$$

$$\begin{aligned} 0 = {} & \tilde{w}_{1,1}(r_k^2 - \tilde{r}_2^2) - \tilde{w}_{2,1}(r_k^2 - \tilde{r}_1^2) + w_{k,1}(\tilde{r}_2^2 - \tilde{r}_1^2) + \tilde{r}_1 r_k w_{k,1} - \tilde{r}_1\tilde{r}_2\tilde{w}_{2,1} - \tilde{r}_2 r_k w_{k,1} \\ & + \tilde{r}_2\tilde{w}_{2,1} r_k - \tilde{r}_1 r_k \tilde{w}_{1,1} + \tilde{r}_1\tilde{r}_2\tilde{w}_{1,1}, \quad k \in \{1,2\}, \end{aligned} \tag{14b}$$

$$\begin{aligned} 0 = {} & (\tilde{w}_{1,1} - \tilde{w}_{2,1})(r_k w_{k,1} - \tilde{r}_2\tilde{w}_{2,1}) + (\tilde{r}_2\tilde{w}_{2,1} - \tilde{r}_1\tilde{w}_{1,1})(w_{k,1} - \tilde{w}_{2,1}) \\ & + (\tilde{r}_1 - \tilde{r}_2)(w_{k,2} - \tilde{w}_{2,2}) + (\tilde{w}_{2,2} - \tilde{w}_{1,2})(r_k - \tilde{r}_2), \quad k \in \{1,2\}. \end{aligned} \tag{14c}$$

Equation (14a) implies that either:

Case 2(a): $\tilde{r}_1 = \tilde{r}_2$; or

Case 2(b): Since $r_1 \neq r_2$ (because $\boldsymbol{\theta} \notin X_{2,2}$), and given normalization 1, $\tilde{r}_1 = r_1$ and $\tilde{r}_2 = r_2$.

Suppose case 2(a) obtains so that $\tilde{r}_1 = \tilde{r}_2$. Then (14b) reduces to

$$0 = r_k(\tilde{w}_{1,1} - \tilde{w}_{2,1})(r_k - \tilde{r}_1), \quad k \in \{1, 2\}. \tag{15}$$

Since $r_k \neq 0$ (because $\boldsymbol{\theta} \notin X_{2,1}$), either $\tilde{w}_{1,1} = \tilde{w}_{2,1}$ or $\tilde{r}_1 = r_1 = r_2$. The latter violates the assumption that $\boldsymbol{\theta} \notin X_{2,2}$. Thus $\tilde{w}_{1,1} = \tilde{w}_{2,1}$. But now (7a) reduces to

$$\begin{aligned}(\mu - \tilde{\mu})(1 - \tilde{r}_1 - \tilde{w}_{1,1}) + (r_1 - \tilde{r}_1)e_{1,t-1} + (w_{1,1} - \tilde{w}_{1,1})e_{1,t} \\ + (r_2 - \tilde{r}_1)e_{2,t-1} + (w_{2,1} - \tilde{w}_{1,1})e_{2,t} = 0, \quad t = 2, \ldots, T.\end{aligned} \tag{16}$$

Since $\boldsymbol{\theta} \notin X_{2,4}$, all of the coefficients in (16) must be zero. Hence $\tilde{r}_1 = r_1 = r_2$, which contradicts $\boldsymbol{\theta} \notin X_{2,2}$. Therefore case 2(a) does not obtain; case 2(b) obtains, and we have

$$\tilde{r}_k = r_k, \quad k \in \{1, 2\}. \tag{17}$$

Substituting (17) into (14b) yields $0 = (\tilde{w}_{1,1} - w_{1,1})r_2(r_1 - r_2) = (\tilde{w}_{2,1} - w_{2,1})r_1(r_1 - r_2)$, which, given $\boldsymbol{\theta} \notin X_{2,1} \cup X_{2,2}$, implies

$$\tilde{w}_{k,1} = w_{k,1}, \quad k \in \{1, 2\}. \tag{18}$$

Substituting (17) and (18) into (14c) for $k = 2$ yields $0 = (r_1 - r_2)(w_{2,2} - \tilde{w}_{2,2})$, which, given $\boldsymbol{\theta} \notin X_{2,2}$, implies

$$\tilde{w}_{2,2} = w_{2,2}. \tag{19}$$

Substituting (17), (18), and (19) into (14c) for $k = 1$ then yields $0 = (r_1 - r_2)(w_{1,2} - \tilde{w}_{1,2})$, which, given $\boldsymbol{\theta} \notin X_{2,2}$, implies

$$\tilde{w}_{1,2} = w_{1,2}. \tag{20}$$

Substituting (9), (17), (18), (19), and (20) into (13a) and collecting terms yields

$$\begin{aligned} 0 = (\mu - \tilde{\mu})[w_{1,1} - w_{2,1} + r_1 - r_2 + (r_2^2 + r_2 w_{2,1} + w_{2,2})(1 - r_1 - w_{1,1}) \\ - (w_{1,2} + r_1 w_{1,1} + r_1^2)(1 - r_2 - w_{2,1})], \end{aligned} \tag{21}$$

which, given $\boldsymbol{\theta} \notin X_{2,6}$, implies

$$\tilde{\mu} = \mu. \tag{22}$$

Substituting (17), (18), and (22) into (7a) yields

$$(r_1 - r_2)(e_{1,t-1} - \tilde{e}_{1,t-1}) + (w_{1,1} - w_{2,1})(e_{1,t} - \tilde{e}_{1,t}) = 0, \quad t = 2, \ldots, T, \tag{23}$$

and since $\boldsymbol{\theta} \notin X_{2,3}$, we have

$$e_{1,t} - \tilde{e}_{1,t} = \left(\frac{r_2 - r_1}{w_{1,1} - w_{2,1}} \right)^{t-1} (e_{1,1} - \tilde{e}_{1,1}), \quad t = 2, \ldots, T. \tag{24}$$

Substituting (17), (18), (19), (20), (22), and (24) into (7b) and collecting terms yields

$$\begin{aligned} [(w_{1,1} - w_{2,1})(r_1 w_{2,1} - r_2 w_{1,1}) + (w_{1,2} - w_{2,2})(r_2 - r_1)] \\ \cdot \left(\frac{r_2 - r_1}{w_{1,1} - w_{2,1}} \right)^{t-2} \frac{1}{w_{1,1} - w_{2,1}} (e_{1,1} - \tilde{e}_{1,1}) = 0, \quad t = 3, \ldots, T, \end{aligned} \tag{25}$$

which, given $\boldsymbol{\theta} \notin X_{2,2} \cup X_{2,5}$, implies

$$\tilde{e}_{1,1} = e_{1,1}. \tag{26}$$

Equation (24) then gives

$$\tilde{e}_{1,t} = e_{1,t}, \quad t = 2, \ldots, T, \tag{27}$$

and equation (6a) then yields

$$\tilde{e}_{2,t} = e_{2,t}, \quad t = 1, \ldots, T. \tag{28}$$

Equations (17), (18), (19), (20), (22), (26), (27), and (28) identify $e_{k,t}$ for $t = 1, \ldots, T$ as well as r_k, $w_{k,1}$, $w_{k,2}$, and μ. It remains to show that $\tilde{e}_{k,t} = e_{k,t}$ for $t \leq 0$ and (if $A > 2$) that $\tilde{w}_{k,a} = w_{k,a}$ for $a > 2$. First consider identification of $w_{k,a}$. Fix any $a > 2$. If $\tilde{w}_{k,a'} = w_{k,a'}$ for $k = 1, 2$ and all $a' < a$, then, using all of the foregoing results, equation (1) at a reduces to

$$\sum_{k=1}^{2} (w_{k,a} - \tilde{w}_{k,a}) e_{k,t} = 0, \quad t = a + 1, \ldots, T. \tag{29}$$

Since $T \geq A + K$, (29) contains at least two equations, and since $\boldsymbol{\theta} \notin X_{2,7}$, these equations have the unique solution $\tilde{w}_{k,a} = w_{k,a}$. By induction, this gives $\tilde{w}_{k,a} = w_{k,a}$ for all $a > 2$.

Finally, consider identification of $e_{k,t}$ for $t \leq 0$. Fix any $s \in \{0, -1, \ldots, 2 - A\}$. If $\tilde{e}_{k,\tau} = e_{k,\tau}$ for $\tau = s + 1, \ldots, T$, then, using all of the foregoing results, equation (1) at

$t=1, a=1-s$ and $t=2, a=2-s$ reduces to

$$0=\sum_{k=1}^{2} r_k^{1-s}(e_{k,s}-\tilde{e}_{k,s})=\sum_{k=1}^{2} r_k^{2-s}(e_{k,s}-\tilde{e}_{k,s}). \tag{30}$$

Since $\boldsymbol{\theta}\notin X_{2,1}\cup X_{2,2}$, $\det\left[\begin{smallmatrix} r_1^{1-s} & r_2^{1-s} \\ r_1^{2-s} & r_2^{2-s}\end{smallmatrix}\right]\neq 0$; therefore the unique solution to (30) is $\tilde{e}_{k,s}=e_{k,s}$. By induction, this gives $\tilde{e}_{k,t}=e_{k,t}$ for $t=0,-1,\ldots,2-A$. Applying normalization (3) and the foregoing results to (1) at $t=1, a=A$ gives $\tilde{e}_{k,1-A}=e_{k,1-A}$. We have now identified all parameters for the case $K=2$.

Case 3: $K=3$**.**

Define the following matrices:

$$\mathbf{B}'_{1,2}=\begin{bmatrix} -[\beta_{1,2,1}\beta_{2,1,0}+\beta_{1,2,0}\beta_{2,1,1}]/\beta_{1,2,0}\beta_{2,1,0} & 1 & 0 & 0 \\ -[\beta_{1,2,0}\beta_{2,1,2}+\beta_{1,2,1}\beta_{2,1,1}]/\beta_{1,2,0}\beta_{2,1,0} & 0 & 1 & 0 \\ -\beta_{1,2,1}\beta_{2,1,2}/\beta_{1,2,0}\beta_{2,1,0} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}'_{1,3}=\begin{bmatrix} -[\beta_{1,2,1}\beta_{3,1,0}-\beta_{1,1,1}\beta_{3,2,0}+\beta_{1,2,0}\beta_{3,1,1}-\beta_{1,1,0}\beta_{3,2,1}]/[\beta_{1,2,0}\beta_{3,1,0}-\beta_{1,1,0}\beta_{3,2,0}] & 1 & 0 & 0 \\ -[\beta_{1,2,0}\beta_{3,1,2}-\beta_{1,1,0}\beta_{3,2,2}+\beta_{1,2,1}\beta_{3,1,1}-\beta_{1,1,1}\beta_{3,2,1}]/[\beta_{1,2,0}\beta_{3,1,0}-\beta_{1,1,0}\beta_{3,2,0}] & 0 & 1 & 0 \\ -[\beta_{1,2,0}\beta_{3,1,3}-\beta_{1,1,0}\beta_{3,2,3}+\beta_{1,2,1}\beta_{3,1,2}-\beta_{1,1,1}\beta_{3,2,2}]/[\beta_{1,2,0}\beta_{3,1,0}-\beta_{1,1,0}\beta_{3,2,0}] & 0 & 0 & 1 \\ -[\beta_{1,2,1}\beta_{3,1,3}-\beta_{1,1,1}\beta_{3,2,3}]/[\beta_{1,2,0}\beta_{3,1,0}-\beta_{1,1,0}\beta_{3,2,0}] & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_{2,1}=-\begin{bmatrix} \beta_{1,2,0}^{-1}\beta_{1,2,1} & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{B}_{2,2} = \begin{bmatrix} -\beta_{2,2,0}^{-1}\beta_{2,2,1} & -\beta_{2,2,0}^{-1}\beta_{2,2,2} \\ 1 & 0 \end{bmatrix},$$

where equation (70) below defines $\beta_{j,k,\ell}$ as polynomials in $\{r_k, w_{k,1}, w_{k,2}, w_{k,3}\}_{k=1}^{3}$. (Equation (34) below also gives definitions of $\beta_{j,k,\ell}$, but they are not the definitions we refer to here.)

Define the following sets:

$$X_{3,1} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \exists k \text{ s.t. } r_k = 0\}$$

$$X_{3,2} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \exists k \neq k' \text{ s.t. } r_k = r_k'\}$$

$$X_{3,3} = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \operatorname{rank} \begin{bmatrix} 1 & \begin{pmatrix} e_{k,j} \\ \vdots \\ e_{k,T-4+j} \end{pmatrix}_{\substack{k \in \{1,2,3\}, \\ j \in \{1,2,3,4,5,6,7,8\}}} \\ \vdots & \\ 1 & \end{bmatrix} < 25 \right\}$$

$$X_{3,4} = \left\{ \boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \operatorname{rank} \begin{bmatrix} 1 & w_{1,1} & w_{1,2} \\ 1 & w_{2,1} & w_{2,2} \\ 1 & w_{3,1} & w_{3,2} \end{bmatrix} < 3 \right\}$$

$$\begin{aligned}
X_{3,5} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon 0 = \\
&[(w_{3,1} - w_{2,1} + r_3 - r_2)(w_{3,2} - w_{1,2} + r_3^2 - r_1^2 + r_3 w_{3,1} - r_1 w_{1,1}) \\
&\quad - (w_{3,1} - w_{1,1} + r_3 - r_1)(w_{3,2} - w_{2,2} + r_3^2 - r_2^2 + r_3 w_{3,1} - r_2 w_{2,1})] \\
&\cdot [(w_{3,1} - w_{2,1} + r_3 - r_2)(1 - r_3^3 - r_3^2 w_{3,1} - r_3 w_{3,2} - w_{3,3}) \\
&\quad - (w_{3,3} - w_{2,3} + r_3 w_{3,2} - r_2 w_{2,2} + r_3^2 w_{3,1} - r_2^2 w_{2,1} + r_3^3 - r_2^3)(1 - r_3 - w_{3,1})] \\
&- [(w_{3,1} - w_{2,1} + r_3 - r_2)(w_{3,3} - w_{1,3} + r_3 w_{3,2} - r_1 w_{1,2} + r_3^2 w_{3,1} - r_1^2 w_{1,1} + r_3^3 - r_1^3) \\
&\quad - (w_{3,1} - w_{1,1} + r_3 - r_1)(w_{3,3} - w_{2,3} + r_3 w_{3,2} - r_2 w_{2,2} + r_3^2 w_{3,1} - r_2^2 w_{2,1} + r_3^3 - r_2^3)] \\
&\cdot [(w_{3,1} - w_{2,1} + r_3 - r_2)(1 - r_3^2 - r_3 w_{3,1} - w_{3,2}) \\
&\quad - (w_{3,2} - w_{2,2} + r_3 w_{3,1} - r_2 w_{2,1} + r_3^2 - r_2^2)(1 - r_3 - w_{3,1})]\}
\end{aligned}$$

$$X_{3,6} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon (w_{3,1} - w_{2,1})(w_{3,2} - w_{1,2}) = 0\}$$

$$X_{3,7} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon (w_{3,1} - w_{2,1})(w_{3,3} - w_{1,3}) - (w_{3,1} - w_{1,1})(w_{3,3} - w_{2,3}) = 0\}$$

$$X_{3,8} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \mathbf{B}_{1,2} = \mathbf{B}_{1,3} \text{ or } \operatorname{rank}(\mathbf{B}_{1,3}) < 4\}$$

$$X_{3,9} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon w_{3,2} - w_{2,2} = 0\}$$

$$X_{3,10} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \mathbf{B}_{2,1} = \mathbf{B}_{2,2} \text{ or } \operatorname{rank}(\mathbf{B}_{2,2}) < 2\}$$

$$X_{3,11} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \operatorname{rank} \begin{bmatrix} e_{1,A+1} & e_{2,A+1} & e_{3,A+1} \\ e_{1,A+2} & e_{2,A+2} & e_{3,A+2} \\ e_{1,A+3} & e_{2,A+3} & e_{3,A+3} \end{bmatrix} < 3\}$$

$$X_{3,12} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \exists k, k' \text{ s.t. } r_k + r_{k'} = 0\}$$

$$X_{3,13} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon (w_{3,1} - w_{2,1})(w_{3,2} - w_{1,2}) - (w_{3,1} - w_{1,1})(w_{3,2} - w_{2,2}) = 0\}$$

$$X_{3,14} = \{\boldsymbol{\theta} \in \boldsymbol{\Theta} \colon \exists a, b \in \mathbb{R} \text{ s.t. } w_{k,2} = a + b w_{k,1},\ k = 1, 2, 3\}$$

Let $X_3 = \cup_{j=1}^{1} 4 X_{3,j}$. Since X_3 is a finite union of sets of measure zero, X_3 is a set of measure zero.

Assume $\boldsymbol{\theta} \in \boldsymbol{\Theta} \setminus X_3$. Specializing (1) to $K = 3$, $a = 0$, $a = 1$, $a = 2$, and $a = 3$ (by hypothesis, $A \geq 3$) and using normalization 2, we have

$$\mu + e_{1,t} + e_{2,t} + e_{3,t} = \tilde{\mu} + \tilde{e}_{1,t} + \tilde{e}_{2,t} + \tilde{e}_{3,t}, \quad t = 1, \ldots, T, \tag{31a}$$

$$\begin{aligned} &\mu + r_1 e_{1,t-1} + w_{1,1} e_{1,t} + r_2 e_{2,t-1} + w_{2,1} e_{2,t} + r_3 e_{3,t-1} + w_{3,1} e_{3,t} \\ &\quad = \tilde{\mu} + \tilde{r}_1 \tilde{e}_{1,t-1} + \tilde{w}_{1,1} \tilde{e}_{1,t} + \tilde{r}_2 \tilde{e}_{2,t-1} + \tilde{w}_{2,1} \tilde{e}_{2,t} + \tilde{r}_3 \tilde{e}_{3,t-1} + \tilde{w}_{3,1} \tilde{e}_{3,t}, \\ &\qquad t = 2, \ldots, T, \end{aligned} \tag{31b}$$

$$
\begin{aligned}
\mu + r_1^2 e_{1,t-2} + r_1 w_{1,1} e_{1,t-1} + w_{1,2} e_{1,t} + r_2^2 e_{2,t-2} + r_2 w_{2,1} e_{2,t-1} + w_{2,2} e_{2,t} \\
+ r_3^2 e_{3,t-2} + r_3 w_{3,1} e_{3,t-1} + w_{3,2} e_{3,t} \\
= \tilde{\mu} + \tilde{r}_1^2 \tilde{e}_{1,t-2} + \tilde{r}_1 \tilde{w}_{1,1} \tilde{e}_{1,t-1} + \tilde{w}_{1,2} \tilde{e}_{1,t} + \tilde{r}_2^2 \tilde{e}_{2,t-2} + \tilde{r}_2 \tilde{w}_{2,1} \tilde{e}_{2,t-1} + \tilde{w}_{2,2} \tilde{e}_{2,t} \\
+ \tilde{r}_3^2 \tilde{e}_{3,t-2} + \tilde{r}_3 \tilde{w}_{3,1} \tilde{e}_{3,t-1} + \tilde{w}_{3,2} \tilde{e}_{3,t}, \\
t = 3, \ldots, T. \quad (31c)
\end{aligned}
$$

$$
\begin{aligned}
\mu + r_1^3 e_{1,t-3} + r_1^2 w_{1,1} e_{1,t-2} + r_1 w_{1,2} e_{1,t-1} + w_{1,3} e_{1,t} \\
+ r_2^3 e_{2,t-3} + r_2^2 w_{2,1} e_{2,t-2} + r_2 w_{2,2} e_{2,t-1} + w_{2,3} e_{2,t} \\
+ r_3^3 e_{3,t-3} + r_3^2 w_{3,1} e_{3,t-2} + r_3 w_{3,2} e_{3,t-1} + w_{3,3} e_{3,t} \\
= \tilde{\mu} + \tilde{r}_1^3 \tilde{e}_{1,t-3} + \tilde{r}_1^2 \tilde{w}_{1,1} \tilde{e}_{1,t-2} + \tilde{r}_1 \tilde{w}_{1,2} \tilde{e}_{1,t-1} + \tilde{w}_{1,3} \tilde{e}_{1,t} \\
+ \tilde{r}_2^3 \tilde{e}_{2,t-3} + \tilde{r}_2^2 \tilde{w}_{2,1} \tilde{e}_{2,t-2} + \tilde{r}_2 \tilde{w}_{2,2} \tilde{e}_{2,t-1} + \tilde{w}_{2,3} \tilde{e}_{2,t} \\
+ \tilde{r}_3^3 \tilde{e}_{3,t-3} + \tilde{r}_3^2 \tilde{w}_{3,1} \tilde{e}_{3,t-2} + \tilde{r}_3 \tilde{w}_{3,2} \tilde{e}_{3,t-1} + \tilde{w}_{3,3} \tilde{e}_{3,t}, \\
t = 4, \ldots, T. \quad (31d)
\end{aligned}
$$

Solving (31a) for $\tilde{e}_{3,t}$, substituting into (31b), (31c), and (31d), and collecting terms yields

$$
\begin{aligned}
(\mu - \tilde{\mu})(1 - \tilde{r}_3 - \tilde{w}_{3,1}) + (r_1 - \tilde{r}_3) e_{1,t-1} + (w_{1,1} - \tilde{w}_{3,1}) e_{1,t} \\
+ (r_2 - \tilde{r}_3) e_{2,t-1} + (w_{2,1} - \tilde{w}_{3,1}) e_{2,t} + (r_3 - \tilde{r}_3) e_{3,t-1} + (w_{3,1} - \tilde{w}_{3,1}) e_{3,t} \\
= (\tilde{r}_1 - \tilde{r}_3) \tilde{e}_{1,t-1} + (\tilde{w}_{1,1} - \tilde{w}_{3,1}) \tilde{e}_{1,t} + (\tilde{r}_2 - \tilde{r}_3) \tilde{e}_{2,t-1} + (\tilde{w}_{2,1} - \tilde{w}_{3,1}) \tilde{e}_{2,t}, \\
t = 2, \ldots, T, \quad (32a)
\end{aligned}
$$

$$(\mu-\tilde{\mu})(1-\tilde{r}_3^2-\tilde{r}_3\tilde{w}_{3,1}-\tilde{w}_{3,2})+(r_1^2-\tilde{r}_3^2)e_{1,t-2}+(r_1w_{1,1}-\tilde{r}_3\tilde{w}_{3,1})e_{1,t-1}+(w_{1,2}-\tilde{w}_{3,2})e_{1,t}$$

$$+(r_2^2-\tilde{r}_3^2)e_{2,t-2}+(r_2w_{2,1}-\tilde{r}_3\tilde{w}_{3,1})e_{2,t-1}+(w_{2,2}\tilde{w}_{3,2})e_{2,t}$$

$$+(r_3^2-\tilde{r}_3^2)e_{3,t-2}+(r_3w_{3,1}-\tilde{r}_3\tilde{w}_{3,1})e_{3,t-1}+(w_{3,2}-\tilde{w}_{3,2})e_{3,t}$$

$$=(\tilde{r}_1^2-\tilde{r}_3^2)\tilde{e}_{1,t-2}+(\tilde{r}_1\tilde{w}_{1,1}-\tilde{r}_3\tilde{w}_{3,1})\tilde{e}_{1,t-1}+(\tilde{w}_{1,2}-\tilde{w}_{3,2})\tilde{e}_{1,t}$$

$$+(\tilde{r}_2^2-\tilde{r}_3^2)\tilde{e}_{2,t-2}+(\tilde{r}_2\tilde{w}_{2,1}-\tilde{r}_3\tilde{w}_{3,1})\tilde{e}_{2,t-1}+)(\tilde{w}_{2,2}-\tilde{w}_{3,2})\tilde{e}_{2,t},$$

$$t=3,\ldots,T. \quad (32b)$$

$$(\mu-\tilde{\mu})(1-\tilde{r}_3^3-\tilde{r}_3^2\tilde{w}_{3,1}-\tilde{r}_3\tilde{w}_{3,2}-\tilde{w}_{3,3})$$

$$+(r_1^3-\tilde{r}_3^3)e_{1,t-3}+(r_1^2w_{1,1}-\tilde{r}_3^2\tilde{w}_{3,1})e_{1,t-2}+(r_1w_{1,2}-\tilde{r}_3\tilde{w}_{3,2})e_{1,t-1}+(w_{1,3}-\tilde{w}_{3,3})e_{1,t}$$

$$+(r_2^3-\tilde{r}_3^3)e_{2,t-3}+(r_2^2w_{2,1}-\tilde{r}_3^2\tilde{w}_{3,1})e_{2,t-2}+(r_2w_{2,2}-\tilde{r}_3\tilde{w}_{3,2})e_{2,t-1}+(w_{2,3}-\tilde{w}_{3,3})e_{2,t}$$

$$+(r_3^3-\tilde{r}_3^3)e_{3,t-3}+(r_3^2w_{3,1}-\tilde{r}_3^2\tilde{w}_{3,1})e_{3,t-2}+(r_3w_{3,2}-\tilde{r}_3\tilde{w}_{3,2})e_{3,t-1}+(w_{3,3}-\tilde{w}_{3,3})e_{3,t}$$

$$=(\tilde{r}_1^3-\tilde{r}_3^3)\tilde{e}_{1,t-3}+(\tilde{r}_1^2\tilde{w}_{1,1}-\tilde{r}_3^2\tilde{w}_{3,1})\tilde{e}_{1,t-2}+(\tilde{r}_1\tilde{w}_{1,2}-\tilde{r}_3\tilde{w}_{3,2})\tilde{e}_{1,t-1}+(\tilde{w}_{1,3}-\tilde{w}_{3,3})\tilde{e}_{1,t}$$

$$+(\tilde{r}_2^3-\tilde{r}_3^3)\tilde{e}_{2,t-3}+(\tilde{r}_2^2\tilde{w}_{2,1}-\tilde{r}_3^2\tilde{w}_{3,1})\tilde{e}_{2,t-2}+(\tilde{r}_2\tilde{w}_{2,2}-\tilde{r}_3\tilde{w}_{3,2})\tilde{e}_{2,t-1}+(\tilde{w}_{2,3}-\tilde{w}_{3,3})\tilde{e}_{2,t},$$

$$t=4,\ldots,T. \quad (32c)$$

Rewrite (32) as

$$-\sum_{\ell=0}^{1}\beta_{1,2,\ell}\tilde{e}_{2,t-\ell} = \delta_1 + \sum_{k=1}^{3}\sum_{\ell=0}^{1}\alpha_{1,k,\ell}e_{k,t-\ell} + \sum_{\ell=0}^{1}\beta_{1,1,\ell}\tilde{e}_{1,t-\ell}, \quad t = 2,\dots,T, \tag{33a}$$

$$0 = \delta_2 + \sum_{k=1}^{3}\sum_{\ell=0}^{2}\alpha_{2,k,\ell}e_{k,t-\ell} + \sum_{\ell=0}^{2}(\beta_{2,1,\ell}\tilde{e}_{1,t-\ell} + \beta_{2,2,\ell}\tilde{e}_{2,t-\ell}), \quad t = 3,\dots,T, \tag{33b}$$

$$0 = \delta_3 + \sum_{k=1}^{3}\sum_{\ell=0}^{3}\alpha_{3,k,\ell}e_{k,t-\ell} + \sum_{\ell=0}^{3}(\beta_{3,1,\ell}\tilde{e}_{1,t-\ell} + \beta_{3,2,\ell}\tilde{e}_{2,t-\ell}), \quad t = 4,\dots,T, \tag{33c}$$

where

$$\begin{aligned}
\delta_1 &= (\mu - \tilde{\mu})(1 - \tilde{r}_3 - \tilde{w}_{3,1}),\\
\delta_2 &= (\mu - \tilde{\mu})(1 - \tilde{r}_3^2 - \tilde{r}_3\tilde{w}_{3,1} - \tilde{w}_{3,2}),\\
\delta_3 &= (\mu - \tilde{\mu})(1 - \tilde{r}_3^3 - \tilde{r}_3^2\tilde{w}_{3,1} - \tilde{r}_3\tilde{w}_{3,2} - \tilde{w}_{3,3}),\\
\alpha_{j,k,j} &= r_k^j - \tilde{r}_3^j, \quad \beta_{j,k,j} = \tilde{r}_3^j - \tilde{r}_k^j, \quad j = 1,\dots,3,\\
\alpha_{j,k,0} &= w_{k,j} - \tilde{w}_{3,j}, \quad \beta_{j,k,0} = \tilde{w}_{3,j} - \tilde{w}_{k,j}, \quad j = 1,\dots,3,\\
\alpha_{2,k,1} &= r_k w_{k,1} - \tilde{r}_3\tilde{w}_{3,1}, \quad \beta_{2,k,1} = \tilde{r}_3\tilde{w}_{3,1} - \tilde{r}_k\tilde{w}_{k,1},\\
\alpha_{3,k,2} &= r_k^2 w_{k,1} - \tilde{r}_3^2\tilde{w}_{3,1}, \quad \alpha_{3,k,1} = r_k w_{k,2} - \tilde{r}_3\tilde{w}_{3,2},\\
\beta_{3,k,2} &= \tilde{r}_3^2\tilde{w}_{3,1} - \tilde{r}_k^2\tilde{w}_{k,1}, \quad \beta_{3,k,1} = \tilde{r}_3\tilde{w}_{3,2} - \tilde{r}_k\tilde{w}_{k,2}.
\end{aligned} \tag{34}$$

Add $\beta_{1,2,1}$ times (33b) at $t-1$ to $\beta_{1,2,0}$ times (33b) at t, then use (33a) to eliminate $\tilde{e}_{2,t}$, and do likewise with (33c):

$$0 = \sum_{j=0}^{1} \beta_{1,2,j} \left[\delta_2 + \sum_{k=1}^{3} \sum_{\ell=0}^{2} \alpha_{2,k,\ell} e_{k,t-\ell-j} + \sum_{\ell=0}^{2} \beta_{2,1,\ell} \tilde{e}_{1,t-\ell-j} \right]$$
$$- \sum_{\ell=0}^{2} \beta_{2,2,\ell} \left[\delta_1 + \sum_{k=1}^{3} \sum_{j=0}^{1} \alpha_{1,k,j} e_{k,t-\ell-j} + \sum_{j=0}^{1} \beta_{1,1,j} \tilde{e}_{1,t-\ell-j} \right]$$
$$t = 4, \ldots, T, \quad \text{(35a)}$$

$$0 = \sum_{j=0}^{1} \beta_{1,2,j} \left[\delta_3 + \sum_{k=1}^{3} \sum_{\ell=0}^{3} \alpha_{3,k,\ell} e_{k,t-\ell-j} + \sum_{\ell=0}^{3} \beta_{3,1,\ell} \tilde{e}_{1,t-\ell-j} \right]$$
$$- \sum_{\ell=0}^{3} \beta_{3,2,\ell} \left[\delta_1 + \sum_{k=1}^{3} \sum_{j=0}^{1} \alpha_{1,k,j} e_{k,t-\ell-j} + \sum_{j=0}^{1} \beta_{1,1,j} \tilde{e}_{1,t-\ell-j} \right],$$
$$t = 5, \ldots, T. \quad \text{(35b)}$$

Collecting terms:

$$- \sum_{\ell=0}^{3} \gamma_\ell \tilde{e}_{1,t-\ell} = \sum_{j=0}^{1} \beta_{1,2,j} \delta_2 - \sum_{\ell=0}^{2} \beta_{2,2,\ell} \delta_1 + \sum_{k=1}^{3} \sum_{\ell=0}^{2} \sum_{j=0}^{1} (\beta_{1,2,j} \alpha_{2,k,\ell} - \beta_{2,2,\ell} \alpha_{1,k,j}) e_{k,t-\ell-j},$$
$$t = 3, \ldots, T, \quad \text{(36a)}$$

$$0 = \sum_{j=0}^{1} \beta_{1,2,j} \delta_3 - \sum_{\ell=0}^{3} \beta_{3,2,\ell} \delta_1 + \sum_{k=1}^{3} \sum_{\ell=0}^{3} \sum_{j=0}^{1} (\beta_{1,2,j} \alpha_{3,k,\ell} - \beta_{3,2,\ell} \alpha_{1,k,j}) e_{k,t-\ell-j}$$
$$+ \sum_{j=0}^{1} \sum_{\ell=0}^{3} (\beta_{1,2,j} \beta_{3,1,\ell} - \beta_{3,2,\ell} \beta_{1,1,j}) \tilde{e}_{1,t-\ell-j}, \quad t = 4, \ldots, T, \quad \text{(36b)}$$

where

$$\begin{aligned}
\gamma_0 &= \beta_{1,2,0}\beta_{2,1,0} - \beta_{1,1,0}\beta_{2,2,0}, \\
\gamma_1 &= \beta_{1,2,0}\beta_{2,1,1} - \beta_{1,1,0}\beta_{2,2,1} + \beta_{1,2,1}\beta_{2,1,0} - \beta_{1,1,1}\beta_{2,2,0}, \\
\gamma_2 &= \beta_{1,2,0}\beta_{2,1,2} - \beta_{1,1,0}\beta_{2,2,2} + \beta_{1,2,1}\beta_{2,1,1} - \beta_{1,1,1}\beta_{2,2,1}, \\
\gamma_3 &= \beta_{1,2,1}\beta_{2,1,2} - \beta_{1,1,1}\beta_{2,2,2}.
\end{aligned} \tag{37}$$

Now sum γ_j times (36b) at $t-j$ for $j = 0, \ldots, 3$, and use (36a) to eliminate $\tilde{e}_{1,t}$:

$$\begin{aligned}
0 = (\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3) &\left(\sum_{\ell=0}^{1} \beta_{1,2,\ell}\delta_3 - \sum_{\ell=0}^{3} \beta_{3,2,\ell}\delta_1 \right) \\
&- \left(\sum_{j=0}^{1}\sum_{\ell=0}^{3} (\beta_{1,2,j}\beta_{3,1,\ell} - \beta_{3,2,\ell}\beta_{1,1,j}) \right) \left(\sum_{\ell=0}^{1} \beta_{1,2,\ell}\delta_2 - \sum_{\ell=0}^{2} \beta_{2,2,\ell}\delta_1 \right) \\
&+ \sum_{k=1}^{3}\sum_{\ell=0}^{3}\sum_{j=0}^{1}\sum_{m=0}^{3} \gamma_m (\beta_{1,2,j}\alpha_{3,k,\ell} - \beta_{3,2,\ell}\alpha_{1,k,j}) e_{k,t-\ell-j-m} \\
&- \sum_{k=1}^{3}\sum_{\ell=0}^{3}\sum_{j=0}^{1}\sum_{\ell'=0}^{2}\sum_{j'=0}^{1} (\beta_{1,2,j}\beta_{3,1,\ell} - \beta_{3,2,\ell}\beta_{1,1,j})(\beta_{1,2,j'}\alpha_{2,k,\ell'} - \beta_{2,2,\ell'}\alpha_{1,k,j'}) e_{k,t-\ell-j-\ell'-j'}, \\
& t = 8, \ldots, T.
\end{aligned} \tag{38}$$

Equation (38) says that a linear combination of $e_{k,t}, \ldots, e_{k,t-7}$, and a constant is zero. With $T \geq 32$, there are at least 25 equations in (38); since $\boldsymbol{\theta} \notin X_{3,3}$, all of the coefficients in the linear combination must therefore be zero. Therefore, for $k \in \{1, 2, 3\}$, we have:

$$\begin{aligned}
0 = \gamma_3(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) \\
- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}),
\end{aligned} \tag{39a}$$

$$
\begin{aligned}
0 = \gamma_2(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) &+ \gamma_3(\beta_{1,2,0}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,1}) \\
&- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1}) \\
&- (\beta_{1,2,0}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (39b)
\end{aligned}
$$

$$
\begin{aligned}
0 = \gamma_1(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) &+ \gamma_2(\beta_{1,2,0}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,1}) \\
&+ \gamma_3(\beta_{1,2,0}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,1}) \\
&- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1}) \\
&-(\beta_{1,2,0}\beta_{3,1,3}-\beta_{3,2,3}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,2}-\beta_{2,2,2}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,1}) \\
&- (\beta_{1,2,0}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (39c)
\end{aligned}
$$

$$
\begin{aligned}
0 = \gamma_0(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) & \\
&+ \gamma_1(\beta_{1,2,0}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,1}) \\
&+ \gamma_2(\beta_{1,2,0}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,1}) \\
&+ \gamma_3(\beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1}) \\
&- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}) \\
&-(\beta_{1,2,0}\beta_{3,1,3}-\beta_{3,2,3}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,0}-\beta_{2,2,0}\alpha_{1,k,1}) \\
&-(\beta_{1,2,0}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,1}-\beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,2}-\beta_{2,2,2}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,1}) \\
&- (\beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (39d)
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \gamma_0(\beta_{1,2,0}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,1}) \\
& + \gamma_1(\beta_{1,2,0}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,1}) \\
& + \gamma_2(\beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1}) \\
& + \gamma_3(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0}) \\
& - (\beta_{1,2,0}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}) \\
& - (\beta_{1,2,0}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1}) \\
& - (\beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1}) \\
& - (\beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1} + \beta_{1,2,0}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,0}) \\
& - (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}),
\end{aligned} \tag{39e}
$$

$$
\begin{aligned}
0 = {} & \gamma_0(\beta_{1,2,0}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,1}) \\
& + \gamma_1(\beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1}) + \gamma_2(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0}) \\
& - (\beta_{1,2,0}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}) \\
& - (\beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1}) \\
& - (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1}),
\end{aligned} \tag{39f}
$$

$$
\begin{aligned}
0 = {} & \gamma_1(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0}) + \gamma_0(\beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1} + \beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0}) \\
& - (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1}) \\
& - (\beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1} + \beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}),
\end{aligned} \tag{39g}
$$

$$0 = \gamma_0(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0}) - (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}). \quad (39h)$$

Using the definitions of γ_0, γ_1, γ_2, and γ_3, (39) becomes:

$$0 = (\beta_{1,2,1}\beta_{2,1,2} - \beta_{1,1,1}\beta_{2,2,2})(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) - (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (40a)$$

$$\begin{aligned}0 &= (\beta_{1,2,0}\beta_{2,1,2} - \beta_{1,1,0}\beta_{2,2,2} + \beta_{1,2,1}\beta_{2,1,1} - \beta_{1,1,1}\beta_{2,2,1})(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) \\ &+ (\beta_{1,2,1}\beta_{2,1,2} - \beta_{1,1,1}\beta_{2,2,2})(\beta_{1,2,0}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,1}) \\ &- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1}) \\ &- (\beta_{1,2,0}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (40b)\end{aligned}$$

$$\begin{aligned}0 &= (\beta_{1,2,0}\beta_{2,1,1} - \beta_{1,1,0}\beta_{2,2,1} + \beta_{1,2,1}\beta_{2,1,0} - \beta_{1,1,1}\beta_{2,2,0})(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) \\ &+(\beta_{1,2,0}\beta_{2,1,2}-\beta_{1,1,0}\beta_{2,2,2}+\beta_{1,2,1}\beta_{2,1,1}-\beta_{1,1,1}\beta_{2,2,1})(\beta_{1,2,0}\alpha_{3,k,3}-\beta_{3,2,3}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{3,k,2}-\beta_{3,2,2}\alpha_{1,k,1}) \\ &+ (\beta_{1,2,1}\beta_{2,1,2} - \beta_{1,1,1}\beta_{2,2,2})(\beta_{1,2,0}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,1}) \\ &- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1}) \\ &-(\beta_{1,2,0}\beta_{3,1,3}-\beta_{3,2,3}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,2}-\beta_{2,2,2}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,1}) \\ &- (\beta_{1,2,0}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (40c)\end{aligned}$$

$$
\begin{aligned}
0 = &(\beta_{1,2,0}\beta_{2,1,0} - \beta_{1,1,0}\beta_{2,2,0})(\beta_{1,2,1}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,1}) \\
&+(\beta_{1,2,0}\beta_{2,1,1}-\beta_{1,1,0}\beta_{2,2,1}+\beta_{1,2,1}\beta_{2,1,0}-\beta_{1,1,1}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,3}-\beta_{3,2,3}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{3,k,2}-\beta_{3,2,2}\alpha_{1,k,1}) \\
&+(\beta_{1,2,0}\beta_{2,1,2}-\beta_{1,1,0}\beta_{2,2,2}+\beta_{1,2,1}\beta_{2,1,1}-\beta_{1,1,1}\beta_{2,2,1})(\beta_{1,2,0}\alpha_{3,k,2}-\beta_{3,2,2}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{3,k,1}-\beta_{3,2,1}\alpha_{1,k,1}) \\
&+ (\beta_{1,2,1}\beta_{2,1,2} - \beta_{1,1,1}\beta_{2,2,2})(\beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1}) \\
&- (\beta_{1,2,1}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}) \\
&-(\beta_{1,2,0}\beta_{3,1,3}-\beta_{3,2,3}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,0}-\beta_{2,2,0}\alpha_{1,k,1}) \\
&-(\beta_{1,2,0}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,1}-\beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,2}-\beta_{2,2,2}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,1}) \\
&- (\beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (40d)
\end{aligned}
$$

$$
\begin{aligned}
0 = &(\beta_{1,2,0}\beta_{2,1,0} - \beta_{1,1,0}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,3} - \beta_{3,2,3}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,1}) \\
&+(\beta_{1,2,0}\beta_{2,1,1}-\beta_{1,1,0}\beta_{2,2,1}+\beta_{1,2,1}\beta_{2,1,0}-\beta_{1,1,1}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,2}-\beta_{3,2,2}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{3,k,1}-\beta_{3,2,1}\alpha_{1,k,1}) \\
&+(\beta_{1,2,0}\beta_{2,1,2}-\beta_{1,1,0}\beta_{2,2,2}+\beta_{1,2,1}\beta_{2,1,1}-\beta_{1,1,1}\beta_{2,2,1})(\beta_{1,2,0}\alpha_{3,k,1}-\beta_{3,2,1}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{3,k,0}-\beta_{3,2,0}\alpha_{1,k,1}) \\
&+ (\beta_{1,2,1}\beta_{2,1,2} - \beta_{1,1,1}\beta_{2,2,2})(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0}) \\
&- (\beta_{1,2,0}\beta_{3,1,3} - \beta_{3,2,3}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}) \\
&-(\beta_{1,2,0}\beta_{3,1,2}-\beta_{3,2,2}\beta_{1,1,0}+\beta_{1,2,1}\beta_{3,1,1}-\beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1}-\beta_{2,2,1}\alpha_{1,k,0}+\beta_{1,2,1}\alpha_{2,k,0}-\beta_{2,2,0}\alpha_{1,k,1}) \\
&- (\beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1}) \\
&- (\beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1})(\beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1} + \beta_{1,2,0}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,0}) \\
&- (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,1}\alpha_{2,k,2} - \beta_{2,2,2}\alpha_{1,k,1}), \quad (40e)
\end{aligned}
$$

$$0 = (\beta_{1,2,0}\beta_{2,1,0} - \beta_{1,1,0}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,2} - \beta_{3,2,2}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,1})$$

$$+(\beta_{1,2,0}\beta_{2,1,1} - \beta_{1,1,0}\beta_{2,2,1} + \beta_{1,2,1}\beta_{2,1,0} - \beta_{1,1,1}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1})$$

$$+ (\beta_{1,2,0}\beta_{2,1,2} - \beta_{1,1,0}\beta_{2,2,2} + \beta_{1,2,1}\beta_{2,1,1} - \beta_{1,1,1}\beta_{2,2,1})(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0})$$

$$- (\beta_{1,2,0}\beta_{3,1,2} - \beta_{3,2,2}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0})$$

$$-(\beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0} + \beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1})$$

$$- (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,1}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,1}), \quad (40f)$$

$$0 = (\beta_{1,2,0}\beta_{2,1,1} - \beta_{1,1,0}\beta_{2,2,1} + \beta_{1,2,1}\beta_{2,1,0} - \beta_{1,1,1}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0})$$

$$+ (\beta_{1,2,0}\beta_{2,1,0} - \beta_{1,1,0}\beta_{2,2,0})(\beta_{1,2,1}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,1} + \beta_{1,2,0}\alpha_{3,k,1} - \beta_{3,2,1}\alpha_{1,k,0})$$

$$- (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,1} - \beta_{2,2,1}\alpha_{1,k,0} + \beta_{1,2,1}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,1})$$

$$- (\beta_{1,2,1}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,1} + \beta_{1,2,0}\beta_{3,1,1} - \beta_{3,2,1}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}), \quad (40g)$$

$$0 = (\beta_{1,2,0}\beta_{2,1,0} - \beta_{1,1,0}\beta_{2,2,0})(\beta_{1,2,0}\alpha_{3,k,0} - \beta_{3,2,0}\alpha_{1,k,0})$$

$$- (\beta_{1,2,0}\beta_{3,1,0} - \beta_{3,2,0}\beta_{1,1,0})(\beta_{1,2,0}\alpha_{2,k,0} - \beta_{2,2,0}\alpha_{1,k,0}). \quad (40h)$$

Equations (40a)–(40h) are a system of 24 polynomial equations in 12 unknowns, $(\tilde{r}_k, \tilde{w}_{k,1}, \tilde{w}_{k,2}, \tilde{w}_{k,3})_{k=1}^{3}$. We now solve these equations for the unknowns.

Substituting (34) into (40a) and collecting terms yields:

$$0 = (\tilde{r}_3 - \tilde{r}_2)^2(\tilde{r}_3 - \tilde{r}_1)(\tilde{r}_1 - \tilde{r}_2)(r_k - \tilde{r}_1)(r_k - \tilde{r}_2)(r_k - \tilde{r}_3). \quad (41)$$

Equation (41) shows that any solution to (40) falls into one of the following cases:

Case 3(a): $\tilde{r}_2 = \tilde{r}_3 \neq \tilde{r}_1$;

Case 3(b): $\tilde{r}_1 = \tilde{r}_3 \neq \tilde{r}_2$;

Case 3(c): $\tilde{r}_1 = \tilde{r}_2 \neq \tilde{r}_3$;

Case 3(d): $\tilde{r}_1 = \tilde{r}_2 = \tilde{r}_3$; or

Case 3(e): Since $r_1 \neq r_2$, $r_2 \neq r_3$, and $r_1 \neq r_3$ (because $\boldsymbol{\theta} \notin X_{3,2}$), and given normalization 1, $\tilde{r}_1 = r_1$, $\tilde{r}_2 = r_2$ and $\tilde{r}_3 = r_3$.

First, suppose that case 3(a) applies. Then (40c) reduces to

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{2,1})\tilde{r}_1, \tag{42}$$

which implies that the solution to (40) satisfies one of the following:

Case 3(a)1: $\tilde{w}_{3,1} = \tilde{w}_{2,1}$; or

Case 3(a)2: $\tilde{w}_{3,1} \neq \tilde{w}_{2,1}$ and $\tilde{r}_1 = 0$.

Suppose case 3(a)1 applies. Then (33a) reduces to

$$0 = \delta_1 + \sum_{k=1}^{3}\sum_{\ell=0}^{1} \alpha_{1,k,\ell} e_{k,t-\ell} + \sum_{\ell=0}^{1} \beta_{1,1,\ell}\tilde{e}_{1,t-\ell}, \quad t = 2, \ldots, T. \tag{43}$$

Since $\boldsymbol{\theta} \notin X_{3,3}$, $(\tilde{e}_{1,t}, \tilde{e}_{1,t-1})$ cannot span the same space as $(e_{k,t}, e_{k,t-1})_{k=1}^{3}$; therefore all the coefficients in (43) are zero. In particular, $\beta_{1,1,1} = 0$, which implies $\tilde{r}_3 = \tilde{r}_1$, contrary to the assumption that case 3(a) applies. Therefore case 3(a)1 does not apply. Alternatively, suppose case 3(a)2 applies. Then, using $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$ and noting that $\tilde{r}_2 \neq 0$ in case 3(a)2,

(40) reduces to

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{2,1})\tilde{w}_{1,1} - (\tilde{w}_{3,2} - \tilde{w}_{2,2}), \tag{44a}$$

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{1,1})\tilde{w}_{1,1} - (\tilde{w}_{3,2} - \tilde{w}_{1,2}), \tag{44b}$$

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{2,1})\tilde{w}_{1,2} - (\tilde{w}_{3,3} - \tilde{w}_{2,3}), \tag{44c}$$

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{1,1})\tilde{w}_{1,2} - (\tilde{w}_{3,3} - \tilde{w}_{1,3}). \tag{44d}$$

Using (44), in case 3(a)2 equations (32a) and (32b) simplify to:

$$\begin{aligned}(\mu - \tilde{\mu})(1 - \tilde{r}_2 - \tilde{w}_{3,1}) &+ (r_1 - \tilde{r}_2)e_{1,t-1} + (w_{1,1} - \tilde{w}_{3,1})e_{1,t} \\ &+ (r_2 - \tilde{r}_2)e_{2,t-1} + (w_{2,1} - \tilde{w}_{3,1})e_{2,t} + (r_3 - \tilde{r}_2)e_{3,t-1} + (w_{3,1} - \tilde{w}_{3,1})e_{3,t} \\ &= -\tilde{r}_2\tilde{e}_{1,t-1} + (\tilde{w}_{1,1} - \tilde{w}_{3,1})\tilde{e}_{1,t} + (\tilde{w}_{2,1} - \tilde{w}_{3,1})\tilde{e}_{2,t}, \\ &\qquad t = 2, \ldots, T, \end{aligned} \tag{45a}$$

$$\begin{aligned}(\mu - \tilde{\mu})(1 - \tilde{r}_2^2 - \tilde{r}_2\tilde{w}_{3,1} - \tilde{w}_{3,2}) &+ (r_1^2 - \tilde{r}_2^2)e_{1,t-2} + (r_1 w_{1,1} - \tilde{r}_2\tilde{w}_{3,1})e_{1,t-1} + (w_{1,2} - \tilde{w}_{3,2})e_{1,t} \\ &+ (r_2^2 - \tilde{r}_2^2)e_{2,t-2} + (r_2 w_{2,1} - \tilde{r}_2\tilde{w}_{3,1})e_{2,t-1} + (w_{2,2} - \tilde{w}_{3,2})e_{2,t} \\ &+ (r_3^2 - \tilde{r}_2^2)e_{3,t-2} + (r_3 w_{3,1} - \tilde{r}_2\tilde{w}_{3,1})e_{3,t-1} + (w_{3,2} - \tilde{w}_{3,2})e_{3,t} \\ &= -\tilde{r}_2^2\tilde{e}_{1,t-2} - \tilde{r}_2\tilde{w}_{3,1}\tilde{e}_{1,t-1} - (\tilde{w}_{3,1} - \tilde{w}_{1,1})\tilde{w}_{1,1}\tilde{e}_{1,t} \\ &+ \tilde{r}_2(\tilde{w}_{2,1} - \tilde{w}_{3,1})\tilde{e}_{2,t-1} + (\tilde{w}_{2,1} - \tilde{w}_{3,1})\tilde{w}_{1,1}\tilde{e}_{2,t}, \\ &\qquad t = 3, \ldots, T. \end{aligned} \tag{45b}$$

Substituting (45a) into (45b) yields

$$
\begin{aligned}
(\mu - \tilde{\mu})[(1 - \tilde{w}_{1,1})(1 - \tilde{r}_2) - \tilde{w}_{3,2} + \tilde{w}_{1,1}\tilde{w}_{3,1}] \\
+ r_1(r_1 - \tilde{r}_2)e_{1,t-2} + (r_1 - \tilde{r}_2)(w_{1,1} - \tilde{w}_{1,1})e_{1,t-1} \\
+ [(w_{1,2} - \tilde{w}_{3,2}) - \tilde{w}_{1,1}(w_{1,1} - \tilde{w}_{3,1})]e_{1,t} \\
+ r_2(r_2 - \tilde{r}_2)e_{2,t-2} + (r_2 - \tilde{r}_2)(w_{2,1} - \tilde{w}_{1,1})e_{2,t-1} \\
+ [(w_{2,2} - \tilde{w}_{3,2}) - \tilde{w}_{1,1}(w_{2,1} - \tilde{w}_{3,1})]e_{2,t} \\
+ r_3(r_3 - \tilde{r}_2)e_{3,t-2} + (r_3 - \tilde{r}_2))(w_{3,1} - \tilde{w}_{1,1})e_{3,t-1} \\
+ [(w_{3,2} - \tilde{w}_{3,2}) - \tilde{w}_{1,1}(w_{2,1} - \tilde{w}_{3,1})]e_{3,t} \\
= 0, \quad t = 3, \ldots, T. \qquad (46)
\end{aligned}
$$

Since $\boldsymbol{\theta} \notin X_{3,3}$, all of the coefficients in (46) must equal zero. But since $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$, the coefficients on $e_{1,t-2}$, $e_{2,t-2}$, and $e_{3,t-2}$ cannot all be zero. Therefore case 3(a)2 does not apply, and we have ruled out case 3(a). Cases 3(b) and 3(c) are symmetric to case 3(a) and are likewise ruled out.

Alternatively, suppose case 3(d) applies. Then, applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$, (40b)–(40h) reduce to

$$0 = [(\tilde{w}_{3,1} - \tilde{w}_{2,1})(\tilde{w}_{3,2} - \tilde{w}_{1,2}) - (\tilde{w}_{3,1} - \tilde{w}_{1,1})(\tilde{w}_{3,2} - \tilde{w}_{2,2})](\tilde{w}_{3,1} - \tilde{w}_{2,1}), \qquad (47a)$$

$$0 = [(\tilde{w}_{3,1} - \tilde{w}_{2,1})(\tilde{w}_{3,3} - \tilde{w}_{1,3}) - (\tilde{w}_{3,3} - \tilde{w}_{2,3})(\tilde{w}_{3,1} - \tilde{w}_{1,1})]\tilde{r}_1(\tilde{w}_{3,1} - \tilde{w}_{2,1}), \qquad (47b)$$

$$0 = [(\tilde{w}_{3,1} - \tilde{w}_{2,1})(\tilde{w}_{3,2} - \tilde{w}_{1,2}) - (\tilde{w}_{3,1} - \tilde{w}_{1,1})(\tilde{w}_{3,2} - \tilde{w}_{2,2})](r_k - \tilde{r}_1)(\tilde{w}_{3,3} - \tilde{w}_{2,3})$$
$$+ [(\tilde{w}_{3,1} - \tilde{w}_{2,1})(\tilde{w}_{3,3} - \tilde{w}_{1,3}) - (\tilde{w}_{3,3} - \tilde{w}_{2,3})(\tilde{w}_{3,1} - \tilde{w}_{1,1})]$$
$$\cdot (r_k - \tilde{r}_1)[(\tilde{w}_{3,1} - \tilde{w}_{2,1})w_{k,1} - (\tilde{w}_{3,2} - \tilde{w}_{2,2})], \quad (47c)$$

$$0 = [(\tilde{w}_{3,1} - \tilde{w}_{2,1})(\tilde{w}_{3,2} - \tilde{w}_{1,2}) - (\tilde{w}_{3,1} - \tilde{w}_{1,1})(\tilde{w}_{3,2} - \tilde{w}_{2,2})](\tilde{w}_{3,3} - \tilde{w}_{2,3})(w_{k,1} - \tilde{w}_{3,1})$$
$$+ [(\tilde{w}_{3,1} - \tilde{w}_{2,1})(\tilde{w}_{3,3} - \tilde{w}_{1,3}) - (\tilde{w}_{3,3} - \tilde{w}_{2,3})(\tilde{w}_{3,1} - \tilde{w}_{1,1})]$$
$$\cdot [(\tilde{w}_{3,1} - \tilde{w}_{2,1})\alpha_{2,k,0} - (\tilde{w}_{3,2} - \tilde{w}_{2,2})(w_{k,1} - \tilde{w}_{3,1})]. \quad (47d)$$

Suppose (47) has a solution in which $\tilde{w}_{3,1} \neq \tilde{w}_{2,1}$. Then (36a) reduces to

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{2,1})\delta_2 - \sum_{\ell=0}^{1} \beta_{2,2,\ell}\delta_1$$
$$+ \sum_{k=1}^{3} \left(\sum_{\ell=0}^{2} (\tilde{w}_{3,1} - \tilde{w}_{2,1})\alpha_{2,k,\ell} e_{k,t-\ell} - \sum_{\ell=0}^{1} \sum_{j=0}^{1} \beta_{2,2,\ell}\alpha_{1,k,j} e_{k,t-\ell-j} \right)$$
$$, \quad t = 3, \ldots, T, \quad (48)$$

and since $\boldsymbol{\theta} \notin X_{3,3}$, all the coefficients in (48) must be zero. In particular, this requires the coefficient on $e_{k,t}$ to be zero, or

$$(\tilde{w}_{3,1} - \tilde{w}_{2,1})(w_{k,2} - \tilde{w}_{3,2}) - (\tilde{w}_{3,2} - \tilde{w}_{2,2})(w_{k,1} - \tilde{w}_{3,1}) = 0. \quad (49)$$

Rearranging, we require

$$w_{k,2} = \tilde{w}_{3,2} - \frac{\tilde{w}_{3,2} - \tilde{w}_{2,2}}{\tilde{w}_{3,1} - \tilde{w}_{2,1}}\tilde{w}_{3,1} + \frac{\tilde{w}_{3,2} - \tilde{w}_{2,2}}{\tilde{w}_{3,1} - \tilde{w}_{2,1}}w_{k,1}, \quad k = 1, 2, 3. \tag{50}$$

But since $\boldsymbol{\theta} \notin X_{3,14}$, (50) cannot hold for all k. Hence if (47) has a solution, it satisfies $\tilde{w}_{3,1} = \tilde{w}_{2,1}$. In that case, however, (36) reduces to

$$0 = -\sum_{\ell=0}^{1} \beta_{2,2,\ell}\delta_1 + \sum_{k=1}^{3}\sum_{\ell=0}^{1}\sum_{j=0}^{1} \beta_{2,2,\ell}\alpha_{1,k,j}e_{k,t-\ell-j} - (\tilde{w}_{3,1} - \tilde{w}_{1,1})(\tilde{w}_{3,2} - \tilde{w}_{2,2})\tilde{e}_{1,t}, \quad t = 3, \ldots, T. \tag{51a}$$

$$0 = -\sum_{\ell=0}^{2} \beta_{3,2,\ell}\delta_1 - \sum_{k=1}^{3}\sum_{\ell=0}^{2}\sum_{j=0}^{1} \beta_{3,2,\ell}\alpha_{1,k,j}e_{k,t-\ell-j} - \sum_{\ell=0}^{2} \beta_{3,2,\ell}\beta_{1,1,0}\tilde{e}_{1,t-\ell}, \quad t = 4, \ldots, T. \tag{51b}$$

Equations (51a) and (51b) cannot both hold unless all of the coefficients are zero. In particular, from the coefficients on $e_{k,t}$ and $\tilde{e}_{1,t}$ in (51a), we must have

$$0 = (\tilde{w}_{3,1} - \tilde{w}_{1,1})(\tilde{w}_{3,2} - \tilde{w}_{2,2}) = (\tilde{w}_{3,2} - \tilde{w}_{2,2})(w_{k,1} - \tilde{w}_{3,1}), \quad k = 1, 2, 3, \tag{52}$$

from which it follows that either $w_{k,1} = \tilde{w}_{1,1}$ for all k or $\tilde{w}_{3,2} = \tilde{w}_{2,2}$. The hypothesis that $\boldsymbol{\theta} \notin X_{3,4}$ rules out the former possibility; hence any solution in case 3(d) satisfies both

$\tilde{w}_{3,1} = \tilde{w}_{2,1}$ and $\tilde{w}_{3,2} = \tilde{w}_{2,2}$. Now, however, (32) simplifies to

$$
\begin{aligned}
&(\mu - \tilde{\mu})(1 - \tilde{r}_1 - \tilde{w}_{2,1}) + (r_1 - \tilde{r}_1)e_{1,t-1} + (w_{1,1} - \tilde{w}_{2,1})e_{1,t} \\
&\qquad + (r_2 - \tilde{r}_1)e_{2,t-1} + (w_{2,1} - \tilde{w}_{2,1})e_{2,t} + (r_3 - \tilde{r}_1)e_{3,t-1} + (w_{3,1} - \tilde{w}_{2,1})e_{3,t} \\
&\qquad\qquad = (\tilde{w}_{1,1} - \tilde{w}_{2,1})\tilde{e}_{1,t}, \quad t = 2, \ldots, T, \quad \text{(53a)}
\end{aligned}
$$

$$
\begin{aligned}
&(\mu - \tilde{\mu})(1 - \tilde{r}_1^2 - \tilde{r}_1\tilde{w}_{2,1} - \tilde{w}_{2,2}) + (r_1^2 - \tilde{r}_1^2)e_{1,t-2} + (r_1 w_{1,1} - \tilde{r}_1\tilde{w}_{2,1})e_{1,t-1} + (w_{1,2} - \tilde{w}_{2,2})e_{1,t} \\
&\qquad + (r_2^2 - \tilde{r}_1^2)e_{2,t-2} + (r_2 w_{2,1} - \tilde{r}_1\tilde{w}_{2,1})e_{2,t-1} + (w_{2,2} - \tilde{w}_{2,2})e_{2,t} \\
&\qquad + (r_3^2 - \tilde{r}_1^2)e_{3,t-2} + (r_3 w_{3,1} - \tilde{r}_1\tilde{w}_{2,1})e_{3,t-1} + (w_{3,2} - \tilde{w}_{2,2})e_{3,t} \\
&\qquad\qquad = \tilde{r}_1(\tilde{w}_{1,1} - \tilde{w}_{2,1})\tilde{e}_{1,t-1} + (\tilde{w}_{1,2} - \tilde{w}_{2,2})\tilde{e}_{1,t}, \quad t = 3, \ldots, T. \quad \text{(53b)}
\end{aligned}
$$

$$
\begin{aligned}
&(\mu - \tilde{\mu})(1 - \tilde{r}_1^3 - \tilde{r}_1^2\tilde{w}_{2,1} - \tilde{r}_1\tilde{w}_{2,2} - \tilde{w}_{3,3}) \\
&\quad + (r_1^3 - \tilde{r}_1^3)e_{1,t-3} + (r_1^2 w_{1,1} - \tilde{r}_1^2\tilde{w}_{2,1})e_{1,t-2} + (r_1 w_{1,2} - \tilde{r}_1\tilde{w}_{2,2})e_{1,t-1} + (w_{1,3} - \tilde{w}_{3,3})e_{1,t} \\
&\quad + (r_2^3 - \tilde{r}_1^3)e_{2,t-3} + (r_2^2 w_{2,1} - \tilde{r}_1^2\tilde{w}_{2,1})e_{2,t-2} + (r_2 w_{2,2} - \tilde{r}_1\tilde{w}_{2,2})e_{2,t-1} + (w_{2,3} - \tilde{w}_{3,3})e_{2,t} \\
&\quad + (r_3^3 - \tilde{r}_1^3)e_{3,t-3} + (r_3^2 w_{3,1} - \tilde{r}_1^2\tilde{w}_{2,1})e_{3,t-2} + (r_3 w_{3,2} - \tilde{r}_1\tilde{w}_{2,2})e_{3,t-1} + (w_{3,3} - \tilde{w}_{3,3})e_{3,t} \\
&\qquad = \tilde{r}_1^2(\tilde{w}_{1,1} - \tilde{w}_{2,1})\tilde{e}_{1,t-2} + \tilde{r}_1(\tilde{w}_{1,2} - \tilde{w}_{2,2})\tilde{e}_{1,t-1} + (\tilde{w}_{1,3} - \tilde{w}_{3,3})\tilde{e}_{1,t} \\
&\qquad\qquad + (\tilde{w}_{2,3} - \tilde{w}_{3,3})\tilde{e}_{2,t}, \\
&\qquad\qquad\qquad t = 4, \ldots, T. \quad \text{(53c)}
\end{aligned}
$$

If the solution does not satisfy $\tilde{w}_{1,1} = \tilde{w}_{2,1}$, we can use (53a) to eliminate $\tilde{e}_{1,t}$ from (53b) and

(53c), leaving two linear combinations of $e_{k,t}$ whose coefficients, since $\boldsymbol{\theta} \notin X_{3,4}$, must be zero. In particular, examining the coefficients on $e_{k,t-2}$ in (53b), we will require

$$(r_1^2 - \tilde{r}_1^2) - \tilde{r}_1(r_1 - \tilde{r}_1) = (r_2^2 - \tilde{r}_1^2) - \tilde{r}_1((r_2 - \tilde{r}_1) = 0, \tag{54}$$

which implies, since $\boldsymbol{\theta} \notin X_{3,1}$, that

$$r_1 = r_2 = \tilde{r}_1. \tag{55}$$

Equation (55) contradicts $\boldsymbol{\theta} \notin X_{3,2}$, so we conclude that any solution in case 3(d) also satisfies $\tilde{w}_{1,1} = \tilde{w}_{2,1}$. Then, however, all of the coefficients in (53a) must be zero, and examining the coefficients on $e_{k,t-1}$, we again obtain (55), which again contradicts $\boldsymbol{\theta} \notin X_{3,2}$. Therefore, there is no solution in case 3(d).

We have ruled out cases 3(a), 3(b), 3(c) and 3(d), so any solution to (40) falls in case 3(e) and therefore satisfies

$$\tilde{r}_k = r_k, \quad k = 1, \ldots, 3. \tag{56}$$

We now solve the remaining equations in (40) for the remaining unknowns, which are

$$(\tilde{w}_{k,1}, \tilde{w}_{k,2}, \tilde{w}_{k,3})_{k=1}^{3}.$$

Substituting (34) and (56) into (40b) for $k = 3$ and applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$ yields

$$\tilde{w}_{3,1} = w_{3,1}. \tag{57}$$

Substituting (34), (56), and (57) into (40b) for $k = 2$ and applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$ then yields

$$\tilde{w}_{2,1} = w_{2,1}. \tag{58}$$

Substituting (34) and (56)–(58) into (40b) for $k = 1$ and applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$ then yields

$$\tilde{w}_{1,1} = w_{1,1}. \tag{59}$$

Substituting (34) and (56)–(59) into (40c) for $k = 3$ and applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2} \cup X_{3,12}$ yields

$$\tilde{w}_{3,2} = w_{3,2}. \tag{60}$$

Substituting (34) and (56)–(60) into (40c) for $k = 2$ and applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2} \cup X_{3,12}$ yields

$$\tilde{w}_{2,2} = w_{2,2}. \tag{61}$$

Substituting (34) and (56)–(61) into (40c) for $k = 1$ and applying $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2} \cup X_{3,12}$ then yields

$$\tilde{w}_{1,2} = w_{1,2}. \tag{62}$$

Substituting (34) and (57)–(62) into (40h) for $k = 3$ yields

$$\begin{aligned} 0 = {} & [(w_{3,1} - w_{2,1})(w_{3,2} - w_{1,2}) - (w_{3,1} - w_{1,1})(w_{3,2} - w_{2,2})] \\ & \cdot (w_{3,1} - w_{2,1})(w_{3,3} - \tilde{w}_{3,3}). \end{aligned} \tag{63}$$

Since $\boldsymbol{\theta} \notin X_{3,6} \cup X_{3,13}$, (63) implies

$$\tilde{w}_{3,3} = w_{3,3}. \tag{64}$$

Substituting (34), (57)–(62) and (64) into (40h) for $k = 2$ and applying $\boldsymbol{\theta} \notin X_{3,6} \cup X_{3,13}$ yields

$$\tilde{w}_{2,3} = w_{2,3}. \tag{65}$$

Finally, substituting (34), (57)–(62), (64), and (65) into (40h) for $k = 1$ and applying $\boldsymbol{\theta} \notin X_{3,6} \cup X_{3,13}$ yields

$$\tilde{w}_{1,3} = w_{1,3}. \tag{66}$$

We have now solved (40) and identified $(r_k, w_{k,1}, w_{k,2}, w_{k,3})_{k=1}^{3}$. Equation (38) imposes one more restriction on $\tilde{\boldsymbol{\theta}}$: The constant term must be zero, or

$$\begin{aligned} 0 = (\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3) \left(\sum_{\ell=0}^{1} \beta_{1,2,\ell}\delta_3 - \sum_{\ell=0}^{3} \beta_{3,2,\ell}\delta_1 \right) \\ - \left(\sum_{j=0}^{1} \sum_{\ell=0}^{3} (\beta_{1,2,j}\beta_{3,1,\ell} - \beta_{3,2,\ell}\beta_{1,1,j}) \right) \left(\sum_{\ell=0}^{1} \beta_{1,2,\ell}\delta_2 - \sum_{\ell=0}^{2} \beta_{2,2,\ell}\delta_1 \right). \end{aligned} \tag{67}$$

Substituting (56)–(62), (64)–(66), and (37) into (67), we require

$$
\begin{aligned}
0 = (\mu - \tilde{\mu}) \\
\Bigg[[(w_{3,1} - w_{2,1} + r_3 - r_2)(w_{3,2} - w_{1,2} + r_3^2 - r_1^2 + r_3 w_{3,1} - r_1 w_{1,1}) \\
- (w_{3,1} - w_{1,1} + r_3 - r_1)(w_{3,2} - w_{2,2} + r_3^2 - r_2^2 + r_3 w_{3,1} - r_2 w_{2,1})] \\
\cdot [(w_{3,1} - w_{2,1} + r_3 - r_2)(1 - r_3^3 - r_3^2 w_{3,1} - r_3 w_{3,2} - w_{3,3}) \\
- (w_{3,3} - w_{2,3} + r_3 w_{3,2} - r_2 w_{2,2} + r_3^2 w_{3,1} - r_2^2 w_{2,1} + r_3^3 - r_2^3)(1 - r_3 - w_{3,1})] \\
- [(w_{3,1} - w_{2,1} + r_3 - r_2)(w_{3,3} - w_{1,3} + r_3 w_{3,2} - r_1 w_{1,2} + r_3^2 w_{3,1} - r_1^2 w_{1,1} + r_3^3 - r_1^3) \\
- (w_{3,1} - w_{1,1} + r_3 - r_1)(w_{3,3} - w_{2,3} + r_3 w_{3,2} - r_2 w_{2,2} + r_3^2 w_{3,1} - r_2^2 w_{2,1} + r_3^3 - r_2^3)] \\
\cdot [(w_{3,1} - w_{2,1} + r_3 - r_2)(1 - r_3^2 - r_3 w_{3,1} - w_{3,2}) - (w_{3,2} - w_{2,2} + r_3 w_{3,1} - r_2 w_{2,1} + r_3^2 - r_2^2)(1 - r_3 - w_{3,1})] \Bigg].
\end{aligned}
\tag{68}
$$

Hence, since $\boldsymbol{\theta} \notin X_{3,5}$,

$$\tilde{\mu} = \mu. \tag{69}$$

We have now identified $(r_k, w_{k,1}, w_{k,2}, w_{k,3})_{k=1}^3$. Our next step is to identify $e_{k,t}$. Combining (34), (56)–(62), (64)–(66), and (69), we now have

$$
\begin{aligned}
&\delta_1 = \delta_2 = \delta_3 = 0, \\
&\beta_{j,k,j} = r_3^j - r_k^j, \quad \beta_{j,k,0} = w_{3,j} - w_{k,j}, j = 1, \dots, 3, \\
&\beta_{2,k,1} = r_3 w_{3,1} - r_k w_{k,1}, \quad \beta_{3,k,1} = r_3 w_{3,2} - r_k w_{k,2}, \\
&\beta_{3,k,2} = r_3^2 w_{3,1} - r_k^2 w_{k,1}, \quad \alpha_{j,k,\ell} = -\beta_{j,k,\ell}.
\end{aligned}
\tag{70}
$$

Therefore, (35) reduces to

$$0 = \sum_{j=0}^{1}\sum_{\ell=0}^{2} \beta_{1,2,j}\beta_{2,1,\ell}(\tilde{e}_{1,t-\ell-j} - e_{1,t-\ell-j}), \quad t = 4, \ldots, T, \tag{71a}$$

$$0 = \sum_{j=0}^{1}\sum_{\ell=0}^{3}(\beta_{1,2,j}\beta_{3,1,\ell} - \beta_{1,1,j}\beta_{3,2,\ell})(\tilde{e}_{1,t-\ell-j} - e_{1,t-\ell-j}), \quad t = 5, \ldots, T. \tag{71b}$$

It is convenient to define $z_{1,t} = \tilde{e}_{1,t} - e_{1,t}$ and write (71) as

$$\begin{aligned} -\beta_{1,2,0}\beta_{2,1,0}z_{1,t} &= (\beta_{1,2,1}\beta_{2,1,0} + \beta_{1,2,0}\beta_{2,1,1})z_{1,t-1} \\ &\quad + (\beta_{1,2,0}\beta_{2,1,2} + \beta_{1,2,1}\beta_{2,1,1})z_{1,t-2} + \beta_{1,2,1}\beta_{2,1,2}z_{1,t-3} + 0z_{t-4}, \quad t = 4, \ldots, T, \end{aligned} \tag{72a}$$

$$\begin{aligned} -(\beta_{1,2,0}\beta_{3,1,0} - \beta_{1,1,0}\beta_{3,2,0})z_t &= (\beta_{1,2,1}\beta_{3,1,0} - \beta_{1,1,1}\beta_{3,2,0} + \beta_{1,2,0}\beta_{3,1,1} - \beta_{1,1,0}\beta_{3,2,1})z_{1,t-1} \\ &\quad + (\beta_{1,2,0}\beta_{3,1,2} - \beta_{1,1,0}\beta_{3,2,2} + \beta_{1,2,1}\beta_{3,1,1} - \beta_{1,1,1}\beta_{3,2,1})z_{1,t-2} \\ &\quad + (\beta_{1,2,0}\beta_{3,1,3} - \beta_{1,1,0}\beta_{3,2,3} + \beta_{1,2,1}\beta_{3,1,2} - \beta_{1,1,1}\beta_{3,2,2})z_{1,t-3} \\ &\quad + (\beta_{1,2,1}\beta_{3,1,3} - \beta_{1,1,1}\beta_{3,2,3})z_{1,t-4}, \quad t = 5, \ldots, T. \end{aligned} \tag{72b}$$

Since $\boldsymbol{\theta} \notin X_{3,6} \cup X_{3,7}$, we can write (72) as

$$\mathbf{z}_{1,t} = \mathbf{B}_{1,2}\mathbf{z}_{1,t-1} = \mathbf{B}_{1,3}\mathbf{z}_{1,t-1}, \quad t = 5, \ldots, T, \tag{73}$$

where $\mathbf{z}_{1,t} = (z_{1,t}, z_{1,t-1}, z_{1,t-2}, z_{1,t-3})'$ and

$$\mathbf{B}_{1,2} = \begin{bmatrix} -\mathbf{b}_{1,2}/(\beta_{1,2,0}\beta_{2,1,0}) \\ 1 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \end{bmatrix}, \quad \mathbf{B}_{1,3} = \begin{bmatrix} -\mathbf{b}_{1,3}/(\beta_{1,2,0}\beta_{3,1,0} - \beta_{1,1,0}\beta_{3,2,0}) \\ 1 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \end{bmatrix}, \tag{74}$$

where $\mathbf{b}_{1,2}$ and $\mathbf{b}_{1,3}$ are row vectors containing the coefficients in (72a) and (72b), respectively. Since $\boldsymbol{\theta} \notin X_{3,8}$, $\mathbf{B}_{1,3}^{-1}$ exists, and (73) implies

$$\mathbf{B}_{1,3}^{-1}\mathbf{B}_{1,2}\mathbf{z}_{1,t} = \mathbf{z}_{1,t}, \quad t = 4, \ldots, T-1. \tag{75}$$

But $\boldsymbol{\theta} \notin X_{3,8}$ also implies that the upper left 3×3 block of $\mathbf{B}_{1,3}^{-1}\mathbf{B}_{1,2}$ is a nonsingular matrix not equal to the identity. Hence (75) requires that the first three elements of $\mathbf{z}_{1,t} = 0$ equal zero for $t = 4, \ldots, T-1$, and hence

$$\tilde{e}_{1,t} = e_{1,t}, \quad t = 1, \ldots, T. \tag{76}$$

Substituting (56)–(62), (64)–(66), (69), and (76) into (33a) and (33b) yields

$$0 = \sum_{\ell=0}^{1} \beta_{1,2,\ell} z_{2,t-\ell}, \quad t = 2, \ldots, T, \tag{77a}$$

$$0 = \sum_{\ell=0}^{2} \beta_{2,2,\ell} z_{2,t-\ell}, \quad t = 3, \ldots, T, \tag{77b}$$

where $z_{2,t} = \tilde{e}_{2,t} - e_{2,t}$. In matrix form, (77) becomes

$$\mathbf{z}_{2,t} = \mathbf{B}_{2,1}\mathbf{z}_{2,t-1} = \mathbf{B}_{2,2}\mathbf{z}_{2,t-1}, \quad t = 3, \ldots, T, \tag{78}$$

where $\mathbf{z}_{2,t} = (z_{2,t}, z_{t-1})'$ and

$$\mathbf{B}_{2,1} = -\begin{bmatrix} \beta_{1,2,0}^{-1}\beta_{1,2,1} & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}_{2,2} = \begin{bmatrix} -\beta_{2,2,0}^{-1}\beta_{2,2,1} & -\beta_{2,2,0}^{-1}\beta_{2,2,2} \\ 1 & 0 \end{bmatrix}, \tag{79}$$

where we have used the fact that $\boldsymbol{\theta} \notin X_{3,6} \cup X_{3,9}$. By the same argument as applied to (73), since $\boldsymbol{\theta} \notin X_{3,10}$, we have that the first two elements of $\mathbf{z}_{2,t}$ equal zero for $t = 2, \ldots, T-1$, and hence

$$\tilde{e}_{2,t} = e_{2,t}, \quad t = 1, \ldots, T. \tag{80}$$

Substituting (69), (76), and (80) into (31a) then yields

$$\tilde{e}_{3,t} = e_{3,t}, \quad t = 1, \ldots, T. \tag{81}$$

Equations (56)–(62), (64)–(66), (69), (76), (80), and (81) identify $e_{k,t}$ for $t = 1, \ldots, T$ as well as r_k, $w_{k,1}$, $w_{k,2}$, $w_{k,3}$, and μ. It remains to show that $\tilde{e}_{k,t} = e_{k,t}$ for $t \leq 0$ and (if $A > 3$) that $\tilde{w}_{k,a} = w_{k,a}$ for $a > 3$. First consider identification of $w_{k,a}$. Fix any $a > 3$. If $\tilde{w}_{k,a'} = w_{k,a'}$ for $k = 1, 2, 3$ and all $a' < a$, then, using all of the foregoing results, equation (1) at a reduces to

$$\sum_{k=1}^{3}(w_{k,a} - \tilde{w}_{k,a})e_{k,t} = 0, \quad t = a+1, \ldots, T. \tag{82}$$

Since $T \geq A+K$, (82) contains at least three equations, and since $\boldsymbol{\theta} \notin X_{3,11}$, these equations have the unique solution $\tilde{w}_{k,a} = w_{k,a}$. By induction, this gives $\tilde{w}_{k,a} = w_{k,a}$ for all $a > 3$.

Finally, consider identification of $e_{k,t}$ for $t \leq 0$. Fix any $s \in \{0, -1, \ldots, 3-A\}$. If $\tilde{e}_{k,\tau} = e_{k,\tau}$ for $\tau = s+1, \ldots, T$, then, using all of the foregoing results, equation (1) at $t = 1, a = 1-s$; $t = 2, a = 2-s$; and $t = 3, a = 3-s$ reduces to

$$0 = \sum_{k=1}^{3} r_k^{1-s}(e_{k,s} - \tilde{e}_{k,s}) = \sum_{k=1}^{3} r_k^{2-s}(e_{k,s} - \tilde{e}_{k,s}) = \sum_{k=1}^{3} r_k^{3-s}(e_{k,s} - \tilde{e}_{k,s}). \tag{83}$$

Since $\boldsymbol{\theta} \notin X_{3,1} \cup X_{3,2}$,

$$\det \begin{bmatrix} r_1^{1-s} & r_2^{1-s} & r_3^{1-s} \\ r_1^{2-s} & r_2^{2-s} & r_3^{2-s} \\ r_1^{3-s} & r_2^{3-s} & r_3^{3-s} \end{bmatrix} \neq 0. \tag{84}$$

Therefore the unique solution to (83) is $\tilde{e}_{k,s} = e_{k,s}$. By induction, this gives $\tilde{e}_{k,t} = e_{k,t}$ for $t = 0, -1, \ldots, 3-A$. Applying normalization (3) and the foregoing results to (1) at $(t = 1, a = A)$, $(t = 1, a = A-1)$ and $(t = 2, a = A)$ gives $\tilde{e}_{k,1-A} = e_{k,1-A}$ and $\tilde{e}_{k,2-A} = e_{k,2-A}$. We have now identified all parameters for the case $K = 3$. □

2. Proof of Proposition 2

Proposition 2. *Under the hypotheses of proposition 1, any parameter vector* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ *is not identified if either:*

(a) $e_{k,t} = \bar{e}_k$ *for some* k *and all* $t = 1-A, \ldots, T$*, or*

(b) $K > 1$ *and* $r_k = r_{k'}$ *for some* $k \neq k'$*.*

Further, $\boldsymbol{\theta}$ *remains unidentified in each of these cases even if* μ *is known.*

We must show that under each of conditions (a) and (b), (1) has multiple solutions for $(\tilde{\mu}, \tilde{\mathbf{r}}, \tilde{\mathbf{e}}, \tilde{\mathbf{w}})$ in terms of $(\mu, \mathbf{r}, \mathbf{e}, \mathbf{w})$, and that this is so even if we impose $\tilde{\mu} = \mu$.

Condition (a): Without loss of generality, suppose $e_{1,t} = \bar{e}_1$. Choose any $r^* \in [0, 1]$. Let $\{w_a^*\}_{a=1}^A$ be the unique solution to the following nonsingular triangular system of linear equations given r^*, r_1 and $\{w_{1,a}\}_{a=1}^A$:

$$\sum_{a'=1}^{a} (r^*)^{a-a'} w_{a'}^* = -(r^*)^a + \sum_{a'=0}^{a} r_1^{a-a'} w_{1,a'}, \quad a = 1, \ldots, A. \tag{85}$$

Substitution verifies that, given $e_{1,t} = \bar{e}_1$, the following solves (1):

$$\tilde{\mu} = \mu; \quad \tilde{e}_{j,t} = e_{j,t} \;\forall\, j, t; \quad \tilde{r}_1 = r^*; \quad \tilde{r}_j = r_j \;\forall\, j > 1;$$
$$\tilde{w}_{1,a} = w_a^* \;\forall\, a; \quad \tilde{w}_{j,a} = w_{j,a} \;\forall\, j > 1, a. \tag{86}$$

Specifically, substituting $e_{1,t} = \bar{e}_1$ and (86) into (1) yields

$$\sum_{a'=0}^{a} r_1^{a-a'} w_{1,a'} \bar{e}_1 = \sum_{a'=0}^{a} (r^*)^{a-a'} w_{a'}^* \bar{e}_1 \quad \forall\, a, t. \tag{87}$$

If $\bar{e}_1 = 0$, then (87) holds trivially; otherwise, (87) is identical to (85) and holds by definition of w_a^*. Therefore, (1) has a continuum of solutions indexed by $r^* \in [0, 1]$, and the parameter vector is not identified.

Condition (b): Without loss of generality, suppose $r_1 = r_2$. Choose any $x \in (1/2, 1]$.

Substitution verifies that, given $r_1 = r_2$, the following solves (1):

$$\tilde{\mu} = \mu; \quad \tilde{r}_j = r_j \ \forall\ j; \quad \tilde{w}_{1,a} = xw_{1,a} + (1-x)w_{2,a} \ \forall\ a; \quad \tilde{w}_{2,a} = (1-x)w_{1,a} + xw_{2,a} \ \forall\ a;$$
$$\tilde{e}_{1,t} = \frac{(1-x)e_{2,t} - xe_{1,t}}{1-2x} \ \forall\ t; \quad \tilde{e}_{2,t} = \frac{(1-x)e_{1,t} - xe_{2,t}}{1-2x} \ \forall\ t;$$
$$\tilde{e}_{j,t} = e_{j,t} \ \forall\ j > 2, t; \quad \tilde{w}_{j,a} = w_{j,a} \ \forall j > 2, a. \tag{88}$$

Specifically, substituting $r_1 = r_2$ and (88) into (1) yields

$$\sum_{a'=0}^{a} r_k^{a-a'} \sum_{k=1}^{2} (w_{k,a'} e_{k,t-a+a'} - \tilde{w}_{k,a'} \tilde{e}_{k,t-a+a'}) \quad \forall\ a, t, \tag{89}$$

which holds because

$$\begin{aligned} & w_{1,a}e_{1,s} + w_{2,a}e_{2,s} - \tilde{w}_{1,a}\tilde{e}_{1,s} - \tilde{w}_{2,a}\tilde{e}_{2,s} \\ & \quad = w_{1,a}e_{1,s} + w_{2,a}e_{2,s} - [xw_{1,a} + (1-x)w_{2,a}] \frac{(1-x)e_{2,s} - xe_{1,s}}{1-2x} \\ & \qquad - [(1-x)w_{1,a} + xw_{2,a}] \frac{(1-x)e_{1,s} - xe_{2,s}}{1-2x} \\ & \quad = 0 \end{aligned} \tag{90}$$

for all a, s. Therefore, (1) has a continuum of solutions indexed by $x \in (1/2, 1]$, and the parameter vector is not identified. □