

Chapter 1

Borrowing Constraints and Transactions Costs

In this chapter and the next, we will discuss models in which there exist market incompleteness and other market frictions.¹ There are several reasons for examining such models. We have already seen that different versions of the representative consumer model have been unable to rationalize such asset pricing anomalies as the equity premium puzzle, the average real risk-free rate puzzle, and the behavior of the term premiums. In a representative agent model, all asset returns are driven by a common stochastic discount factor which suggests that, to some extent, stocks and bonds should tend to move together.² Yet the empirical evidence appears to be at odds with this requirement. By introducing market incompleteness, borrowing constraints, and other sorts of frictions, this close link can be broken.

The first topic we consider in this chapter is analyzing the equilibrium allocations and prices in an economy with idiosyncratic risk. We examine two cases: one in which markets are complete and the other in which borrowing constraints with asymmetric information so that markets are incomplete. Our discussion follows Scheinkman and Weiss [?]. We examine the complete markets case to highlight the role that *ex ante* heterogeneity plays in the economy. In the borrowing constraint model, idiosyncratic income risk is nondiversifiable because these shocks are not publicly observed. Hence agents have limited opportunities to borrow against future income and cannot totally insure against all types of risks. Using this framework, we characterize the equilibrium in a model with heterogeneous consumers and borrowing constraints.

¹This chapter was written to be included in the text *Dynamic Choice and Asset Markets*, with Pamela Labadie, 1994, Academic Press. The material was developed jointly by Pamela Labadie and myself. I have posted this chapter because I believe that the material retains its relevance and originality. If you make use of the material in your work, please provide the appropriate citation.

²This point has been explored by Barsky [?], among others.

This model also has implications for the behavior of individual consumption and leisure/labor supply allocations. Even in the absence of aggregate shocks, the model generates random fluctuations in aggregate output, the labor input, and the relative price of the asset that is traded in equilibrium. Another implication is that the cross-sectional distribution of nonhuman wealth is an important determinant of aggregate economic activity. By contrast, representative consumer models imply that fluctuations in aggregate unemployment arise solely from the intertemporal substitution of labor and have been rejected in alternative tests based on aggregate data. (See, for example, Mankiw, Rotemberg and Summers [?], Eichenbaum, Hansen and Singleton [?], and Altug [?], among others.)

In Section 7.2, we present a model with bid-ask spreads and review the literature associated with transactions costs. As the final topic of this chapter, we present a method for calculating volatility bounds for the intertemporal MRS in consumption when there are borrowing constraints and transactions costs. These volatility bounds provide a nonparametric method for examining the restrictions of dynamic asset pricing models without specifying consumers' utility functions or explicitly characterizing the equilibrium.

1.1 A Model with Idiosyncratic Risk

We start our discussion with the paper by Scheinkman and Weiss [?]. For consistency with the remainder of the material, we present a discrete-time version of their model. Because we wish to appeal to the law of large numbers in describing the aggregate properties of individual risk, we will introduce individual risk based on Feldman and Gilles [?].

Assume that there is a countable infinity of agents and that the index set of agents is $A = \{1, 2, \dots\}$. Let ν denote the probability measure defined over A such that $\nu(A) = 1$. Let $B \subset A$ be an infinite subset consisting of type 1 agents³ and assume that $\nu(B) = \alpha$ where α is the proportion of agents that are type 1. Let B^c be the infinite subset of agents that are type 2 such that $\nu(B^c) = 1 - \alpha$.

Uncertainty is introduced through a productivity shock. A stochastic process is a collection of random variables $\{s(t, \omega), t \in T\}$ defined on the same probability space (Ω, \mathcal{F}, P) , where $T = \{0, 1, 2, \dots\}$ and $s : T \times \Omega \rightarrow S$. For a fixed $\omega \in \Omega$, $s(\cdot, \omega)$ is the sample path or the realization. For a fixed $t \in T$, $s(t, \cdot)$ is a random variable. Let F be the transition probability function which is assumed to have the Feller property. Let the state space be $S = \{1, 2\}$. To conserve on notation, we let $s_t(\omega) = s(t, \omega)$.

We now relate the aggregate uncertainty and individual risk through the

³Examples of infinite subsets are the set of even numbers $\{2, 4, 6, \dots\}$ or a Fibonacci sequence $\{1, 2, 3, 5, \dots\}$ in which the $(i+1)$ element equals the sum of elements i and $i-1$.

productivity shock. Each agent, whether a type 1 or type 2, has time-additive preferences over consumption and leisure streams. An agent can produce one unit of the consumption good per one unit of labor when he is productive, but he may suffer random spells of nonproductivity. In the absence of *ex ante* heterogeneity, agents become differentiated from each other as a result of their histories of productivity. Define a function $\theta : S \rightarrow \{0, 1\}$ indexed by $a \in A$. Assume that if $s_t(\omega) = 1$, then

$$\theta_{a,t}(s) = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{otherwise,} \end{cases} \quad (1.1)$$

while if $s_t(\omega) = 2$, then

$$\theta_{a,t}(s) = \begin{cases} 1 & \text{if } a \in B^c \\ 0 & \text{otherwise.} \end{cases} \quad (1.2)$$

The production function for an agent of type i is

$$y_i = \theta_i \ell_i, \quad (1.3)$$

where ℓ_i is the labor supply.

A typical type i consumer has preferences over stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ of the form

$$U^i = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - W(\ell_{i,t})] \right\}. \quad (1.4)$$

We make the following assumptions on preferences.

Assumption 1.1 (i) Let $U : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ be concave, increasing, and thrice continuously differentiable with $U''' > 0$, and that

$$\lim_{c \rightarrow 0} U'(c) = +\infty, \quad \lim_{c \rightarrow \infty} U'(c) = 0.$$

Also assume that $-U''(c)c/U'(c) \leq 1$ and $-U'''/U''$ is decreasing.

(ii) Let $W : [0, L) \rightarrow \mathfrak{R}_+$ be continuous, strictly increasing, thrice continuously differentiable, and convex with $W(0) = 0$, $W'(0) = 0$ and $\lim_{\ell \rightarrow L} W'(\ell) = \infty$.

The assumption that $-U''(c)c/U'(c) \leq 1$ ensures that the substitution effect is greater than the income effect. The assumption that $-U'''/U''$ is decreasing implies that the utility function U displays decreasing absolute risk prudence. The concept of absolute risk prudence is studied by Kimball [?]. Decreasing absolute risk prudence implies increasing absolute risk aversion when agents are strictly risk averse.

When a type i agent is productive, he chooses the labor input to maximize his expected discounted present value of utility, trading the gain in the utility of consumption made available by producing against the disutility of labor. An unproductive agent chooses $\ell = 0$ to avoid any disutility from labor.

1.1.1 Complete Contingent Claims Equilibrium

We will discuss the complete markets case in two ways. First, we describe the equilibrium allocations under the assumption that all trades are made at time zero. In the second case, we provide a sequential definition of the equilibrium. For the sequential equilibrium, we describe several schemes. The first is to allow borrowing while the second is to treat each type of individual as a firm issuing claims on its output stream.

In our discussion below, we assume that $W(\ell) = \ell$. In equilibrium, all agents of the same type will be identical and so we will describe the behavior of the representative type i agent. All variables are expressed as per-capita.

At time zero, all agents trade in the market for claims to consumption and labor supply contingent on state ω at time t . In other words, define $\{\mathcal{F}_t\}_{t=1}^{\infty}$ as an increasing sequence of σ -algebras generated by $\{s_s(\omega), s \leq t\}$. Agents trade for claims contingent on a given history $\{s_s(\omega), s \leq t\}$ for all $w \in \Omega$. Here $(\Omega, \mathcal{F}, \mathcal{P})$, Ω is the set of sample points, $\mathcal{F} \supset \mathcal{F}_t$ for all t is the set of possible events, which are subsets of Ω . \mathcal{F} is closed under the taking of complements and countable unions.

Define $p_t(\omega)$ as the price of a right to delivery of 1 unit of consumption in state ω at time t for all t and for $\omega \in \Omega$. In a complete markets equilibrium, a representative type i consumer maximizes

$$E[U^i | F_0] = \sum_{t=0}^{\infty} \int_{\Omega} \beta^t [U(c_{i,t}(\omega)) - \ell_{i,t}(\omega)] \mathcal{P}(d\omega) \quad (1.5)$$

subject to the single budget constraint

$$\sum_{t=0}^{\infty} \int_{\Omega} [p_t(\omega)(c_{i,t}(\omega) - \theta_{i,t}(\omega)\ell_{i,t}(\omega))] \mathcal{P}(d\omega) \leq 0, \quad (1.6)$$

and the constraints that $c_{i,t}(\omega) \geq 0$ and $\ell_{i,t}(\omega) \geq 0$. Notice that in (??) and (??), we integrate over all possible realizations ω and sum over time.

The equilibrium allocations must also satisfy the market-clearing conditions, which indicate that the consumption of the two types of individuals at any given date and in a given state must equal the total amount produced at that date and in that state. More precisely,

$$\begin{aligned} \alpha c_{1,t}(\omega) + (1 - \alpha)c_{2,t}(\omega) = \\ \alpha \theta_{1,t}(\omega)\ell_{1,t}(\omega) + (1 - \alpha)\theta_{2,t}(\omega)\ell_{2,t}(\omega). \end{aligned} \quad (1.7)$$

Now let us characterize the complete markets equilibrium. Let λ_i denote the Lagrange multiplier associated with the budget constraint for a type i agent. The first-order conditions with respect to $c_{i,t}(\omega)$ and $\ell_{i,t}(\omega)$ for an agent of type i are

$$\beta^t U'(c_{i,t}(\omega)) = \lambda_i p_t(\omega), \quad (1.8)$$

$$\beta^t = \lambda_i \theta_{i,t}(\omega) p_t(\omega). \quad (1.9)$$

When $\theta_{i,t}(\omega) = 1$, the first-order conditions for agent i imply that $\lambda_i = \beta^t/p_t(\omega)$, and $U'(c_{i,t}(\omega)) = 1$. Define the function g as

$$g(x) = (U')^{-1}(x),$$

which is well-defined because marginal utility is strictly concave. Define \bar{c} as $\bar{c} \equiv g(1)$. The consumption of an agent i with $\theta_{i,t}(\omega) = 1$ (a productive agent) equals \bar{c} . When $s_t(\omega) = i$, the price satisfies $p_t(\omega) = \beta^t/\lambda_i$. Notice that λ_i does not vary over time or over realizations ω .

An unproductive agent (one with $\theta_{j,t}(\omega) = 0$) chooses $\ell_{j,t}(\omega) = 0$ and sets consumption to satisfy $\beta^t U'(c_{j,t}(\omega)) = \lambda_j p_t(\omega)$. Substituting in the price $p_t(\omega)$, which we related earlier to the multiplier for the productive agent λ_i , we have

$$\beta^t U'(c_{j,t}(\omega)) = \beta^t \lambda_j / \lambda_i, \quad (1.10)$$

so that the consumption of the unproductive agent satisfies

$$\hat{c}_{j,t} = g(\lambda_j / \lambda_i). \quad (1.11)$$

We now look at the market-clearing conditions. For any t, ω such that $\theta_{1,t}(\omega) = 1$,

$$\begin{aligned} \alpha c_{1,t}(\omega) + (1 - \alpha) c_{2,t}(\omega) &= \alpha g(1) + (1 - \alpha) g(\lambda_2 / \lambda_1) \\ &= \alpha \ell_{1,t}(\omega). \end{aligned} \quad (1.12)$$

For any t, ω such that $\theta_{2,t}(\omega) = 2$, market-clearing requires

$$\begin{aligned} \alpha c_{1,t}(\omega) + (1 - \alpha) c_{2,t}(\omega) &= \alpha g(\lambda_1 / \lambda_2) + (1 - \alpha) g(1) \\ &= (1 - \alpha) \ell_{2,t}(\omega). \end{aligned} \quad (1.13)$$

To proceed further, we make a simplifying assumption. Assume that the random variable $s_t(\omega)$ is i.i.d and that the probability that $s_t(\omega) = 1$ is π so that the probability that $s_t(\omega) = 2$ is $1 - \pi$. The expected present value of lifetime earnings of a type 1 agent are

$$\sum_{t=0}^{\infty} \int_{\Omega} p_t(\omega) \theta_{1,t}(\omega) \ell_{1,t}(\omega) = \frac{1}{1 - \beta} \frac{\pi \ell_1}{\lambda_1},$$

where we have substituted $p_t(\omega) = \beta^t / \lambda_1$ for $t \geq 0$. The expected present value of the type 1 agent's consumption stream is

$$\sum_{t=0}^{\infty} \int_{\Omega} p_t(\omega) c_{1,t}(\omega) = \frac{1}{1 - \beta} \frac{\pi g(1)}{\lambda_1} + \frac{1}{1 - \beta} \left[\frac{(1 - \pi)}{\lambda_2} g(\lambda_1 / \lambda_2) \right].$$

Equating the two expressions and using the market-clearing condition (??) to solve for $\ell_1 - g(1)$, we have

$$\left(\frac{1 - \alpha}{\alpha} \right) g \left(\frac{\lambda_2}{\lambda_1} \right) = \left(\frac{1 - \pi}{\pi} \right) \left(\frac{\lambda_1}{\lambda_2} \right) g \left(\frac{\lambda_1}{\lambda_2} \right). \quad (1.14)$$

We can repeat the same steps for a type 2 agent; this results in

$$\left(\frac{\alpha}{1-\alpha}\right)g\left(\frac{\lambda_1}{\lambda_2}\right) = \left(\frac{\pi}{1-\pi}\right)\frac{\lambda_2}{\lambda_1}g\left(\frac{\lambda_2}{\lambda_1}\right).$$

Define $x \equiv \lambda_1/\lambda_2$; then the equilibrium condition (??) becomes

$$\left(\frac{1-\alpha}{\alpha}\right)g\left(\frac{1}{x}\right) = \left(\frac{1-\pi}{\pi}\right)xg(x). \quad (1.15)$$

We now consider an important case corresponding to the complete market equilibrium studied in Scheinkman and Weiss. Suppose that $\pi = 1/2$ and that $\alpha = 1/2$. Then each period half of the agents are productive and each type of agent expects to be productive with the same probability as any other agent. Under these assumptions, a stationary solution is $\lambda_1 = \lambda_2 = 1$. In this case, individual $a \in A$ consumes a constant amount equal to \bar{c} at all dates and in all states. Output is constant and equal to $2\bar{c}$. Prices are also constant and the real interest rate r satisfies

$$E_t\left(\frac{p_{t+1}}{p_t}\right) = \frac{1}{1+r} = \beta.$$

This is the case of complete insurance in which the opportunities to pool risks enable all agents to consume a fixed amount regardless of the particular realization ω which determines their earnings stream.

Suppose now that $\pi = 2/3$ but retain the assumption that $\alpha = 1/2$. Then each period, one half of the agents are productive just as before. But now notice that the expected present value of the lifetime earnings for a type 1 agent is greater than that of a type 2 agent. Equation (??) now becomes

$$\frac{1}{2}g(x)x = g\left(\frac{1}{x}\right). \quad (1.16)$$

Suppose utility displays constant relative risk aversion so that $U'(c) = c^{-\gamma}$. Then the solution is $x = (\frac{1}{2})^{\frac{\gamma}{2-\gamma}}$. The real interest rate r_1 when type 1 agents are productive ($s_t(\omega) = 1$) is

$$\frac{1}{1+r_1} = \pi\beta + \beta(1-\pi)x,$$

and the real interest rate r_2 when type 2 agents are productive ($s_t(\omega) = 2$) is

$$\frac{1}{1+r_2} = \pi x\beta + \beta(1-\pi).$$

Hence both agents experience fluctuations in consumption over time, depending on the realization of the random variable. The economy experiences aggregate fluctuations in output, prices and real interest rates because agents are no longer identical in expected present value of expected lifetime earnings. There is no market incompleteness here and risks are pooled.

When agents have the same discounted present value of labor income, then they can borrow and lend to smooth consumption to an extent that the agent's consumption is no longer dependent on the particular time path of his wealth. When agents are no longer identical in expected present value, the ability of each agent to smooth consumption is affected. When $\pi = 2/3$, agent 1 is better off with fluctuating consumption than in the case where consumption is constant at \bar{c} . The same result would hold if $\pi = 1/2$ and $\alpha \neq 1/2$. In that case, although any individual agent expects to be productive with the same probability as any other agent, the proportion of agents that are productive varies so that aggregate output fluctuates because of the concentration of the productivity shock.

A feature of the complete market equilibrium that emerges in both of the examples considered above is that the marginal rate of substitution for consumption and at any future date and state is equated across consumers. To see this, consider the ratios of (??) for any dates t and τ , with $\tau > t$, and any possible realized state $\omega \in \Omega$:

$$\frac{\beta^\tau U'(c_{i,\tau}(\omega))}{\beta^t U'(c_{i,t}(\omega))} = \frac{p_\tau(\omega)}{p_t(\omega)}.$$

Since this condition holds for any state, it also holds in expectation. For simplicity, let $\tau = t + 1$. Then:

$$E_t \left(\frac{\beta U'(c_{i,\tau}(\omega))}{U'(c_{i,t}(\omega))} \right) = E_t \left(\frac{p_{t+1}(\omega)}{p_t(\omega)} \right) \equiv \frac{1}{1 + r_t}.$$

Thus, r_t varies in equilibrium if there is *ex ante* heterogeneity among agents, which translates into *ex post* heterogeneity. It will also vary if there are aggregate shocks in the economy. In the next section, we consider a sequential interpretation of the complete markets equilibrium, and describe how to derive the stochastic discount factor used to value risky payoffs.

1.1.2 Sequential Equilibrium

Instead of assuming that all trades take place at time 0 in terms of contingent claims, imagine instead that agents make their consumption and labor decisions sequentially. We discuss two cases. First we permit borrowing and lending. This can take place through a financial intermediary or by transactions among individual agents. We then look at the case where a household issues an equity share which is a claim to some portion of its earnings. When the household is productive, it pays a dividend to shareholders while when it is not productive, no dividend is paid.

In all cases, we search for a stationary equilibrium.

Borrowing and Lending

Suppose that borrowing and lending are permitted. Assume that there is a

durable and nondepletable asset that is fixed in per-capita supply at one unit. The asset is bought and sold at a real price q_t at time t . Let $z_{a,t}$, where $a \in A$, denote the asset holdings of agent a at time t . If $a \in B$ then agent a is a type 1 and a type 2 otherwise. Let $x_{i,t}$ denote the holdings of the asset by the representative type i agent at the beginning of time t .

The supply of the asset is fixed at unity. Thus, market-clearing requires that

$$\alpha x_{1,t} + (1 - \alpha)x_{2,t} = 1 \quad (1.17)$$

We can determine x_2 if we know x_1 so that we need only keep track of the per-capita asset holdings of one type of agent. We will find it convenient later on to let x be the vector (x_1, x_2) and to let the state of the system be described by the pair (x, s) .

The representative type i agent, for $i = 1, 2$ chooses stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - \ell_{i,t}] \right\}, \quad (1.18)$$

subject to the set of constraints

$$z_{i,t+1} - z_{i,t} = (y_{i,t} - c_{i,t})/q_t, \quad (1.19)$$

$$y_{i,t} = \theta_i(s_t)\ell_{i,t}, \quad (1.20)$$

$$\ell_{i,t} \geq 0, \quad c_{i,t} \geq 0, \quad (1.21)$$

and given the initial distribution of the asset, which satisfies

$$1 = \alpha x_{1,0} + (1 - \alpha)x_{2,0}.$$

We will assume that all agents of the same type are identical so that in equilibrium $z_a = x_1$ if $a \in B$ and $z_a = x_2$ otherwise. We allow z and x to be negative; this can be interpreted as a debt (or borrowing). Let $\xi_{i,t}$ be the multiplier associated with the constraint. The state variables clearly consist of s and the vector x . The state of an individual agent depends on his asset holdings z_a and the system state variables, (x, s) .

We now set this up as a dynamic programming problem. The problem is

$$V_i(z, x, s) = \max_{\{c, \ell, z'\}} [U(c) - \ell + \beta \int_S V_i(z', x', s') F(s, ds')] \quad (1.22)$$

subject to the constraints (??)-(??) and the law of motion for x . We assume that $z \in \mathcal{Z} = [-\bar{z}, \bar{z}]$ where $1 < \bar{z} < \infty$, $\ell \in [0, L]$ where $L < \infty$, and $c \in [0, \bar{Y}]$ where $\bar{Y} < \infty$. The upper bound \bar{Y} can be justified by setting $\bar{Y} = 2L$ which is the maximum possible output that could be attained. Also assume that $x_i \in \mathcal{Z}$

for $i = 1, 2$. Define $\mathcal{S} \equiv \mathcal{Z} \times \mathcal{S}$. Let Q be the set of functions $q : \mathcal{S} \rightarrow \mathfrak{R}_+$ such that $\{q : 0 < q(x, s) < \infty, (x, s) \in \mathcal{S}\}$.

Notice that if $q(x, s)$ is strictly positive, then the set of values $\{c, \ell, z'\}$ satisfying (?? – ??) can be denoted $\phi(z, x, s)$; this set is compact and convex-valued. If q is continuous, then under assumption (??), ϕ is continuous in s . Let \mathcal{V} be the space of bounded, continuous, real-valued functions $V_i(z, x, s)$ on $\mathcal{Z} \times \mathcal{S}$ with the norm $\|V_i\| = \sup |V_i(z, x, s)|$. Given any continuous, strictly positive price, it is straightforward to show that there exists a unique value function satisfying (??). This summarizes the information we need for the individual agent $a \in A$. Let us define an equilibrium for this economy.

Definition 1.1 *A stationary equilibrium is a set of functions $\hat{q} : \mathcal{Z} \times \mathcal{S} \rightarrow \mathfrak{R}_+$, $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, and $\hat{z}_i(z, x, s)$, defined on $\mathcal{Z} \times \mathcal{S}$ and measurable with respect to \mathcal{F}_t , such that*

(i) $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, and $\hat{z}_i(z, x, s)$ solve (??) subject to the constraints (??), (??) and (??);

(ii) markets clear:

$$1 = \alpha \hat{z}_1(z_1, x, s) + (1 - \alpha) \hat{z}_2(z_2, x, s), \quad (1.23)$$

and

$$\alpha \hat{y}_1 + (1 - \alpha) \hat{y}_2 = \alpha \hat{c}_1(z_1, x, s) + (1 - \alpha) \hat{c}_2(z_2, x, s), \quad (1.24)$$

where $\hat{y}_i = \theta_i(s) \hat{\ell}_i(z_i, x, s)$, and

(iii) the laws of motion for the system variables x evolve as

$$x'_i = \hat{z}_i(x_i, x, s), \quad i = 1, 2. \quad (1.25)$$

Recall that if we know x_1 , then we can determine x_2 from the market-clearing condition (??). Without any loss of information, we can define $\xi_i(x_i, s)$ as the equilibrium multiplier on the constraint (??) for the representative type i agent when the average holdings of the durable asset by the type i agents at the beginning of the period is x_i . The pair (x_i, s) completely describes the state of the system. Similarly, let $c_i(x_i, s)$ denote the equilibrium consumption for the representative type i agent, and let $\ell_i(x_i, s)$ denote the equilibrium labor supply. Finally, let φ_i denote the partial derivative of the value function with respect to its first argument, or $\varphi_i(x_i, s) = V'_i(x_i, x, s)$.

The equilibrium first-order conditions for the representative type i agent are

$$U'(c_i(x, s)) = \frac{\xi_i(x_i, s)}{q(x, s)} \quad (1.26)$$

$$1 = \frac{\theta_i(s) \xi_i(x_i, s)}{q(x, s)} \quad (1.27)$$

$$\xi_i(x_i, s) = \beta E_s[\varphi_i(x'_i, s')]. \quad (1.28)$$

The envelope condition is $\varphi_i(x_i, s) = U'(c_i(x_i, s))q(x, s)$.

If $\theta_i(s) = 1$, then $\xi_i(x_i, s)/q(x, s) = 1$ and $c_i(x_i, s) = \bar{c}$, where \bar{c} was defined earlier. Suppose that agents of type i are productive while type j agents are not. Then the first-order condition for the representative type j agent is

$$U'(c_j(x_j, s)) = \xi_j(x_j, s)/q(x, s) = \frac{\xi_j(x_j, s)}{\xi_i(x_i, s)},$$

where we have substituted $q(x, s) = \xi_i(x_i, s)$. It becomes apparent when we compare the first-order conditions for the sequential equilibrium with the contingent claims equilibrium, that $\xi_i = \lambda_i$ and that the price of the asset equals the Lagrange multiplier for the productive agent, which is independent over dates and alternative states of the economy.

In the discussion above, the borrowing and lending takes place between individuals of different types. Instead of borrowing and lending, suppose that agents act as if they were firms and issue equities shares that are claims to their earnings stream. We will show that the allocation is the same as that under the borrowing and lending.

Equities Trading

We assume that there is one outstanding claim to each earnings stream. Let $z_i^j(s_t)$ denote the shares to the j th earnings stream in state s_t held by agent i . The share sells at the price Q_t^j . The sum of shares satisfies

$$\alpha x_{i,t}^j + (1 - \alpha)x_{j,t}^j = 1, \quad i, j = 1, 2. \quad (1.29)$$

An agent of type i buys shares of the equity issued by agents of type j . The type i agent also issues shares and pays dividends. Agent i maximizes (??) subject to the current period budget constraint

$$\begin{aligned} c_{i,t} + Q_t^i x_{i,t+1}^i + Q_t^j x_{i,t+1}^j &\leq (Q_t^i + d_{i,t})x_{i,t}^i \\ &+ (Q_t^j + d_{j,t})x_{i,t}^j + \theta_{i,t}\ell_{i,t} - d_{i,t}(\alpha x_{i,t}^i + (1 - \alpha)x_{j,t}^i). \end{aligned} \quad (1.30)$$

We have written this constraint assuming that the representative type i also buys and sells claims to his own earnings stream. To be consistent, we also assume that dividends equal output, or $d_{i,t} = \theta_{i,t}\ell_{i,t}$. We will see shortly that many of these terms will drop out. To start, we look for a stationary equilibrium in which $x_{i,t}^j = x_{i,t+1}^j$ and (??) is satisfied. There may be other stationary solutions, but we examine only this one because, as we will show it has a natural interpretation. Under these assumptions, the budget constraint becomes

$$c_{i,t} \leq d_{i,t}x_{i,t}^i + d_{j,t}x_{i,t}^j. \quad (1.31)$$

Let $\xi_{i,t}$ denote the multiplier associated with the constraint (??) for consumer i . If $s_t = i$, then $\theta_j(s_t) = 0$ for $j \neq i$, and no output is produced by type

j agents so that $d_{j,t} = 0$. The first-order conditions imply that when $s_t = 1$, type 1 agents consume \bar{c} , and type 2 agents consume $g(\xi_{2,t})$. Recall that in the contingent claims equilibrium, the share of output consumed by a type 1 agent when $\theta_1 = 1$ was $\bar{c}/\theta_1\ell_1$ and the share consumed by type 2 agents was

$$\frac{1-\alpha}{\alpha\theta_1\ell_1}g(\lambda_2/\lambda_1).$$

When $\theta_2 = 1$, the share of output consumed by a type 2 agent was $\bar{c}/\theta_2\ell_2$ and the share consumed by type 1 agents was

$$\frac{\alpha}{1-\alpha\theta_2\ell_2}g(\lambda_1/\lambda_2).$$

Setting the equity shares equal to the consumption shares,

$$\begin{aligned} x_1^1 &= \frac{g(1)}{\theta_1\ell_1}, & x_2^1 &= \frac{1-\alpha}{\alpha} \frac{g(\lambda_2/\lambda_1)}{\theta_1\ell_1}, \\ x_1^2 &= \frac{g(1)}{\theta_2\ell_2}, & x_2^2 &= \frac{\alpha}{1-\alpha} \frac{g(\lambda_1/\lambda_2)}{\theta_2\ell_2}. \end{aligned}$$

One can verify that this allocation is market-clearing and satisfies the first-order conditions. To show that these distribution of shares, together with the implied consumption and labor supply allocations, can be used to replicate the complete markets equilibrium, merely set $\xi_{i,t}(\omega) = \lambda_i p_t(\omega)$ for each $\omega \in \Omega$.

The first-order conditions can be used to find expressions for equity prices as⁴

$$Q_t^j = E_t \left[\frac{\beta U'(c_{i,t+1})}{U'(c_{i,t})} (Q_{t+1}^j + d_{j,t+1}) \right], \quad i, j = 1, 2. \quad (1.32)$$

Under the complete markets assumption, consumers set their intertemporal MRS's equal to the common ratio $p_{t+1}(\omega)/p_t(\omega)$. Since this ratio varies with ω , the price of a claim to type i 's earning stream is the expected discounted value of that stream. We could introduce aggregate uncertainty into this setup, and allow for a production technology that yields an exogenous output stream $\{d_t\}$. Then the price of a claim to this output stream would be determined as in (??), with the common intertemporal MRS used as the stochastic discount factor.

We already studied the pricing of such claims in the representative consumer pure exchange economy of Chapter 2. There the stochastic discount factor is equal to the random intrtemporal MRS of the representative consumer and can be evaluated using a parametric specification of preferences

⁴In this case, we use the budget constraint (??) with the last two terms equal to each other, and differentiate with respect to $x_{i,t}^j$ for $i, j = 1, 2$.

and aggregate or per-capita consumption data.⁵ With incomplete markets, there is in general no common stochastic discount factor, and asset pricing relations based on an intertemporal MRS evaluated with aggregate or per-capita consumption data are not valid. Likewise, market frictions such as short sales constraints and bid-ask spreads will alter the relationship between individual intertemporal MRS's and the common stochastic discount factor used to value random payoffs. The results of Luttmer [?] and He and Modest [?] suggest that we can construct volatility bounds for stochastic discount factors in the presence of various forms of market frictions, provided there exist a complete set of contingent claims. We describe their methods in a later section. For the incomplete markets case, we address the issue of characterizing the stochastic discount factor by analyzing a specific model with frictions.

1.1.3 Borrowing Constraints

From our review of the empirical evidence based on representative consumer models, we know that such models fail to account for the temporal behavior of asset returns due to the lack of correlation of aggregate consumption growth with asset returns. Thus, explaining the empirical facts requires that the link between the intertemporal MRS and asset returns be loosened. As we noted before, market incompleteness and borrowing constraints are frictions that can potentially accomplish this. In a later section, we discuss other types of frictions, such as transactions costs, that may have the same effect.

Several authors, including Bewley [?] and by Mankiw [?], have noted that introducing market frictions can help to explain the equity premium. Mankiw uses a two-period model in which the risk-free rate is fixed and shows how the concentration of idiosyncratic shocks throughout the population affects the equity premium. Under certain circumstances, an econometrician who uses per-capita consumption series and a representative agent framework will over-predict the degree of risk aversion required to generate an equity premium of the magnitude observed. Hence, a potential explanation of the equity premium puzzle is that it is an artifact of the representative agent model. Problem 3 below is based on the Mankiw paper. This point has been further studied by Weil [?] who also models the risk-free rate. Kahn [?] also develops a two-period model with moral hazard and imperfect risk sharing.

There is a recent literature studying these issues using infinite-lived agent models. While the results are very preliminary, there appear to be some general conclusions. In a model with no aggregate uncertainty and with i.i.d. shocks for individuals, Aiyagari and Gertler [?] have found in simulations that the borrowing constraints did not generate enough volatility of asset returns.

⁵The Euler equation tests that we described in Chapter 4 typically assume *ex ante* heterogeneity among consumers but rely on the aggregation conditions specified by Rubenstein [?] in order to derive representations involving aggregate or per-capita consumption.

To improve their results, they also included transactions costs. This is similar to the results of the papers by Heaton and Lucas [?], who work with a three-period model and incorporate transactions costs, short sales constraints and borrowing constraints. Telmer [?] develops a model in which there is both aggregate and individual uncertainty. While he is unable to prove formally existence and uniqueness of equilibrium, he does have a computational algorithm which allows him to simulate the model. He finds that introducing a risk-free asset allows the agents to do a great deal of consumption smoothing.⁶ Constantinides and Duffie [?] have pointed out that in most of these models, the idiosyncratic labor income shocks are i.i.d. and hence, transient so that the permanent income of agents is almost equal across agents despite imperfect risk sharing. Hence, the consumption smoothing opportunities afforded by a risk-free bond are almost enough to allow risk sharing and that this is the reason transactions costs and short sales constraint are needed.

We now study equilibrium with incomplete markets and borrowing constraints. We first describe how to prove existence and uniqueness of the competitive equilibrium for this model, and then study some of its implications. To construct the equilibrium, we start by fixing the marginal valuation function for the asset, which is equal to the Lagrange multiplier on the budget constraint. We then determine the price that clears the market, holding the Lagrange multiplier fixed. The market-clearing price that results is then held fixed as we solve for the marginal value function. The method of proof in this step follows that of Deaton and Laroque [?]. We then show that the marginal value functions are increasing and concave in the market-clearing price. In the final step, we show that there exists a unique price function that clears the market that is also used to construct the marginal valuation function.

All variables are measured as per-capita. We retain the assumption that there is one unit of the durable asset. We will search for a stationary equilibrium. The state of the system at time t is described by the Markov process s_t and the distribution of the durable asset across the type 1 and 2 agents. As before, we assume that agents within a class – type 1 or type 2 – are identical. The proportion of each type of agent in the population is fixed. Let x_i denote the amount of the asset held by the average type i agent and let x be the vector (x_1, x_2) . At the beginning of the period, the distribution of the durable asset across agents satisfies

$$1 = \alpha x_1 + (1 - \alpha)x_2. \tag{1.33}$$

The state of the economy is summarized by (x, s) .

The representative type i agent, for $i = 1, 2$, chooses stochastic sequences

⁶Other related papers are by Brown [?] and Danthine, Donaldson, and Mehra [?].

$\{c_{i,t}, \ell_{i,t}\}$ to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - W(\ell_{i,t})] \right\}, \quad (1.34)$$

subject to the set of constraints

$$z_{i,t+1} - z_{i,t} = (y_{i,t} - c_{i,t})/q_t, \quad (1.35)$$

$$y_{i,t} = \theta_i(s_t)\ell_{i,t}, \quad (1.36)$$

the nonnegativity constraints

$$z_{i,t+1} \geq 0, \ell_{i,t} \geq 0, c_{i,t} \geq 0, \quad (1.37)$$

and initial conditions $x_{i,0} = z_{i,0}$ with $1 = \alpha x_{1,0} + (1 - \alpha)x_{2,0}$.

There are two features worth noting about this problem. First, it rules out complete insurance of idiosyncratic risk by ruling out the existence of prices to consumption contingent on any possible history $\{s_s(\omega), s < t\}$. This can be described in terms of the underlying probability space. One possible reason for idiosyncratic risk to be uninsurable may be that the shocks to individuals' productivity are not publicly observable. Second, the above problem assumes that individuals in this economy face borrowing constraints. The borrowing constraints are introduced through the constraint that the asset holdings of the consumer must be nonnegative at all dates and all states; i.e., $z_{i,t+1} \geq 0$. Notice that if the agent is productive, he chooses both consumption and labor supply. Otherwise, he chooses only consumption. Also, when the agent is unproductive, he is able to consume a positive amount by running down his asset holdings.

We now study the consumer's problem as a dynamic programming problem. The average type i agent who begins the period with asset holdings z solve

$$V_i(z, x, s) = \max_{\{c, \ell, z'\}} \left[U(c) - W(\ell) + \beta \int_S V_i(z', x', s') F(s, ds') \right] \quad (1.38)$$

subject to the constraints (??)–(??) and the law of motion for x . We assume that $z \in \mathcal{Z} = [0, \bar{z}]$ where $1 < \bar{z} < \infty$, $\ell \in [0, L]$ where $L < \infty$, and $c \in [0, \bar{Y}]$ where $\bar{Y} < \infty$. Also assume that $x_i \in \mathcal{Z}$ for $i = 1, 2$.

The equilibrium price is a function $q : \mathcal{S} \rightarrow \mathbb{R}_+$ such that $\{q : 0 < q(x, s) < \infty, (x, s) \in \mathcal{S}\}$. Notice that if q is strictly positive, then the set of values $\{c, \ell, z'\}$ satisfying (?? – ??), denoted $\phi(z, x, s)$, is compact and convex-valued. If q is continuous, then under Assumption ??, ϕ is continuous in s . Let \mathcal{V} be the space of bounded, continuous, real-valued functions $V_i(z, x, s)$ on $\mathcal{Z} \times \mathcal{S}$ with the norm $\|V_i\| = \sup |V_i(z, x, s)|$. Given any continuous, strictly positive

price q , it is straightforward to show that there exists a unique value function satisfying (??). This summarizes the information we need for the individual agent $a \in A$. Let us define an equilibrium for this economy.

Definition 1.2 A stationary equilibrium is a set of functions $\hat{q} : \mathcal{Z} \times \mathcal{S} \rightarrow \mathfrak{R}_+$, $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, and $\hat{z}_i(z, x, s)$, defined on $\mathcal{Z} \times \mathcal{S}$ and measurable with respect to \mathcal{F}_t , such that

(i) $\hat{c}_i(z, x, s)$, $\hat{\ell}_i(z, x, s)$, and $\hat{z}_i(z, x, s)$ solve (??) subject to the constraints (??)–(??);

(ii) markets clear:

$$1 = \alpha \hat{z}_1(z_1, x, s) + (1 - \alpha) \hat{z}_2(z_2, x, s), \quad (1.39)$$

and

$$\alpha \hat{y}_1 + (1 - \alpha) \hat{y}_2 = \alpha \hat{c}_1(z_1, x, s) + (1 - \alpha) \hat{c}_2(z_2, x, s), \quad (1.40)$$

where $\hat{y}_i = \theta_i(s) \hat{\ell}_i(z_i, x, s)$, and

(iii) the laws of motion for the system variables x evolve as

$$x'_i = \hat{z}_i(x_i, x, s), \quad i = 1, 2. \quad (1.41)$$

Recall that if we know x_1 , then we can determine x_2 from the market-clearing condition (??). Without any loss of information, we can define $\xi_i(x_i, s)$ as the equilibrium multiplier on the constraint (??) for the representative type i agent when the average holdings of the durable asset by the type i agents at the beginning of the period is x_i . The pair (x_i, s) completely describes the state of the system. Similarly, let $c_i(x_i, s) = \hat{c}_i(x_i, x, s)$ denote the equilibrium consumption for the representative type i agent, and let $\ell_i(x_i, s) = \hat{\ell}_i(x_i, x, s)$ denote the equilibrium labor supply. Finally, let φ_i denote the partial derivative of the value function with respect to its first argument, or $\varphi_i(x_i, s) = V'_i(x_i, x, s)$.

The equilibrium first-order conditions for the representative type i agent are

$$U'(c_i(x_i, s)) = \frac{\xi_i(x_i, s)}{q(x, s)}, \quad (1.42)$$

$$W'(\ell_i(x_i, s)) = \frac{\theta_i(s) \xi_i(x_i, s)}{q(x, s)}, \quad (1.43)$$

$$\xi_i(x_i, s) = \beta E_s[\varphi_i(x'_i, s')] + \mu_i, \quad (1.44)$$

where μ is the multiplier on the nonnegativity constraint for $z_{i,t+1}$ so that $\mu_i = 0$ only if $z_{i,t+1} > 0$. The envelope condition is $\varphi_i(x_i, s) = U'(c_i(x_i, s))q(x, s)$. When $\theta_i = 1$, the agent always produces enough so that $z_{i,t+1} > 0$ and $\mu = 0$. When $\theta_i = 0$, the maximum that the agent can consume is $c_i = z_i q$. Hence the multiplier ξ_i obeys

$$\xi_i(x_i, s) = \max [U'(x_i q(x, s))q(x, s), \beta E_s \xi_i(x'_i, s')]. \quad (1.45)$$

Define the function h by

$$h(k) \equiv (W')^{-1}(k)$$

for $k \geq 0$ so that $h : \mathfrak{R}_+ \rightarrow [0, L]$. Recall that the definition of the function g is $g \equiv (U')^{-1}$ so that $g : \mathfrak{R}_+ \rightarrow [0, \bar{Y}]$. Given (x, s) , for fixed $\xi_i \geq 0$ and $q > 0$, equations (??-??) are four equations ($i = 1, 2$) in four unknowns $(c_1, c_2, \ell_1, \ell_2)$ which are the value of the functions $c_i(x_i, s)$ and $\ell_i(x_i, s)$ when $\xi_i = \xi_i(x_i, s)$ and $q = q(x, s)$. The values (c_i, ℓ_i) satisfy $c_i = g(\xi_i/q)$ and $\ell_i = h(\theta\xi_i/q)$. For notational convenience, define the function H as

$$H(k, \theta) \equiv \theta h(\theta k) - g(k).$$

Proposition 1.1 *Under Assumption ??, $H_1 > 0$ and $H_{1,1} < 0$, where H_i denotes the partial derivative with respect to the i th argument. Also, $\lim_{k \rightarrow 0} H(k, \theta) = -B = -\infty$ and $\lim_{k \rightarrow \infty} H(k, \theta) = L$.*

It is straightforward to verify these properties under Assumption ??. Notice that g is a function satisfying $U'(g(k)) = k$ such that $g'(k) = (U'')^{-1} < 0$ and $g''(k) = -U'''/(U'')^2 < 0$. The function h satisfies $W'(h(k)) = k$. The solution (c_i, ℓ_i) for $i = 1, 2$ to equations (??-??) can be used in the budget constraints (??) to solve for the average asset holdings next period x'_i of type i agents.

So far we have established that for fixed (x, s) and given $q > 0$ and $\xi_i \geq 0$, equations (??) and (??-??) form a system of six equations in six unknowns. We now fix only the functions ξ_i and determine the value of the price q such that markets clear; essentially we are adding one more equation and one unknown. Substituting for x'_i into (??) and using $\ell_i = \ell_i(x_i, s)$, $c_i = c_i(x_i, s)$ and $\theta_i = \theta_i(s)$, the market-clearing price satisfies

$$\begin{aligned} 1 &= \alpha \left[\frac{\theta_1 \ell_1 - c_1}{q} + x_1 \right] + (1 - \alpha) \left[\frac{\theta_2 \ell_2 - c_2}{q} + x_2 \right] \\ &= 1 + \alpha[\theta_1 \ell_1 - c_1] + (1 - \alpha)[\theta_2 \ell_2 - c_2] \end{aligned}$$

Subtracting 1 from both sides and substituting H and taking as given the values $\xi_i = \xi_i(x_i, s)$ for fixed x_i, s , the market-clearing price $q > 0$ satisfies

$$\alpha H(\xi_1/q, \theta_1) = -(1 - \alpha) H(\xi_2/q, \theta_2). \quad (1.46)$$

We have the following result.

Theorem 1.1 *Under Assumption ??, for fixed $x_i \in \mathcal{Z}$ and $s \in S$ and given $\xi_i = \xi_i(x_i, s)$ such that $\xi_i > 0$, there exists a unique solution $\hat{q} : \mathcal{Z} \times S \times \mathfrak{R}_+ \times \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ that is strictly positive and continuous.*

PROOF.

Under Assumption ??, $\lim_{q \rightarrow 0} H(\xi/q, \theta) = L$ and $\lim_{q \rightarrow \infty} H(\xi/q, \theta) = -B$.

From Proposition ??, it follows that the left-side of (??) is decreasing in q and the right-side is increasing. At $q = 0$, $\lim_{\psi \rightarrow \infty} h(\psi) = L$ and $\lim_{\psi \rightarrow \infty} g(\psi) = 0$ so that $\lim_{\psi \rightarrow \infty} h(\psi) - g(\psi) = L$. As q increases, $h - g$ decreases. For each $s \in S$, either $\theta_1 = 0$ or $\theta_2 = 0$ so that the labor supply is constant at zero for one type of agent. Hence, there exists a unique q solving (??). Because H is continuous in ξ , it follows that q is continuous in ξ . ■

It is straightforward to show by differentiation that \hat{q} is increasing in both ξ_1 and ξ_2 . Given $\xi_i(x_i, s)$, define $q(x, s) \equiv \hat{q}(x, s, \xi_1(x_1, s), \xi_2(x_2, s))$.

Next we hold q fixed such that $0 < q \leq \infty$. In our discussion below, we drop the index of the agent type for convenience. Let $\mathcal{S} \equiv \mathcal{Z} \times S$ and define $\mathcal{C}(\mathcal{S})$ denote the space of continuous bounded functions defined on the state space and let $\mathcal{D}(\mathcal{S}) \in \mathcal{C}(\mathcal{S})$ be the subspace of continuous functions that are nonincreasing in their first argument. The space $\mathcal{D}(\mathcal{S})$ is a Banach space.

We have the following lemma.

Lemma 1.1 *Let $\xi \in \mathcal{D}(\mathcal{S})$ so that ξ is nonincreasing in its first argument. Fix $q \in \mathbb{R}_+$ such that $0 < q < \infty$. Let $\psi \in \mathbb{R}_+$ and let the function $G : \mathbb{R}_+ \times \mathcal{S} \rightarrow \mathbb{R}_+$ be defined by*

$$G(\psi, x, s) \equiv \max \left[U'(xq)q, \beta \int_S \xi \left(x + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right]. \quad (1.47)$$

Then G is nonincreasing in ψ and x . Furthermore, $\lim_{\psi \rightarrow 0} G(\psi, x, s) = \infty$ and $\lim_{\psi \rightarrow \infty} G(\psi, x, s) = \underline{G} \geq \bar{U} > 0$.

PROOF.

The assumptions on ξ and q ensure that G is nonincreasing in its first argument. An increase in ψ increases $H(\psi/q, \theta)$ which decreases ξ , hence G is nonincreasing in its second argument. As $\psi \rightarrow 0$, $H(\psi, \theta) = -B$ and as $\lim_{\psi \rightarrow \infty} H(\psi, \theta) = L$. As $\psi \rightarrow \infty$, $H(\psi, \theta) = L$. Hence,

$$\underline{G}(x, s) \equiv \lim_{\psi \rightarrow \infty} G(\psi, x, s) = \beta \int_S \xi(x + L/q, s') F(s, ds') \geq 0.$$

■

Let q be fixed as before and let f be a solution to

$$\begin{aligned} f(x, s) &= G(f(x, s), x, s) \\ &= \max \left[U'(xq)q, \beta \int_S \xi \left(x + \frac{1}{q} H(f(x, s)/q, \theta), s' \right) F(s, ds') \right] \end{aligned} \quad (1.48)$$

Define T_1 as the operator that assigns the solution f to the function G so that $f = T_1 G$. We have the following lemma.

Lemma 1.2 Assume that $\xi \in \mathcal{D}(\mathcal{S})$ so that G as defined in (??) satisfies the conditions of Lemma ?.?. For fixed q such that $0 < q < \infty$, let $f : \mathcal{S} \rightarrow \mathbb{R}_+$ be the solution to (??). Then:

- i) There exists a unique $f^* \in \mathcal{C}(\mathcal{S})$ satisfying (??).
- ii) The solution function satisfies $f^* \in \mathcal{D}(\mathcal{S})$ so that $T_1 : \mathcal{D}(\mathcal{S}) \rightarrow \mathcal{D}(\mathcal{S})$.
- iii) If $G_1 \geq G_2$ for all (ψ, x, s) , then $T_1 G_1 \geq T_1 G_2$.

PROOF.

Under the conditions of Lemma ??, $G(\psi, x, s) - \psi$ is continuous and strictly decreasing in ψ . For a fixed (x, s) , $H(\psi/q, \theta)$ is increasing in ψ . For $G \in \mathcal{D}(\mathcal{S})$, let ψ satisfy

$$\begin{aligned} \psi &= G(\psi, x, s) \\ &= \max \left[U'(xq)q, \beta \int_{\mathcal{S}} \xi \left(z + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right]. \end{aligned} \quad (1.49)$$

Clearly the left-side is increasing in ψ which, under the assumption that $G \in \mathcal{D}(\mathcal{S})$, implies that the right-side is decreasing in ψ . As $\psi \rightarrow 0$, the left-side tends to 0 and the right-side tends to ∞ . As ψ increases the left-side increases and the right-side tends to \underline{G} . Hence, there exists a unique ψ that satisfies (??).

It also follows that

$$\max \left[U'(xq)q, \beta \int_{\mathcal{S}} \xi \left(z + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right] - \psi$$

is continuous and strictly decreasing in ψ . Therefore, f^* is continuous and $f^*(x, s)$ is decreasing in its first argument.

Suppose that $G_1 \in \mathcal{D}(\mathcal{S})$ and $G_2 \in \mathcal{D}(\mathcal{S})$ and that $G_1 > G_2$ where

$$\begin{aligned} G_i(\psi, x, s) &\equiv \\ &\max \left[U'(xq)q, \beta \int_{\mathcal{S}} \xi_i \left(z + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right], \end{aligned}$$

for $i = 1, 2$.

Let ψ_1 be the solution to the equation $[0 = G_1(\psi_1, x, s) - \psi_1]$ and let ψ_2 be the solution to $[0 = G_2(\psi_2, x, s) - \psi_2]$. It follows that

$$G_1(\psi_2, x, s) - \psi_2 \geq G_2(\psi_2, x, s) - \psi_2 = 0$$

so that

$$\psi_1 - \psi_2 \geq G_1(\psi_1, x, s) - G_2(\psi_2, x, s).$$

Therefore, $f_1^* = T G_1 \geq T G_2 = f_2^*$. ■

For a fixed $0 < q < \infty$, let the operator $T_2 : \mathcal{D}(\mathcal{S}) \rightarrow \mathcal{D}(\mathcal{S})$ be defined by

$$\begin{aligned} \xi^{n+1}(x, s) &= (T_2 \xi^n)(x, s) \\ &= \max \left[U'(xq)q, \beta \int_S \xi^n \left(x + \frac{1}{q} H \left(\frac{T_2 \xi^n}{q}, \theta \right), s' \right) F(s, ds') \right]. \end{aligned} \quad (1.50)$$

We have the following theorem.

Theorem 1.2 *Let $0 < q < \infty$ be fixed and let $T_2 : \mathcal{D}(\mathcal{S}) \rightarrow \mathcal{D}(\mathcal{S})$ be defined by (1.50). Under Assumption ??, T_2 is a contraction.*

PROOF.

For an initial guess $\xi^0 \in \mathcal{D}(\mathcal{S})$, it is clear that

$$\beta \int_S \xi^0 \left(x + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds')$$

is an element of $\mathcal{D}(\mathcal{S})$ for fixed ψ such that $0 \leq \psi < \infty$. Under the conditions of Lemma ??, the solution

$$\begin{aligned} f^*(x, s) &= \\ &= \max \left[U'(xq)q, \beta \int_S \xi^0 \left(x + \frac{1}{q} H(f^*(x, s)/q, \theta), s' \right) F(s, ds') \right] \end{aligned}$$

is an element of $\mathcal{D}(\mathcal{S})$.

Let $\xi_1, \xi_2 \in \mathcal{D}(\mathcal{S})$ and assume that $\xi_1 > \xi_2$. For fixed ψ , it follows that

$$\begin{aligned} \max \left[U'(xq)q, \beta \int_S \xi_1 \left(x + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right] &\geq \\ \max \left[U'(xq)q, \beta \int_S \xi_2 \left(x + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right]. & \end{aligned}$$

Under the conditions of Lemma ??, it follows that

$$\begin{aligned} T_2 \xi_1(x, s) &= \\ &= \max \left[U'(xq)q, \beta \int_S \xi_1 \left(x + \frac{1}{q} H(T_2 \xi_1/q, \theta), s' \right) F(s, ds') \right] \\ &\geq \max \left[U'(xq)q, \beta \int_S \xi_2 \left(x + \frac{1}{q} H(T_2 \xi_2/q, \theta), s' \right) F(s, ds') \right]. \end{aligned}$$

Hence, $T_2 \xi_1 \geq T_2 \xi_2$ so that T_2 is monotone.

Let $0 < a < \infty$. To show that H has the discounting property, notice that

$$\begin{aligned} \beta \int_S (\xi + a) \left(x + \frac{1}{q} H((\xi + a)(x, s)/q, \theta), s' \right) F(s, ds') &\leq \\ \beta \int_S (\xi + a) \left(x + \frac{1}{q} H(\xi(x, s)/q, \theta), s' \right) F(s, ds') &\leq \\ \beta \int_S \xi \left(x + \frac{1}{q} H(\xi(x, s)/q, \theta), s' \right) F(s, ds') + \beta a, & \end{aligned}$$

so that T_2 has the discounting property. Hence, T_2 has a unique fixed point ξ^* . \blacksquare

To find the equilibrium, the first step is to study the behavior of ξ^* as a function of q . Before we can do this, we show that if U' is convex and W' is concave, then ξ^* is convex in x . The argument is basically as follows. We have established that there exists a unique fixed point in the space $\mathcal{D}(\mathcal{S})$. If we start in the subspace of $\mathcal{D}(\mathcal{S})$ consisting of nonincreasing convex functions and show that the operator T_2 maps those functions into other functions in the same subspace, then, because of uniqueness, we know that the fixed point is a function that is convex in x .

Proposition 1.2 *If $U''' > 0$ and $W''' < 0$, then ξ^* is convex in x .*

PROOF.

Fix q such that $0 < q < \infty$. Suppose that $\xi \in \mathcal{D}(\mathcal{S})$ and that ξ is convex in its first argument. Then the function G defined in (??)

$$G(\psi, x, s) = \max \left[U'(xq)q, \beta \int_{\mathcal{S}} \xi \left(x + \frac{1}{q} H(\psi/q, \theta), s' \right) F(s, ds') \right]$$

is convex in (ψ, x) . For fixed (ψ, s) , it is straightforward to verify convexity in x . To show that G is convex in ψ , notice that

$$\frac{\beta}{q^2} \int_{\mathcal{S}} \left[\xi_{11}(x', s') H'(\psi/q, \theta) + \xi_1(x', s') \frac{1}{q} H''(\psi/q, \theta) \right] F(s, ds') > 0,$$

where the convexity of U' and concavity of W' are used to show that $H'' < 0$. Because G is convex in (ψ, x) , the solution to $[G(\psi, s) - \psi = 0]$ is also convex. Recall that $T_2\xi$ is the solution ψ to the equation $G(\psi, x, s) - \psi = 0$. Let $\lambda \in [0, 1]$ and let $x_1, x_2 \in X$. Define $\psi_1 = T_2\xi(x_1, s)$ and $\psi_2 = T_2\xi(x_2, s)$. Then $G(\psi_1, x_1, s) - \psi_1 = G(\psi_2, x_2, s) - \psi_2 = 0$. Because G is convex,

$$\begin{aligned} & G(\lambda T_2\xi(x_1, s) + (1 - \lambda)T_2\xi(x_2, s), \lambda x_1 + (1 - \lambda)x_2) \\ & - [\lambda T_2\xi(x_1, s) + (1 - \lambda)T_2\xi(x_2, s)] \leq 0. \end{aligned}$$

Since G is decreasing in its first argument, it follows that $T_2\xi$ is convex in x if ξ is convex in x . \blacksquare

For fixed x, s , notice that $x' = x + (1/q)H(\xi^*/q)$ is a function of q ; for notational convenience, we write $x' = a(x, q, s)$. It follows that the fixed point ξ^* is also a function of q ; to emphasize this, we will write $\xi^* = \Omega(q)$. If we can show how x' changes as q varies, we can determine how Ω changes as q varies. The results are summarized in the next proposition.

Proposition 1.3 *Let $0 < q < \infty$ and let $\Omega(q) = \xi_q^*$ be a fixed point of T_2 where T_2 is defined in (??). Then ξ_q^* is continuous, increasing and concave in q .*

PROOF.

For a fixed $0 < \psi < \infty$, notice that

$$\frac{\partial a(x, q, s)}{\partial q} = -\frac{1}{q^2} \left[H(\psi/q) + \frac{\psi}{q} H'(\psi/q) \right] < 0.$$

For this to be true for the unproductive agent, we have used the condition $-U''c/U' < 1$ in Assumption ???. It follows that ξ^* is increasing in q .

Next,

$$\frac{\partial^2 a(x, q, s)}{\partial q^2} = \frac{1}{q^3} \left[\frac{\psi}{q} \left(H' + \frac{H''\psi}{q} \right) + H + \frac{2H'\psi}{q} \right] > 0$$

because, under Assumption ???,

$$H' + \frac{H''\psi}{q} = \frac{W'}{(W'')^2} \left[\frac{W''}{W'} - \frac{W'''}{W''} \right] - \frac{U'}{(U'')^2} \left[\frac{U''}{U'} - \frac{U'''}{U''} \right] > 0.$$

Hence x' is decreasing and convex in q . Because ξ^* is nonincreasing and convex in x' and

$$\frac{\partial}{\partial q} \left[\frac{\partial \xi_1^*(x', s)}{\partial x'} \frac{\partial x'}{\partial q} \right] = \xi_{11}^* a_1(q, x, s) + \xi_1^* a_{11}(q, x, s),$$

it follows that ξ^* is nonincreasing and concave in q ; or $\Omega' > 0$ and $\Omega'' < 0$. \blacksquare

Although we have found a fixed point ξ_q^* for a given q and determined the market clearing price \hat{q} for given ξ_1 and ξ_2 , we have not shown that $q = \hat{q}$. In fact, we must address the issues of whether a solution exists and if it exists, whether it is unique. Fix (x, s) and define the function $\Omega_i(q) = \xi_{i,q}^*$, which expresses the fixed point ξ^* as a function of q . Define the function $\nu : [0, \infty) \rightarrow [0, \infty)$ as the solution to

$$\alpha \left[H \left(\frac{\Omega_1(q)}{\nu(q)}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{(q)}, \theta_2 \right) \right] = 0. \quad (1.51)$$

For notational convenience, let H'_i denote the partial derivative with respect to the first argument of H for an agent of type i . We have the following proposition.

Proposition 1.4 *Let ν be as defined in (??). Under Assumption ???, ν is increasing and concave.*

PROOF.

Differentiating (??) with respect to q and solving for ν' , we have

$$\begin{aligned} \nu'(q) &= \left[\alpha H'_1 \frac{\Omega_1(q)}{\nu(q)} + (1 - \alpha) H'_2 \frac{\Omega_2(q)}{\nu(q)} \right]^{-1} \times \\ &\quad \alpha H'_1 \Omega'_1(q) + (1 - \alpha) H'_2 \Omega'_2(q) > 0. \end{aligned} \quad (1.52)$$

For notational convenience, define $A_i \equiv \Omega'_i(q) - \Omega_i(q)\nu'(q)/\nu(q)$. Then (??) can be written as $\alpha H'_1 A_1 + (1 - \alpha)H'_2 A_2 = 0$. Differentiating the preceding equation with respect to q and simplifying, we have

$$\begin{aligned} \alpha \left[H''_1 (A_1)^2 + H'_1 \Omega''_1 \right] + (1 - \alpha) \left[H''_2 (A_2)^2 + H'_2 \Omega''_2 \right] = \\ \left[\alpha H'_1 \frac{\Omega_1}{\nu(q)} + (1 - \alpha) H'_2 \frac{\Omega_2}{\nu(q)} \right] \nu''(q). \end{aligned}$$

Because $H''_i < 0$ and $\Omega''_i < 0$, the left-side is negative. The coefficient on ν'' is positive. Hence $\nu'' < 0$. \blacksquare

We have the following proposition.

Proposition 1.5 *Under Assumption ??, there is a unique fixed point $q = \nu(q)$.*

PROOF.

Notice that if

$$\alpha \left[H \left(\frac{\Omega_1(q)}{q}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{q}, \theta_2 \right) \right] > 0,$$

then $\nu(q) < q$, while if

$$\alpha \left[H \left(\frac{\Omega_1(q)}{q}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{q}, \theta_2 \right) \right] < 0,$$

then $\nu(q) > q$. Recall that $U'(xq)q$ is increasing in q and that

$$(1/q)H(\psi/q)$$

is decreasing in q . Because U' is unbounded, we can show that for all $c \leq c^*$, $cU'(c) \geq \underline{U}$ and $\underline{U} > 0$. Notice that ξ is bounded below by 0 and bounded above by some $\bar{\xi}$. Hence as $q \rightarrow 0$, $\xi^* \rightarrow \underline{U}$. Then, for some $0 < \epsilon < \infty$, as $\lim_{q \rightarrow 0}$, we have

$$\begin{aligned} \lim_{q \rightarrow 0} \left[\alpha \left[H \left(\frac{\Omega_1(q)}{q}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{q}, \theta_2 \right) \right] \right] \\ = \left[\alpha \left[H \left(\underline{U}/0, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\underline{U}/0, \theta_2 \right) \right] \right] = \alpha L > 0. \end{aligned}$$

Hence $\nu(0) > 0$. As $q \rightarrow \infty$, ξ^* is bounded and $\lim_{q \rightarrow \infty} U'(xq)q = \infty$. Recall that as $\nu \rightarrow \infty$, $H(\nu) \rightarrow L$. We show that

$$\begin{aligned} \lim_{q \rightarrow \infty} \left[\alpha \left[H \left(\frac{\Omega_1(q)}{q}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{q}, \theta_2 \right) \right] \right] \\ = \left[\alpha \left[H \left(\frac{L}{\infty}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{L}{\infty}, \theta_2 \right) \right] \right] = -2B < 0. \end{aligned}$$

Hence there is some Q such that for $q \geq Q$,

$$\alpha \left[H \left(\frac{\Omega_1(q)}{q}, \theta_1 \right) \right] + (1 - \alpha) \left[H \left(\frac{\Omega_2(q)}{q}, \theta_2 \right) \right] < 0.$$

Hence, $\nu(q) < q$ for $q \geq Q$. Because $\nu(0) > 0$, $\nu' > 0$, $\nu'' < 0$ and there exists some Q such that $\nu(Q) < Q$, a fixed point exists and is unique. ■

The fixed point $\nu(q) = q$ was constructed holding the state (x, s) fixed, so that we can define the function $q(x, s)$. This function has the properties that markets clear and the fixed point of the marginal valuation function was constructed holding the function q fixed. This is the unique stationary equilibrium for which we have been searching.

We now wish to study some of the implications of the equilibrium. Notice that in equilibrium, the first-order condition of a type i agent with respect to asset holdings can be written

$$1 = \beta E_s \left[\frac{U'(c_i(x', s')) q(x', s')}{U'(c_i(x, s)) q(x, s)} \right],$$

which can be rewritten as

$$\beta E_s \left[\frac{U'(c_i(x', s'))}{U'(c_i(x, s))} \right] = \frac{q(x, s)}{E_s(q(x', s'))} \times \left[1 - \text{Cov}_s \left(\beta \frac{U'(c_i(x', s'))}{U'(c_i(x, s))}, \frac{q(x', s')}{q(x, s)} \right) \right].$$

The covariance of the asset return with an agent's intertemporal marginal rate of substitution will depend on what type the agent is, and there is no reason to believe that agents will set the *ex ante* intertemporal MRS equal. This occurs because we have not introduced a risk-free asset into the model. As Scheinkman and Weiss note, introducing additional assets into the model may change this result. It is still the case, however, that the *ex post* MRS will be different across agents because of the limits to pooling risk resulting from the borrowing constraints.

Another feature of the equilibrium is that the borrowing constraints do not bind in equilibrium. The argument goes as follows. Recall that we assumed $\lim_{c \rightarrow 0} U'(c) = \infty$. For a fixed price $0 < q < \infty$, suppose that the constraint was binding for some agent so that $\xi(x, s) = U'(xq)q$, implying that $c = xq$. Then the multiplier next period when there is no savings and the agent is unproductive, which is always a possibility, is equal to

$$\xi(0, s') = \max[U'(0)q', \beta E_{s'} \xi(0, s'')] = \infty.$$

In that case, $\beta E_s \xi(0, s') = \infty$ so that $\xi(x, s) = \infty$. If $\xi(x, s) = \infty$, then for the first-order conditions to hold,

$$U'(c) = \frac{\infty}{q}$$

so that $c = 0$, which is a contradiction. Thus, the solution to equation (??) is satisfied with $\mu_i = 0$ and

$$\xi_i(x, s) = \beta E_s[U'(c_i(x'_i, s'))q(x', s')]. \quad (1.53)$$

This feature also arises in the models considered by Bewley [?] and Deaton [?]. Nevertheless, the nonnegativity constraint on the accumulation of nonhuman wealth alters the ensuing equilibrium because, as we showed in Section 7.1.2, allowing unrestricted borrowing yields allocations that are identical to the complete markets equilibrium.

The intertemporal MRS of both agents are used to price the asset in equilibrium. An econometrician using aggregated consumption data would not be able to evaluate the equilibrium Euler equation. One of the model's implications is that the distribution of asset holdings across consumers affects the asset price and output. Using this implication may provide a way of testing the model's restrictions. We postpone discussion of some of these issues until the next chapter.

1.2 Transactions Costs

During the process of buying or selling most assets, some kind of transactions cost is incurred. Often, these costs take the form of a difference between the price at which the asset is sold and the price at which it can be purchased, commonly known as the 'bid-ask' spread. Transactions costs can take other forms such as up-front fees on load mutual funds and brokerage commission costs. Aiyagari and Gertler [?] report that the ratio of the bid-ask spread to the price is .52% for actively traded stocks and that this ratio increases as firm size declines, reaching 6.55% for the average firm with assets under ten million dollars. For the buyer or seller, there are additional costs associated with managing a portfolio such as information costs and bookkeeping costs. For the financial intermediary, which may take the form of an exchange or an organized market, the fees, commissions, and the bid-ask spread paid by the buyer and seller of assets are charges for the services provided by the intermediary. Three kinds of costs faced by an intermediary have been emphasized in the literature: order processing costs, which can include research and information gathering costs and costs of providing financial counseling; inventory holding costs, which take the form of price risk because there may be a time lag between the time the dealer buys an asset and the time he sells it; and adverse information costs. Adverse information costs may be incurred when there is asymmetric information. Current prices may signal negative information about the value of the asset which changes its equilibrium price. If the dealer is the asset holder, then he may suffer a loss from the price change. A general discussion on the components of the bid-ask spread is by Stoll [?] and Glosten and Harris [?]. The inventory risk has been studied by Amihud and Mendelson [?] and

Stoll [?], among others, while the adverse information costs has been studied by Copeland and Galai [?], Glosten and Milgrom [?] and Easley and O'Hara [?].

If the liquidity of an asset is measured by the cost of immediate execution of a transaction, then the quoted ask price can include a premium for immediate purchase and the bid price can include a discount for immediate sale. The bid-ask spread can be interpreted as a measure of liquidity; the spread is smaller for more liquid assets. Several empirical studies, such as that by Amihud and Mendelson [?], have concluded that average risk-adjusted returns increase with their bid-ask spread. An empirical study of liquidity and yields is by Amihud and Mendelson [?].

Another type of cost affecting trading volume is a securities transactions tax. This type of tax has been considered in the U.S. and exists in many other countries; see the survey by Schwert and Seguin [?] and the article by Umlauf [?] for examples. Proponents of the tax argue that the tax would reduce excess price volatility caused by excessive speculation, generate tax revenues, and increase the planning horizons of managers; arguments for this sort of tax are contained in the articles by Stiglitz [?] and Summers and Summers [?]. The notion that there is excess volatility in financial markets because of destabilizing speculation is discussed by DeLong, Shleifer, Summers, and Waldman [?]. Critics of the tax proposal argue that it would increase the costs of capital, distort optimal portfolio decisions, reduce market efficiency and drive markets to lower tax countries; see the papers by Grundfest and Shoven [?], and Kupiec [?] [?], Roll [?], Ross [?], Schwert [?] and the article by Grundfest [?] for examples.

While there is an extensive literature studying transactions costs in asset markets, there has not been a great deal of work on the effects of these costs on equilibrium interest rates. One approach is to assume price processes and then derive the effect of transactions costs on optimal consumption and portfolio decisions. This is the approach taken by Constantinides [?], Duffee and Sun [?], Dumas and Luciano [?], among others. Grossman and Laroque [?] study optimal portfolio and consumption choices in the presence of an illiquid durable consumption good such as housing. In their model, optimal consumption is not a smooth function of wealth. It is optimal for a consumer to wait until a large change in wealth occurs before changing his consumption. A rise in transactions cost increases the average time between the sale of durable goods. They conclude that the standard consumption CAPM does not hold.

Aiyagari and Gertler [?], Heaton and Lucas [?], and Vayanos and Vila [?] are examples of general equilibrium models with transactions costs. The papers by Aiyagari and Gertler and Vayanos and Vila have no aggregate uncertainty although there is individual-specific risk. The Heaton and Lucas model has aggregate uncertainty but is a three-period model. They find that, if trading in some assets is costless, then agents substitute almost entirely away from

assets that are costly to trade. Agents would prefer to alter the composition of their portfolio rather than pay transactions costs or tolerate more volatile consumption. Because agents tend to specialize in holdings of assets that are costless to trade, they conclude that small changes in transactions costs do not have significant price effects.

We do not attempt to construct a general equilibrium model here. To study the effect of transactions costs on trading volume and equilibrium asset prices requires the use of a model with heterogeneous agents, which has proven to be analytically difficult.⁷ Instead, a basic description of the dynamic programming problem faced by an agent is provided.

1.2.1 A Model with Bid-Ask Spreads

Suppose that there is a financial intermediary, such as an organized exchange who facilitates trade but charges a proportional fee in an amount depending on whether the client is buying or selling an asset. The fees may reflect the costs of processing the order, price risks associated with the transactions, and informational asymmetries. For simplicity, we assume that the profits of the intermediary are distributed lump-sum to the agents of the economy. At time t , agent i has random income y_t^i and holds a portfolio comprised of an equity share z_t^i , which pays a fixed dividend d , and risk-free bonds issued by the government which sell at discount at price $1/(1+r_t)$. If agent i sells an equity share, he receives the price $q_t(1-\alpha_s)$ and if he buys an equity share, he pays the price $q_t(1+\alpha_b)$. The difference in the prices at which the equity is sold and bought is the “bid-ask” spread, which equals

$$q_t(\alpha_b + \alpha_s).$$

Notice that we make no attempt to explain the origins of the spread and instead treat α_b and α_s as parameters. We can view this spread times the number of transaction as the profit of the financial intermediary; let π_f denote the per capita profit of the intermediary. This is described more fully below. Let s_t denote the vector of exogenous states variables that agent i needs to make a forecast of returns, dividends, income and consumption next period. The budget constraint of agent i at time t takes the form

$$\begin{aligned} y_{i,t} + b_{i,t} + z_{i,t}d_t - \tau_t - \max\{\alpha_b(z_{i,t+1} - z_{i,t}), \alpha_s(z_{i,t} - z_{i,t+1})\} \\ + \pi_{f,t} \geq c_{i,t} + q_t(z_{i,t+1} - z_{i,t}) + \frac{b_{i,t+1}}{1+r_t}. \end{aligned} \quad (1.54)$$

⁷It is possible to incorporate transactions costs in the production technology. For example, Marshall [?] incorporates money into a general equilibrium model assuming that holding real balances lowers the resource costs of consuming.

We set this up as a dynamic programming problem. The representative type i agent solves

$$V(y_i, z_i, b_i, s) = \max[U(c_i) + \beta E_s V(y'_i, z'_i, b'_i, s')] \quad (1.55)$$

subject to constraints described below. Notice that the first-order conditions depend on whether the agent decides to buy, sell or hold the equity share $z_{i,t}$.

To study the properties of the dynamic programming problem under transactions costs, we split the problem into three subproblems. Define V_s as the value of selling equity shares. The problem is

$$V_s(y, z, b, s) = \max_{\{c, b', z'\}} [U(c) + \beta E_s V(y', z', b', s')] \quad (1.56)$$

subject to (??) and the constraint

$$z \geq z'. \quad (1.57)$$

Next, define the value function if the agent decides to sell, V_b . The problem is

$$V_b(y, z, b, s) = \max_{\{c, b', z'\}} [U(c) + \beta E_s V(y', z', b', s')] \quad (1.58)$$

subject to (??) and the constraint

$$z \leq z'. \quad (1.59)$$

Finally, the value of holding onto the existing equity shares,

$$V_h(y, z, b, s) = \max_{\{c, b'\}} [U(c) + \beta E_s V(y', z, b', s')] \quad (1.60)$$

subject to

$$c + \frac{b'}{1+r} \leq y + zd - \tau + \pi_f. \quad (1.61)$$

We can then write the dynamic programming problem as

$$V(y, z, b, s) = \max[V_s(y, z, b, s), V_h(y, z, b, s), V_b(y, z, b, s)] \quad (1.62)$$

Under this formulation, we have still retained the recursive structure of the problem. As an example, we solve one of the subproblems. Consider the solution to V_s . The first-order conditions are

$$U'(c) = \xi_s \quad (1.63)$$

$$\frac{\xi_s}{1+r} = \beta E_s V_3(y'_i, z'_i, b'_i, s'), \quad (1.64)$$

$$\xi_s q(1 - \alpha_s) = \beta E_s V_2(y'_i, z'_i, b'_i, s') + \mu_s, \quad (1.65)$$

where μ_s is the multiplier attached to the constraint (??), and ξ_s the multiplier on the budget constraint. If $\mu_s = 0$, then the constraint is nonbinding and

$$\xi_s q(1 - \alpha_s) = \beta E_s V_2(y'_i, z'_i, b'_i, s'),$$

otherwise,

$$\xi_s q(1 - \alpha_s) \geq \beta E_s V_2(y'_i, z'_i, b'_i, s').$$

We can derive a similar equation for the subproblem of buying the equity share with the result that

$$\xi_b q(1 + \alpha_b) \leq \beta E_s V_2(y'_i, z'_i, b'_i, s').$$

which holds with equality if the constraint (??) is nonbinding. Notice that under this formulation, the function V denotes the value function assuming that the agent behaves optimally at all future dates. The slope of the value function with respect to equity shares is given, and the agent must choose the optimal course of action - buy, sell, or hold - in the current period.

We can define an operator T by

$$TV^n(y, z, b, s) = \max[V_s^n(y, z, b, s), V_h^n(y, z, b, s), V_b^n(y, z, b, s)] \quad (1.66)$$

where V_i^n is defined for $i = s, b, h$. Notice that T is monotonic. If $W > V$ for all (y, z, b, s) , then notice that $TW \geq TV$. Furthermore, T discounts. Each of the V_i is concave and the maximization operator preserves concavity so that V is concave.

Our discussion is incomplete in the sense that the agent takes as given the equity price function q and the return on the risk-free asset r . As we mentioned earlier, constructing an equilibrium with heterogeneous agents is analytically difficult.

1.2.2 Volatility Bounds with Frictions

In Chapter 4, we described how to derive the mean-standard deviation region for intertemporal MRS's that are used to price random payoffs in dynamic asset pricing models. We now extend this discussion to account for short sales constraints, transaction costs and borrowing constraints. As in our earlier discussion, the volatility bounds we derive here can be used as a diagnostic tool for determining the class of asset pricing models that are consistent with asset market data.

We derive restrictions for intertemporal MRS's with various forms of frictions by using a sequential interpretation of the complete contingent claims equilibrium that we described in Section 7.1. Let's define $\tilde{q}_{t+1}(\omega)\tilde{z}_{t+1}(\omega)$ as the payoff on securities purchased at time t that is realized at time $t + 1$.

We assume consumers can purchase securities that pay off for each possible realization of the economy. Portfolios with such payoffs can be purchased at the price $p_{t+1}(\omega)/p_t(\omega)$ in period t . Using this notation, notice that the single budget constraint facing agent i can be written in terms of a sequence of one-period constraints:

$$c_{i,t}(\omega) + E_t \left[\frac{p_{t+1}(\omega)}{p_t(\omega)} \tilde{q}_{t+1}(\omega) \tilde{z}_{t+1}(\omega) \right] \leq \ell_{i,t}(\omega) + \tilde{q}_t(\omega) \tilde{z}_t(\omega), \quad (1.67)$$

for $t \geq 0$ where $E_t(\cdot)$ denotes expectation conditional on the history of shocks, $\{s_s(\omega), s \leq t\}$. We obtain the single budget constraint (??) by solving (??) forward, where we implicitly impose a condition that the value of limiting portfolio payoff goes to zero.

Volatility bounds with frictions have been derived by Luttmer [?] and He and Modest [?] who consider different types of constraints. Luttmer considers a *solvency* constraint of the form:

$$\tilde{q}_{t+1}(\omega) \tilde{z}_{i,t+1}(\omega) \geq 0. \quad (1.68)$$

According to this constraint, any contingent contract that allows debt in some state of the world is prohibited. A weaker version of the constraint is employed by He and Modest who require that

$$E_t \left[\frac{p_{t+1}(\omega)}{p_t(\omega)} \tilde{q}_{t+1}(\omega) \tilde{z}_{i,t+1}(\omega) \right] > 0. \quad (1.69)$$

This states that the value of the portfolio payoff today must be nonnegative. It does not preclude $\tilde{q}_{t+1}(\omega) \tilde{z}_{i,t+1}$ from being negative in some states of the world. We refer to it as the *market-wealth* constraint. We can show that the borrowing constraint in Scheinkman and Weiss is a market-wealth constraint. Consider the Kuhn-Tucker condition for the nonnegativity constraint, $z_{i,t+1} \geq 0$ given by $\mu_{i,t} z_{i,t+1} = 0$. Using the first-order conditions (??-??), this can be expressed as

$$E_t [\beta U'(c_{i,t+1}) q_{t+1} - U'(c_{i,t}) q_t] z_{i,t+1} = 0.$$

Using the above result, we can substitute for $q_t z_{i,t+1}$ in the budget constraint $z_{i,t+1} - z_{i,t} = (\theta_{i,t} \ell_{i,t} - c_{i,t})/q_t$ as

$$c_t + E_t \left[\frac{\beta U'(c_{i,t+1})}{U'(c_{i,t})} q_{t+1} z_{i,t+1} \right] = \theta_{i,t} \ell_{i,t} + q_t z_t.$$

Since q_t is strictly positive, the requirement that $z_{i,t+1} \geq 0$ is equivalent to the form of the market-wealth constraint (??) postulated by He and Modest.

Now we analyze the implications of these constraints for individuals intertemporal MRS's with complete markets. Let $\xi_{i,t}(\omega)$ denote the multiplier on the single-period budget constraints (??). Using the same preferences as in

Section 10.1.1, the first-order condition with respect to the portfolio weights $z_{i,t+1}(\omega)$ imply that

$$E_t \left[-\frac{p_{t+1}(\omega)}{p_t(\omega)} q_{t+1}(\omega) \xi_{i,t}(\omega) - \tilde{q}_{t+1}(\omega) \xi_{i,t+1}(\omega) - \mu_{i,t}(\omega) \right] = 0,$$

where the elements of the vector $\mu_{i,t}(\omega)$ equal zero if and only if the corresponding elements of $\tilde{z}_{i,t+1}(\omega)$ are strictly positive. Substituting for $\xi_{i,t}(\omega)$,

$$E_t \left[\frac{\beta U'(c_{i,t+1}(\omega))}{U'(c_{i,t}(\omega))} \tilde{q}_{t+1}(\omega) \right] \leq E_t \left[\frac{p_{t+1}(\omega)}{p_t(\omega)} \tilde{q}_{t+1}(\omega) \right]. \quad (1.70)$$

Let \mathcal{M}^i denote the individual intertemporal MRS in the above expression and \wp the ratio of the contingent claims prices. Since we assumed complete markets in the construction of the payoffs of the traded securities, and given that both \mathcal{M}^i and \wp are nonnegative, we also have that

$$\mathcal{M}^i \leq \wp. \quad (1.71)$$

Thus, with solvency constraints, the individual intertemporal MRS is downward biased relative to the market-determined stochastic discount factor that is used to value payoffs on one-period securities. For certain classes of utility functions (including exponential and power utility functions), we can show that the intertemporal MRS evaluated with per-capita consumption data also inherits this downward bias:

$$\mathcal{M}^a \leq \wp \quad (1.72)$$

where $\mathcal{M}^a \equiv \beta U'(\bar{c}_{t+1}(\omega))/U'(\bar{c}_t(\omega))$, and U is a function of the average subsistence levels, $\bar{\gamma}$, and per-capita consumption, \bar{c}_t . (See Problem 7.5 at the end of the chapter.)

Now let us consider the implications of the less restrictive market-wealth constraint. Consumers can now form portfolios in addition to those described above. Let Z denote the set of one-period security payoffs with zero market prices, or equivalently, the set of excess returns. Any payoff in Z satisfies the market-wealth constraint. Furthermore,

$$E_t[\mathcal{M}^i z] = E_t[\wp z] \quad \text{for } z \in Z. \quad (1.73)$$

The payoff $\mathcal{M}^i - \wp E_t(\wp \mathcal{M}^i)/E_t(\wp^2)$ has a zero market price, that is, $E_t[\wp(\mathcal{M}^i - \wp E_t(\wp \mathcal{M}^i)/E_t(\wp^2))] = 0$. Using this payoff for z in relation (??), we have

$$\mathcal{M}^i = \psi^i \wp \quad \text{for } \psi^i = E_t(\wp \mathcal{M}^i)/E_t(\wp^2). \quad (1.74)$$

Furthermore, (??) implies that $0 < \psi^i \leq 1$. For the the power utility function, we can show that

$$\mathcal{M}^a = \psi^a \wp, \quad (1.75)$$

where $0 < \psi^a \leq 1$. (See Problem 10.5.) Recall that the market-wealth constraint is less restrictive than the solvency constraint. As the above results demonstrate, the less restrictive constraint imposes the more stringent proportionality requirement on the aggregate intertemporal MRS.

Cochrane and Hansen [?] describe in detail how to compute the boundary of the mean-standard deviation region for intertemporal MRS's or stochastic discount factors that satisfy (??) and (??) in the case of two limited liability securities. They consider quarterly value-weighted stock returns on the NYSE and T-bill returns. Let \mathbf{x} denote a random vector formed by stacking these two returns, and let P^+ denote the cone of random variables or limited-liability payoffs that can be constructed from constant-weighted portfolios of these returns:

$$P^+ \equiv \{p : p = \mathbf{c} \cdot \mathbf{x} \text{ for } \mathbf{c} \in \mathbb{R}^2, p \geq 0\}.$$

Define the region of stochastic discount factors that satisfy (??) by B^+ . It turns out the region B^+ is an expanded version of the region S^+ that we calculated in Chapter 4 for any intertemporal MRS or stochastic discount factor satisfying the the asset pricing relation without a solvency constraint, or short sales constraints. We can also construct a region, denoted W^+ for the set of random random variables \mathcal{M}^a such that the proportionality restriction in (??) holds. W^+ is also an expanded version of S^+ , but it is smaller than B^+ .

The mean-standard region for random variables or stochastic discount factors that satisfy (??) can be constructed by using two so-called edge portfolios, denoted p_1 and p_2 . Any other payoff in P^+ is a convex combination of these edges with nonnegative portfolio weights. Since the original two returns have nonnegative payoffs, each edge has a positive portfolio weight on one of the securities and a nonpositive weight on the other. Let us normalize these edge payoffs so that their price is one, that is, $E(\wp_1) = E(\wp_2) = 1$, and order them so that $E(p_1) \geq E(p_2)$.

The boundary of B^+ has three segments. To see how the first segment arises, notice that for any constant discount factor \mathcal{M}^a such that $0 \leq \mathcal{M}^a \leq 1/E(p_1)$, the inequalities in (??) are satisfied, that is,

$$E(\mathcal{M}^a p_i) \leq E(\wp_i) = 1, \quad i = 1, 2,$$

Let $\sigma(\mathcal{M})$ denote the standard deviation of some stochastic discount factor \mathcal{M} . Then there is a horizontal segment at $\sigma(\mathcal{M}^a) = 0$ from $E(\mathcal{M}^a) = 0$ to $E(\mathcal{M}^a) = 1/E(p_1)$. Furthermore, as long as the constant discount factor \mathcal{M}^a is strictly less than $1/E(p_1)$, the inequalities in (??) will be strict. When $\mathcal{M}^a = 1/E(p_1)$, (??) will hold with equality for p_1 . The second segment begins at this point.

Now consider the mean-standard region region for the set of (strictly positive) stochastic discount factors that correctly price p_1 . Following the notation

of Chapter 4, we denote this region S_1^+ . Notice that the point $(1/E(p_1), 0)$ is on the boundary of this set because the constant discount factor $1/E(p_1)$ prices p_1 correctly. Also, it is easy to see that any other frontier random variable for S_1^+ will also be on the boundary of B^+ provided (??) is satisfied for p_2 . In other words, any \mathcal{M}^1 on the boundary of S_1^+ such that $1 = E(\mathcal{M}^1 p_1)$ will also be on the boundary of B^+ provided $E(\mathcal{M}^1 p_2) \leq E(\wp p_2)$. Thus, we follow the right boundary of S_1^+ until we find a frontier discount factor that prices p_2 correctly.

The third segment of the boundary for B^+ coincides with the mean-standard deviation region of stochastic discount factors, S^+ , which correctly price both p_1 and p_2 . Thus, we can find the minimum variance random variables \mathcal{M} such that $E(\mathcal{M} p_i) = 1$ for $i = 1, 2$. For these random, (??) will hold with equality for all possible payoffs in P^+ .

To construct the boundary of W^+ , multiply both sides of (??) by some payoff x . Taking expectations (first conditional on the information set at time t , and then unconditionally), we obtain $E(\mathcal{M}^a x) = E(\psi^a \tilde{Q})$ where $0 < E(\psi^a) \leq 1$ and \tilde{Q} is the price of the random payoff x . Thus, for any \mathcal{M}^a satisfying (??), we can find a stochastic discount factor $\mathcal{M}^a/E(\psi^a)$ that prices the payoffs x correctly. Since the mean and standard deviation of random variables that are scale multiples the same scaling, we can construct the stochastic discount factors in W^+ by scaling the discount factors in S^+ by arbitrary numbers between zero and one. We have reproduced table 4.1 from Cochrane and Hansen to illustrate the various regions.

1.3 Exercises

7.1 Consider an economy populated by equal numbers of two types of consumers. The preferences of consumers of type $i, i = 1, 2$, over stochastic streams of consumption and labor hours are defined by

$$E_0 \sum_{t=0}^{\infty} \beta^t (\log(c_t^i) - n_t^i),$$

where $0 < \beta < 1$, $c_t^i \geq 0$, $n_t^i \geq 0$ and $E_0(\cdot)$ denotes expectation conditional on information at time zero.

Consumer of type i at date t can transform one unit of time into π_{it} units of date t consumption good. The π_{it} are identically and independently distributed over time with

$$Pr(\pi_{it} = 1) = 1/2 = Pr(\pi_{it} = 0).$$

π_{1t} and π_{2t} are *not* independent. If one type is productive (i.e. $\pi_{it} = 1$), then the other type is unproductive (i.e. $\pi_{jt} = 0$ for $j \neq i$). Thus, $\pi_{1t} + \pi_{2t} = 1$ for all t .

- a) Find the consumption and labor allocations of type $i = 1, 2$ if there exists a full set of contingent claims markets.
- b) Suppose there is not a complete set of contingent claims markets. Instead consumers may trade in a single asset, the quantity of which has been normalized to one. Initially, type 1 consumers hold y_0 units and type 2 consumers hold $1 - y_0$ units. There is a borrowing constraint in that individuals' asset holdings must be nonnegative. Let $\{q_t\}$ denote the stochastic process for the price of the asset.
- Specify the first-order conditions that type i 's consumption, labor supply, and asset holdings must satisfy.
- c) Suppose an econometrician uses the average real return on a risk-free nominal bill to measure the rate of time preference. Comment on this procedure in light of your answer to part (b).

7.2 There are types of consumers, three states of the world. Each agent is endowed with one unit of labor in each state and each period. The following table relates output per unit of labor of consumer $i = 1, 2$ in state of the world $j = 1, 2, 3$:

j	1	2	3
i			
1	4	3	2
2	2	3	4

Output might be stored by individual consumers at no cost and the utility function of consumer i is given by

$$E_0 \sum_{t=1}^2 (\log(c_{it}) - \ell_{it}),$$

where $E_0(\cdot)$ denotes expectation conditional on information at time 0 and ℓ_{it} denotes the amount of labor used in the production of the consumption good. States are i.i.d. and $\pi_1 = \pi_3 = .25$ and $\pi_2 = .5$, where π_i denotes the probability of state i in any period.

- a) Find the consumption and labor supply allocations of type $i = 1, 2$ if there exists a full set of contingent claims markets. Characterize the behaviour of aggregate consumption in equilibrium.
- b) Find the autarkic allocations.
- c) Using your answers to parts a) and b), discuss whether an econometrician can use aggregate consumption data in order to determine whether markets are complete or not, if s/he does not know the form of the production technology.

7.3⁸ Suppose that agents live two periods and that per-capita consumption takes one of two values, μ or $(1-\phi)\mu$ where $0 < \phi < 1$, with each state occurring with probability $1/2$. At time zero, agents choose their portfolio. At time 1, the uncertain endowment is realized, the payoff on the portfolio is made and then agents consume. The portfolio pays -1 in the bad state and $1 + \pi$ in the good state where π is a risk premium.

Assume that all agents are identical. The representative consumer maximizes $EU(c)$. Let R denote the expected payoff on the portfolio so that the first-order condition is $E(RU'(c)) = 0$ which can be written as

$$(1 + \pi)U'(\mu) - U'((1 - \phi)\mu) = 0.$$

Let $U = c^{1-\gamma}/(1 - \gamma)$.

Solve for the risk premium π under this assumption.

7.4 We now introduce heterogeneity and incomplete markets. Agents are identical *ex ante* but not *ex post*. In the bad state assume that the fall in aggregate consumption equal to $\phi\mu$ is concentrated among a fraction λ of the population. This implies that in the good state, which occurs with probability $1/2$, the agent consumes μ and the portfolio pays $1 + \pi$. In the bad state the portfolio pays -1 and his consumption is μ with probability $1 - \lambda$ and $(1 - \phi/\lambda)\mu$ with probability λ .

- a) Derive the first-order condition and the premium π .
- b) Show that the premium depends not only on the size of the aggregate shock ϕ but also on its distribution within the population.
- c) Assume that utility is constant relative risk aversion and show that an decrease in λ increases π (so the more concentrated the shock the larger the premium).

7.5⁹ Agents live for 2 periods and preferences are $u(c_1) + v(c_2)$. The first period endowment is certain and equal to y_1 . The second period endowment is random with $\tilde{y} > 0$. All agents face the same distribution. Agents in period 1 are identical *ex ante* but not *ex post*. He assumes that there is no aggregate component to second period endowment uncertainty. Agents are also endowed with x_0 trees, the dividend $d_1 \geq 0$ is known but next period's dividend \tilde{d} is random. The two sources of risk are independent.

- a) Using the fact that agents are *ex ante* identical, find an expression for the equilibrium expected (gross) rate of return on equities and the equilibrium (gross) risk-free rate. (For simplicity, re-scale utility such that $u'(d_1 + y_1) = 1$.)

⁸This problem is based on Mankiw [?].

⁹This problem is based on Weil [?].

- b) Show that an analyst who uses aggregate data to fit this model data will overpredict the magnitude of the risk-free rate and the equilibrium expected equity return if and only if $v''' > 0$.
- c) Show that if $v(\cdot)$ exhibits decreasing absolute risk aversion (so that $-v''(c)c/v'(c)$ is decreasing), and decreasing absolute risk prudence (so that $-v'''(c)/v''(c)$ is also decreasing), then an analyst who uses aggregate data to fit the model will understate the magnitude of the equity premium.

7.6 Show that the volatility bound in (??) hold for the following utility functions:

$$U(c_i) = ((c_i - \gamma^i)^{1-\sigma} - 1)/(1 - \sigma), \quad \sigma > 0, \quad (1.76)$$

$$U(c_i) = -\exp(-\alpha(c_i - \gamma^i)) \quad \alpha > 0. \quad (1.77)$$

Let $\gamma^i = 0$ in (??), and show that (??) holds with

$$\psi^a = \left[\sum_i \psi^{1/\sigma} (c_{i,t}/\bar{c}_t) \right]^{-\sigma}.$$

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