The Demand for Money During Hyperinflations Under Rational Expectations

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The Demand for Money During Hyperinflations
Under Rational Expectations: I

by
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Note: Research on this paper was supported by the Federal Reserve Bank of Minneapolis, which doesn't necessarily endorse the conclusions. Helpful comments on an earlier draft were received from Christopher Sims, Thomas Turner, John Geweke, Milton Friedman, Jacob Frankel, Robert E. Lucas, Jr., and Rusdu Saracoglu. Rusdu Saracoglu performed the calculations using a computer program that he wrote for estimating bivariate mixed moving average, autoregressive models.
This paper proposes methods for estimating the demand schedule for money that Phillip Cagan used in his famous study of hyperinflation [3]. Wallace and I [8] pointed out that under assumptions that make Cagan's adaptive expectations scheme equivalent with assuming rational expectations, Cagan's estimator of \( \alpha \), which is the slope of the log of the demand for real balances with respect to expected inflation, is not statistically consistent. This is interesting in light of a paradox that emerged when Cagan used his estimates of \( \alpha \) to calculate the sustained rates of inflation associated with the maximum flow of real resources that the creators of money could command by printing money. This "optimal" rate of inflation turns out to be \(-1/\alpha\). For each of the seven hyperinflations, the reciprocal of Cagan's estimate of \(-\alpha\) turned out to be less, and often very much less, than the actual average rate of inflation. The data are shown in Table 1, which reproduces a table of Cagan's. Cagan's estimates imply that the creators of money expanded the money supply at rates that far exceeded the sustained rates which maximized the real revenues they could obtain. A natural first thing to consider in explaining this apparently irrational behavior by the creators of money is the possibility that it is a statistical artifact, namely, a consequence of using bad estimates of \( \alpha \).

This paper aims to complete a task begun by Wallace and me [8], namely, the analysis of Cagan's model of hyperinflation under circumstances in which Cagan's "adaptive" scheme for forming expectations of inflation is equivalent with expectations that are "rational"
Table 1

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Austria</td>
<td>0.117</td>
<td>12</td>
<td>47</td>
</tr>
<tr>
<td>Germany</td>
<td>0.183</td>
<td>20</td>
<td>322</td>
</tr>
<tr>
<td>Greece</td>
<td>0.244</td>
<td>28</td>
<td>365</td>
</tr>
<tr>
<td>Hungary I</td>
<td>0.115</td>
<td>12</td>
<td>46</td>
</tr>
<tr>
<td>Hungary II</td>
<td>0.236</td>
<td>32</td>
<td>19,800</td>
</tr>
<tr>
<td>Poland</td>
<td>0.435</td>
<td>54</td>
<td>81</td>
</tr>
<tr>
<td>Russia</td>
<td>0.327</td>
<td>39</td>
<td>57</td>
</tr>
</tbody>
</table>

Column (1) = \(-1/\hat{\alpha}\) (continuous compounding), rate per month that maximizes revenue of money creator

Column (2) = \((e^{-1/\hat{\alpha}} - 1) \cdot 100\) (neglects compounding)

Column (3) = average actual rate of inflation per month

Source: Cagan's Figure 9, page 81.
in Muth's sense [6]. The model is a very simple simultaneous-equations model of the inflation-money creation process, one equation of which turns out to be identical with Cagan's simple portfolio equilibrium equation. As Wallace and I have argued, Cagan's use of single equations methods exposed him to the possibility of severe simultaneous-equations bias. The present paper uses the full information maximum likelihood estimator to obtain a consistent estimator of Cagan's model. It also obtains an expression for the statistical inconsistency of Cagan's estimator for his model under the circumstances in which adaptive expectations coincide with rational expectations.

One way of justifying imposing rational expectations on Cagan's model is that it enables one to specify a complete model of the inflation-money creation process in a very economical way. This is a virtue, since the time series from the hyperinflations are too short to permit estimating complicated parameterizations. But a more important reason for using the hypothesis of rational expectations to complete Cagan's model is that doing so delivers an econometric model that is seemingly consistent with the exogeneity (or "causal") structure exhibited by the money creation-inflation process during the seven hyperinflations studied by Cagan. Empirical tests by Wallace and me typically indicated substantial evidence of feedback from inflation to money creation, with markedly less feedback from money creation to inflation. Cagan's model under rational expectations predicts a particular extreme version of such a pattern: it predicts that inflation "causes" (in Granger's sense) money creation with no reverse feedback (or "causality") from money creation to inflation. Cagan's model with rational expectations thus seems to provide one way of explaining the Granger-causal structure exhibited in the data.
Cagan's paper is rightly regarded as one of the best pieces of empirical work ever done in economics. His model and his estimation method have been applied with apparent success to a number of additional countries experiencing high inflation rates, but rates falling short of those characterizing hyperinflations.\(^1\) The key substantive conclusion that has been drawn from Cagan's study, and those subsequent studies as well, is that even in the apparently chaotic conditions of rampant inflation it is possible to isolate a stable demand schedule for money having real balances varying inversely with the expected rate of inflation. In the light of the results of this paper, that conclusion must be severely modified. First, it is shown below that under conditions that make Cagan's model equivalent with assuming "rational" expectations, the slope parameter \(\alpha\) is not econometrically identifiable. To identify \(\alpha\) requires imposing a restriction on the covariance of the disturbances to the demand for money and to the supply of money. Neither economic theorizing nor intuition seems to provide a ready restriction on that covariance. Proceeding on the "neutral" assumption that that covariance is zero, one can extract estimates of \(\alpha\). But even then, the estimates of \(\alpha\) are characterized by large standard errors.
From a technical point of view, this paper is an exercise in applying vector time series models. The key references are Granger [4], Sims [9], Wilson [10], Porter [7], and Zellner and Palm [11]. The model studied here is an interesting one from the point of view of the vector time series model, since it is one in which inflation "causes" money creation in Granger's sense, although these two series are supposed to be perfectly in phase, so that neither one "leads" the other. The model thus provides an example that illustrates the difference between Granger's causality and simple notions of the lead of one series over another. The model is also interesting because it illustrates the very important difference between Granger causality and a separate notion of causality often used by economists, namely, that of invariance with respect to an intervention. The present model predicts that money "causes" inflation in the sense that a given change in the stochastic process or "feedback rule" governing the money supply will produce a determinate change in the stochastic process for inflation. The stochastic process for inflation is an invariant function of the stochastic process governing money creation. In Cagan's model with rational expectations imposed, inflation Granger-causes money creation with no reverse Granger causality from money to inflation because the system is operating under a particular money supply rule that in effect prevents the money supply from being of any use in predicting subsequent rates of inflation. If there is a change in monetary regime, that is, a switch in the money supply rule, the economic model predicts that the Granger-causality structure of the money-inflation process will change.

2. The Model

Cagan's model of hyperinflation builds on a demand schedule for real balances of the form
4a

\[(1) \quad m_t - p_t = a\pi_t + u_t\]

where \(m\) is the log of the money supply (which is always equal to the log of the money demand); \(p\) is the log of the price level; \(\pi_t\) is the expected rate of inflation, i.e., the public's psychological expectation of \(p_{t+1} - p_t\); and \(u_t\) is a random variable with mean zero. I have omitted
a constant term from (1), though one would be included in empirical work. Cagan assumed that \( \pi_t \) was formed via the adaptive expectations scheme

\[
\pi_t = \frac{1 - \lambda}{1 - \lambda L} (p_t - p_{t-1})
\]

or

\[
\pi_t = \frac{1 - \lambda}{1 - \lambda L} x_t
\]

where \( x_t = p_t - p_{t-1} \), the rate of inflation, and where \( L \) is the lag operator defined by \( L^n x_t = x_{t-n} \).

Under rational expectations we require that

\[
\pi_t = E_t x_{t+1}
\]

where \( E_t x_{t+1} \) is the mathematical expectation of \( x_{t+1} \) conditional on information available as of time \( t \). Using (3) and recursions on (1), it is straightforward to show that under rational expectations we must have

\[
\pi_t = E_t x_{t+1} = \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \left( \frac{\alpha}{1 - \alpha} \right)^{j-1} E_t \nu_{t+j} - \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \left( \frac{\alpha}{1 - \alpha} \right)^{j-1} (E_t u_{t+j} - E_t u_{t+j-1})
\]

where \( \nu_t = m_t - m_{t-1} \), the percentage rate of increase of the money supply. Equation (4) characterizes the (systematic part of the) stochastic process for inflation as a function of the (systematic part of the) stochastic process for money creation. The model asserts that (4) is invariant with respect to interventions in the form of changes in the stochastic process governing money creation. In this sense, since changes in the stochastic process for money creation are supposed to
produce predictable changes in the stochastic process for inflation, money "causes" inflation.

For Cagan's adaptive expectation scheme (2) to be equivalent to rational expectations we require:

\[
\frac{1 - \lambda}{1 - \lambda L} x_t = \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \left( \frac{-\alpha}{1 - \alpha} \right)^{j-1} E_t \mu_{t+j} - \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \left( \frac{-\alpha}{1 - \alpha} \right)^{j} (E_t u_{t+j} - E_t u_{t+j-1}).
\]

The necessary and sufficient condition for (5) to hold for all \( \alpha \) and all \( t \) is

\[
E_t u_{t+j} - E_t (u_{t+j} - u_{t+j-1}) = \frac{1 - \lambda}{1 - \lambda L} x_t.
\]

For an arbitrary \( \mu \) process, there exists a disturbance process \( u_t \) satisfying the above restriction, one in which \( E_t (u_{t+j} - u_{t+j-1}) \) is a complicated function of lagged \( x \)'s and lagged \( \mu \)'s. From my point of view, however, the most fruitful conditions to impose are the following two that are sufficient (though clearly not necessary) to satisfy (5).

The first condition is

\[
u_t = u_{t-1} + \eta_t
\]

where \( \eta_t \) is a serially uncorrelated random term with mean zero and variance \( \sigma^2 \); I assume that \( E[\eta_t | u_{t-1}, u_{t-2}, \ldots, x_{t-1}, x_{t-2}, \ldots] = 0 \).

According to (6), \( u \) takes a random walk. Equation (6) implies that

\[
E_t u_{t+j} = u_t, \quad j \geq 0
\]

which implies that

\[
E_t u_{t+j} - E_t u_{t+j-1} = 0 \quad \text{for all } j \geq 1.
\]

The second of my pair of sufficient conditions for (5) is
(7) \[ E_t \mu_{t+j} = E_t \mu_{t+1} \quad \text{for } j > 1, \]

so that a constant rate of money creation is expected to occur over the entire future. Assuming (6) and (7) then implies that the appropriate version of (5) is

\[
\left( \frac{1 - \lambda}{1 - \lambda L} \right) x_t = E_t \mu_{t+1} \frac{1}{1 - \alpha} \sum_{j=1}^{\infty} \left( \frac{-\alpha}{1 - \alpha} \right)^{j-1}
\]

or

(8) \[ \frac{1 - \lambda}{1 - \lambda L} x_t = E_t \mu_{t+1}. \]

A process that satisfies (8) is

(9) \[ \mu_t = \left( \frac{1 - \lambda}{1 - \lambda L} \right) x_t + \varepsilon_t \quad (= E_t x_{t+1} + \varepsilon_t) \]

where \( \varepsilon_t \) is a serially uncorrelated random term with mean zero and variance \( \sigma^2 \), and that satisfies
According to (9), the rate of money creation equals the expected rate of inflation plus a random term. Equation (9), which has been arrived at in a purely mechanical fashion by pursuing the implications of the assumption that Cagan's adaptive expectations scheme is rational, is nevertheless of interest as an hypothesis about the government's behavior. For example, if the government is creating money to finance a large part of a roughly fixed rate of real government purchases, then there is a presumption that inflation and expected inflation will feed back into money creation, an implication with which (9) is consistent. Thus, when \( \pi_t \) increases, causing \( m_t - p_t \) to fall and thereby causing \( p_t \) to rise with a fixed \( m_t \), money depreciates in value, prompting the creators of money to increase the rate of printing money in order to maintain their command over the flow of real resources (see Sargent and Wallace [8]). Alternatively, equation (9) is compatible with a "real bills" regime in which the monetary authority sets out to supply whatever money the public demands at some fixed nominal interest rate or some fixed real money supply. Equation (9) looks like a rule in which the monetary authority is attempting to peg the (rate of growth of the) real money supply. During the German hyperinflation, German monetary officials in effect repeatedly acknowledged that they were operating under a real-bills regime, acknowledgments made in efforts to argue that their actions were not causing the inflation but were merely responses to it.

The foregoing establishes that if equations (6) and (9) obtain, Cagan's adaptive expectations scheme is compatible with rational expectations and with the portfolio balance condition that he assumed. Under these assumptions, inflation and money creation form a bivariate stochastic process given by

\[ E(e_t | x_{t-1}, x_{t-2}, \ldots, \mu_{t-1}, \mu_{t-2}, \ldots) = 0. \]
Equation (10) was obtained by first differencing (1) and then substituting for \( \eta_t \) from (2) and for \( u_t - u_{t-1} \) from (6). The process (10) - (9) can be rewritten as

\[
(11) \quad (1 - L) x_t = (\lambda + a(1 - \lambda))^{-1} (1 - \lambda L)(\varepsilon_t - \eta_t)
\]
(12) \( (1 - L) u_t = [(\lambda + \alpha(1 - \lambda))^{-1}](1 - \lambda)(e_t - \eta_t) - e_{t-1} + \epsilon_t \).

Equations (11) and (12) can be derived directly from (10) and (9); alternatively, see Sargent and Wallace for a somewhat different but equivalent way of deriving (11) and (12).

The statistical model (11) - (12) was constructed in a fashion to guarantee the condition

\[ E_t \frac{x_{t+1}}{1-\lambda} = \frac{1-\lambda}{1-\lambda L} x_t \]

a condition that implies that \( \mu \) does not Granger cause \( x \). For the above equation states that once lagged \( x \)'s are taken into account, lagged \( \mu \)'s don't help predict current \( x \), which is Granger's definition of \( \mu \)'s not causing \( x \). It bears mentioning that the statistical model inherits its Granger-causal structure in large part from the particular conditions (6) and (7). The statistical model (11) - (12) is not invariant with respect to an intervention in the form of a change in the money supply rule. Rather, it is equation (4) that is supposed to be invariant with respect to interventions in the form of changes in monetary regime. According to (4), changes in the \( \mu_t \) process—which show up in changes in the (functions) \( E_t \mu_{t+j} \)—result in changes in the systematic part of the inflation process, \( E_t x_{t+1} \). Thus, one cannot expect the Granger-causal structure of the present model to survive interruptions in monetary regimes.

3. The Bias in Cagan's Estimator

A convenient way to evaluate the (asymptotic) bias in Cagan's estimator is first to obtain a bivariate Wold representation for \( (\Delta x_t, \Delta \mu_t) \). Write (11) and (12) as

(13) \( (1 - L) x_t = \phi(1 - \lambda L)(e_t - \eta_t) \)
\[ (14) \quad (1 - L) \mu_t = \theta(1 - \lambda)(\varepsilon_t - \eta_t) + (1 - L) \varepsilon_t \]

where \( \theta = (\lambda + \alpha(1 - \lambda))^{-1} \). Next decompose \( \varepsilon_t \) according to

\[ \varepsilon_t = E[\varepsilon_t | \varepsilon_t - \eta_t] + \nu_t \]

or

\[ (15) \quad \varepsilon_t = \rho(\varepsilon_t - \eta_t) + \nu_t \]

where \( E[\nu_t | \varepsilon_t - \eta_t] = 0 \) and \( \rho \) is the regression coefficient of \( \varepsilon_t \) on \( (\varepsilon_t - \eta_t) \). Substituting (15) into (14) gives

\[ (16) \quad (1 - L) \mu_t = [\theta(1 - \lambda) + \rho(1 - L)](\varepsilon_t - \eta_t) + (1 - L) \nu_t \]

Since \( \nu_t \) is orthogonal to \( (\varepsilon_t - \eta_t) \) and is serially uncorrelated by construction (recall that \( \nu_t - \varepsilon_t - \rho(\varepsilon_t - \eta_t) \), where \( \varepsilon_t \) and \( \eta_t \) are serially uncorrelated), it follows that (13) and (16) form a (triangular) bivariate Wold representation for \((\Delta x_t, \Delta \mu_t)\) with fundamental noises
(ε_t - η_t) and v_t. The existence of a triangular bivariate Wold representation verifies that Δx is econometrically exogenous with respect to Δμ and that Δμ does not cause Δx in Granger's sense (see Sims [9]). It also makes it very easy to determine the population projection of Δμ on current and past Δx's, from which the asymptotic bias in Cagan's estimator is calculable.

From (13) notice that

\[ \varepsilon_t - \eta_t = \phi^{-1} \frac{(1 - L)}{1 - \lambda L} x_t. \]

To obtain the projection of Δμ on against current and past (and future) Δx's, substitute (17) into (16) to get

\[ (1 - L) \mu_t = (\phi(1 - \lambda) + (1 - L)\rho)\phi^{-1} \frac{(1 - L)}{1 - \lambda L} x_t + (1 - L) v_t \]

Dividing through by (1 - L) gives

\[ \mu_t = \left( \frac{1 - \lambda}{1 - \lambda L} + \frac{(1 - L)\phi^{-1}}{1 - \lambda L} \right) x_t + v_t \]

Recall from (13) that the v_t process is orthogonal to the x process. Therefore, equation (18) gives the projection of μ_t on x. Subtracting x_t from both sides gives the projection of μ_t - x_t against x_t:

\[ \mu_t - x_t = \left( \frac{1 - \lambda + \rho(\lambda + \alpha(1 - \lambda))(1 - L) - (1 - \lambda L)}{1 - \lambda L} \right) x_t + v_t \]

or

\[ \mu_t - x_t = \left[ \frac{-\lambda + \rho(\lambda + \alpha(1 - \lambda))(1 - L)}{(1 - \lambda L)} \right] (1 - L) x_t + v_t \]

Operating on (19) with the "summation" operator (1 - L)^{-1} gives

\[ m_t - p_t = \left[ \frac{-\lambda + \rho(\lambda + \alpha(1 - \lambda))}{1 - \lambda L} \right] x_t + \xi_t \]
where $\xi_t = \xi_{t-1} + v_t$. Equation (20) is the projection that Cagan estimated by (nonlinear) least squares regression. Notice that the residuals in (20) follow a random walk. It is noteworthy in this regard that the residuals in Cagan's and Barro's estimates of (20) were highly serially correlated, Barro reporting very low values for Durbin-Watson statistics.

Now Cagan regarded the projection (20) as giving estimates of the equation

$$(21) \quad m_t - p_t = \frac{\alpha(1 - \lambda)}{1 - \lambda L} x_t + u_t.$$

Least squares regression consistently estimates the parameters of the population projection (20)---only those parameters are not in general the same ones Cagan took them to be. Comparison of (20) with (21) shows that Cagan's estimator of $\lambda$ is consistent but that his estimator of $\alpha$ is not in general consistent, and will obey

$$\text{plim} \; \hat{\alpha}(1 - \hat{\lambda}) = \left[-\lambda + \rho(\lambda + \alpha(1 - \lambda))\right]$$

which implies that

$$(22) \quad \text{plim} \; \hat{\alpha} = \rho \alpha + (1 - \rho) \left(\frac{-\lambda}{1 - \lambda}\right).$$

Notice that if $\rho = 0$, which will be true if $\epsilon_t = 0$ for all $t$, (22) implies

$$\text{plim} \; \hat{\alpha} = \frac{-\lambda}{1 - \lambda},$$

which is an expression that Wallace and I derived and used. On the other hand, if $\eta_t = 0$ for all $t$, so that there is no noise in the portfolio balance schedule, from (15) $\rho = 1$, which with (22) implies

$$\text{plim} \; \hat{\alpha} = \alpha,$$
so that in this special case Cagan's estimator of \( \alpha \) is consistent (and furthermore unbiased as it turns out, since \( \nu_t = 0 \) for all \( t \)).

On the special assumption \( \sigma_{\varepsilon \eta} = 0 \), we have

\[
\rho = \frac{E(\varepsilon_t \cdot (\varepsilon_t - \eta_t))}{E(\varepsilon_t - \eta_t)^2} = \frac{\sigma_{\varepsilon}^2}{\sigma_{\varepsilon}^2 + \sigma_{\eta}^2}.
\]

Alternatively, multiplying (15) by \( \eta_t \), taking expectations, and rearranging gives

\[
\rho = \frac{\sigma_{\varepsilon \eta}}{\sigma_{\eta}^2},
\]

so that \( \rho \) is the regression coefficient of the residual \( v \) in the regression of \((\mu - x)\) against current and past \((1 - L)x\) on the disturbance in the demand for money. If \( v = \eta \), then \( \rho = 1 \).

An estimate of \( \rho \) could be obtained in the following way, again on the special assumption \( \sigma_{\varepsilon \eta} = 0 \). Multiplying (15) by \( \varepsilon_t \), taking expectations and rearranging gives

\[
\rho = 1 - \frac{\sigma_{\varepsilon}^2}{\sigma_{\varepsilon}^2}.
\]

The magnitude \( \sigma_{\varepsilon}^2 / \sigma_{\varepsilon}^2 \) is the regression coefficient of \( v \) on \( \varepsilon \). The residual \( v \) can be estimated by the residual in (the first difference of) Cagan's equation. The variable \( \varepsilon_t \) can be extracted using the methods described below in Section 5. Then an estimate of \( \rho \) could be prepared using the above equation. It would be possible to use that estimate of \( \rho \) to correct Cagan's estimate of \( \alpha \) by applying the formula

\[
\text{plim} \ \hat{\alpha} = \rho \alpha + (1 - \rho) \left( \frac{-\lambda}{1 - \lambda} \right).
\]
The calculations in this section provide a useful exercise in interpreting systems in which one variable (x) is econometrically exogenous with respect to (is not Granger-caused by) another variable (u). In such a system, as Sims's theorem 2 assures us and as the preceding calculations verify, the regression of u on past, present, and future x's is one-sided on the present and past. Thus, there exists representations (models) of the (u, x) process in which μ and u·x are each one-sided linear functions of past and present x's with disturbances that are orthogonal to past, present, and future x's—so that in these relations x is strictly exogenous with respect to u and u·x, respectively. But the representation in which x is econometrically exogenous with respect to (u·x)—which is the relation that can be consistently estimated by least squares or generalized least squares—is not the demand function for money, which is the structural relation we are interested in estimating. The reason is that in the structural relation (21), uₜ is not in general orthogonal to the x process. The upshot is that finding that x is exogenous with respect to u·x does not guarantee that the one-sided u·x on x distributed lag regression which is estimable by single equation methods corresponds to the structural relation that we're interested in.
4. A Consistent Estimator

Equations (11) and (12) form a bivariate first-order moving average process in \((1 - L)\mu_t\) and \((1 - L)x_t\). Assuming that the white noises \(\varepsilon_t\) and \(\eta_t\) are jointly normally distributed, the likelihood function of a sample of length \(T\) observations, \(t = 1, \ldots, T\), generated by (11) - (12) can be written down. To apply the method of maximum likelihood, it is most convenient to write the model in its vector autoregressive form.

First note that from (9) we can write

\[
(23) \quad \varepsilon_t = \mu_t - \frac{1 - \lambda}{1 - \lambda L} x_t.
\]

Next from (11) we have

\[
(24) \quad \varepsilon_t - \eta_t = \frac{(\lambda + \alpha(1 - \lambda))(1 - L)}{1 - \lambda L} x_t
\]

Substituting (24) into (23) and rearranging gives

\[
(25) \quad \eta_t = \mu_t - \left( \frac{1 - \lambda + (\lambda + \alpha(1 - \lambda))(1 - L)}{1 - \lambda L} \right) x_t.
\]

In vector notation, equations (23) and (25) can be written

\[
\begin{bmatrix}
\varepsilon_t \\
\eta_t
\end{bmatrix} =
\begin{bmatrix}
\frac{1 - \lambda}{1 - \lambda L} & 1 \\
-(1 - \lambda + (\lambda + \alpha(1 - \lambda))(1 - L)) \frac{1}{1 - \lambda L} & 1
\end{bmatrix}
\begin{bmatrix}
x_t \\
\mu_t
\end{bmatrix}
\]

Multiplying both sides of the equation by \((1 - \lambda L)\cdot I\) where \(I\) is the 2x2 identity matrix, gives

\[
\begin{bmatrix}
(1 - \lambda L)\varepsilon_t \\
(1 - \lambda L)\eta_t
\end{bmatrix} =
\begin{bmatrix}
-(1 - \lambda) & 1 - \lambda L \\
-(1 - \lambda + (\lambda + \alpha(1 - \lambda))(1 - L)) & 1 - \lambda L
\end{bmatrix}
\begin{bmatrix}
x_t \\
\mu_t
\end{bmatrix}
\]
or

\[
\begin{bmatrix}
\varepsilon_t \\
\eta_t
\end{bmatrix}
= (-1) \begin{bmatrix}
(1 - \lambda) & 1 \\
(1 + \alpha(1 - \lambda)) & 1
\end{bmatrix}
\begin{bmatrix}
x_t \\
\mu_t
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & -\lambda \\
\lambda + \alpha(1 - \lambda) & -\lambda
\end{bmatrix}
\begin{bmatrix}
x_{t-1} \\
\mu_{t-1}
\end{bmatrix}
\]
\[ \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} - \lambda I \begin{bmatrix} a_{1t-1} \\ a_{2t-1} \end{bmatrix} = \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} + \begin{bmatrix} 0 & -\lambda \\ \lambda + \alpha(1 - \lambda) & -\lambda \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \mu_{t-1} \end{bmatrix} \]

where

\[ \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = G_0^{-1} \begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} . \]

Computing

\[ G_0^{-1} \begin{bmatrix} 0 & -\lambda \\ \lambda + \alpha(1 - \lambda) & -\lambda \end{bmatrix} \]

explicitly and rearranging the above equation gives

\[ \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (1 - \lambda) & \lambda \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \mu_{t-1} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} - \lambda I \begin{bmatrix} a_{1t-1} \\ a_{2t-1} \end{bmatrix} \]

Equation (27) is a vector first order autoregression, first-order moving average process. The random variables \( a_{1t}, a_{2t} \) are the innovations in the \( x \) and \( \mu \) processes, respectively. They are the one period-ahead forecasting errors for \( x_t \) and \( \mu_t \), respectively. The \( a \)'s are related to the \( \varepsilon \)'s and \( \eta \)'s appearing in the structural equations of the model by

\[ \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} = G_0^{-1} \begin{bmatrix} \varepsilon_t \\ \eta_t \end{bmatrix} \]
Notice that the first equation of (27) can be written as

\[(1 - \lambda)x_t = (1 - \lambda L)a_{1t}.\]

It is straightforward to write this in the autoregressive form

\[(29)\quad x_t = \left(\frac{1 - \lambda}{1 - \lambda L}\right)x_{t-1} + a_{1t}.\]

Since \(E_{t-1}a_{1t} = 0\), we have

\[E_{t-1}x_t = \left(\frac{1 - \lambda}{1 - \lambda L}\right)x_{t-1}.\]

The second equation of (27) can be written as

\[(1 - \lambda L)\mu_t = (1 - \lambda)x_{t-1} + (1 - \lambda L)a_{2t}\]

But from (29) we have \((1 - \lambda)x_{t-1} = (1 - \lambda L)x_t - (1 - \lambda L)a_{1t}\), which when substituted into the above equation gives

\[(1 - \lambda L)\mu_t = (1 - \lambda L)x_t - (1 - \lambda L)a_{1t} + (1 - \lambda L)a_{2t}\]

or

\[(30)\quad \mu_t = x_t + a_{2t} - a_{1t}.\]

From (30), it follows that

\[(31)\quad E_{t-1}\mu_t = E_{t-1}x_t.\]
The triangular character of representation (27) demonstrates that \( u \) does not "cause" in Granger's sense (i.e., help predict, once lagged own values are taken into account) the variable \( x \). That is, \( x \) is econometrically exogenous with respect to \( u \). On the other hand, \( x \) does cause the variable \( \mu_t \). Even stronger, the model implies that

\[
E_{t-1} u_t = E_{t-1} x_t = \frac{1 - \lambda}{1 - \lambda L} x_{t-1}
\]

so that lagged \( \mu \)'s don't help predict \( u \) once lagged \( x \)'s are taken into account. That \( x \) causes \( \mu \) in Granger's sense is not to be confused with \( x \)'s "leading" \( \mu \) in any National Bureau sense. On the contrary, according to (30), \( x_t \) and \( \mu_t \) are "in phase" with one another, neither one leading the other. (The phase of their cross-spectrum equals zero at all frequencies.) Evidence that \( x \) leads \( \mu \) would not be consistent with the model being studied here.

The vector autoregressive, moving average process (27) is in a form that can be estimated by the maximum likelihood estimator described by Wilson [10]. It is essential that the matrices multiplying current \( a_{1t} \) and current \( x_t \) both be identity matrices in order to apply the method, so that each \( a_i \) process can be interpreted as the residual from a vector autoregression either for \( x_t \) or \( \mu_t \). This is by way of getting things in a form in which the likelihood function of \( x_t \) equals the likelihood function of \( \mu_t \).

Let

\[
a_t = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix},
\]

and let \( D_a \) be the covariance matrix of \( a_t \),

\[
D_a = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = E a_t a_{t}',
\]
The likelihood function of the sample \( t=1, \ldots, T \) can now be written as

\[
L(\lambda, \sigma_{11}, \sigma_{12}, \sigma_{22}|\mu_t, x_t) = (2\pi)^{-T}|D_a|^{-T/2}\exp\left(-\frac{1}{2} \sum_{t=1}^{T} a_t'D_a^{-1}a_t'Da_t\right)
\]

Given initial values for \((a_{10}, a_{20})\) or equivalently for \((c_0, \eta_0)\), and given a value of \( \lambda \), equation (26) or (27) can be used to solve for \( a_t \), \( t=1, \ldots, T \). (I will take \( a_{10} = a_{20} = 0 \).)

Wilson notes that maximizing (32) is equivalent with minimizing with respect to \( \lambda \) the determinant of the estimated covariance matrix of the \( a_t \)'s,

\[
|D_{\hat{a}}| \equiv |T^{-1} \sum_{t=1}^{T} \hat{a}_t\hat{a}_t'|\]

where the \( \hat{a}_t \)'s are determined by solving (27) recursively and so depend on \( \lambda \). The covariance matrix of the \( a_t \)'s is estimated as

\[
\hat{D}_a = T^{-1} \sum_{t=1}^{T} \hat{a}_t\hat{a}_t'
\]

evaluated at the value of \( \lambda \) that minimizes (33). The resulting estimates are known to be statistically consistent (see Wilson [10]).

Notice that \( \alpha \) does not appear explicitly in the likelihood function, but only indirectly by way of the elements of \( D_a \), namely, \( \sigma_{11}, \sigma_{12}, \) and \( \sigma_{22} \). That this must be so can be seen by inspecting representation (27), in which \( \lambda \) appears explicitly but \( \alpha \) does not. On the basis of the four parameters \( \lambda, \sigma_{11}, \sigma_{12}, \) and \( \sigma_{22} \) that are identified by (27), i.e., that characterize the likelihood function (32), we can think of attempting to estimate the five parameters of the model \( \alpha, \lambda, \sigma^2_{\varepsilon}, \sigma^2_{\eta}, \) and \( \sigma_{\varepsilon\eta} \). Not surprisingly, some of the parameters are underidentified. In particular, while \( \lambda \) and \( \sigma^2_{\varepsilon} \) are identified, \( \alpha, \sigma^2_{\eta}, \) and \( \sigma_{\varepsilon\eta} \) are not separately identified. To see that \( \alpha \) and \( \sigma_{\varepsilon\eta} \) are not identified consider
the following argument. From equation (28), we know that the identifiable
parameters $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{22}$ are related to the structural parameters
$\sigma_\varepsilon^2$, $\sigma_\eta^2$, $\sigma_{\varepsilon\eta}$, $\alpha$, and $\lambda$ by

$$\sigma_{11} = \left(\frac{1}{\lambda + \alpha (1 - \lambda)}\right)^2 (\sigma_\varepsilon^2 + \sigma_\eta^2 - 2\sigma_{\varepsilon\eta})$$

$$\sigma_{12} = \frac{(1 - \lambda)}{(\lambda + \alpha (1 - \lambda))^2} (\sigma_\varepsilon^2 + \sigma_\eta^2 - 2\sigma_{\varepsilon\eta}) + \frac{1}{\lambda + \alpha (1 - \lambda)} (\sigma_\varepsilon^2 - \sigma_\eta^2)$$

$$\sigma_{22} = \left(\frac{(1 - \lambda)}{\lambda + \alpha (1 - \lambda)}\right)^2 (\sigma_\varepsilon^2 + \sigma_\eta^2 - 2\sigma_{\varepsilon\eta}) + \sigma_\varepsilon^2 + 2 \left(\frac{1 - \lambda}{\lambda + \alpha (1 - \lambda)}\right)(\sigma_\varepsilon^2 - \sigma_\eta^2).$$

These equations imply

$$\sigma_{12} = (1 - \lambda)\sigma_{11} + \frac{1}{\lambda + \alpha (1 - \lambda)} (\sigma_\varepsilon^2 - \sigma_\eta^2)$$

$$\sigma_{22} = (1 - \lambda)^2\sigma_{11} + \sigma_\varepsilon^2 + \frac{2(1 - \lambda)}{\lambda + \alpha (1 - \lambda)} (\sigma_\varepsilon^2 - \sigma_\eta^2).$$

Do there exist offsetting changes in $\alpha$ and $\sigma_{\varepsilon\eta}$ that leave both of these
equations satisfied with $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{12}$ unchanged? That is, holding
$\lambda$ and $\sigma_\varepsilon^2$ constant, can we change $\alpha$ and $\sigma_{\varepsilon\eta}$ in offsetting ways that leave
$\sigma_{11}$, $\sigma_{12}$, and $\sigma_{22}$ constant? The answer is yes, as can be seen by
differentiating (37) and (38) and setting $d\sigma_{12} = d\sigma_{11} = d\sigma_{22} = d\alpha = d\sigma_\varepsilon = 0$:

$$0 = (1 - \lambda)(\lambda + \alpha (1 - \lambda))^{-2}(\sigma_\varepsilon^2 - \sigma_\eta^2)d\alpha$$

$$+ (\lambda + \alpha (1 - \lambda))^{-1}d\sigma_{\varepsilon\eta} = 0$$
Dividing the second equation by \(2(1 - \lambda)\) gives the first equation, which proves that if \(d_\alpha\) and \(d_\sigma\) obey equation (39), both equations (37) and (38) will remain satisfied. Thus, there exist offsetting changes in \(\alpha\) and \(\sigma_\varepsilon\eta\) that leave the identifiable parameters \(\lambda, \sigma_{11}, \sigma_{12},\) and \(\sigma_{22}\) unaltered. It follows that \(\sigma_\varepsilon\eta\) and \(\alpha\) are not separately identifiable.

It is evident from (27) or (32) that \(\lambda\) is identified. To see that \(\sigma_\varepsilon^2\) is identifiable, simply recall that \(\varepsilon_t\) obeys the feedback rule

\[
\mu_t = \frac{1 - \lambda}{1 - \lambda L} x_t + \varepsilon_t,
\]

so that given \(\lambda\), and samples of \(\mu_t\) and \(x_t\), \(\sigma_\varepsilon^2\) is identifiable as the variance of the residual in the above equation.

To proceed to extract estimates of \(\alpha\) it is necessary to impose a value of \(\sigma_\varepsilon\eta\). I propose to impose the condition \(\sigma_\varepsilon\eta = 0\), so that shocks to the money supply rule and shocks to portfolio balance are uncorrelated. It is straightforward to calculate

\[
\begin{bmatrix}
\sigma_\varepsilon^2 & \sigma_\varepsilon\eta \\
\sigma_\varepsilon\eta & \sigma_\eta^2
\end{bmatrix} = E \begin{bmatrix}
\varepsilon_t \\
\eta_t
\end{bmatrix} = G_0 D G_0',
\]

\[
\begin{bmatrix}
(1 - \lambda)^2 \sigma_{11} + 2(1 - \lambda)\sigma_{12} + \sigma_{22}, (1 - \lambda)(1 + \alpha(1 - \lambda))\sigma_{11} - (2 - \lambda + \alpha(1 - \lambda))\sigma_{12} + \sigma_{22} \\
(1 - \lambda)(1 + \alpha(1 - \lambda))\sigma_{11} - (2 - \lambda + \alpha(1 - \lambda))\sigma_{11} + \sigma_{22}, (1 + \alpha(1 - \lambda))^2 \sigma_{11} - 2(1 + \alpha(1 - \lambda))\sigma_{12} + \sigma_{22}
\end{bmatrix}
\]
Imposing \( \varepsilon = 0 \), we have

\[
0 = \varepsilon = (1 - \lambda)(1 + \alpha(1 - \lambda))\sigma_{11} - (2 - \lambda + \alpha(1 - \lambda))\sigma_{12} + \sigma_{22},
\]

which implies that \( \alpha \) is to be estimated by

\[
(41) \quad \alpha = \frac{-\sigma_{11}}{(1 - \lambda)\sigma_{11} - \sigma_{12}} + \frac{(2 - \lambda)\sigma_{12}}{(1 - \lambda)((1 - \lambda)\sigma_{11} - \sigma_{12})} - \frac{\sigma_{22}}{(1 - \lambda)((1 - \lambda)\sigma_{11} - \sigma_{12})}.
\]

Let this estimator of \( \alpha \) be

\[
\hat{\alpha} = g(\lambda, \sigma_{11}, \sigma_{12}, \sigma_{22}) = g(\theta)
\]

where \( \theta = (\lambda, \sigma_{11}, \sigma_{12}, \sigma_{22}) \). Let \( \Sigma_\theta \) be the estimated asymptotic covariance matrix of \( \theta \). Then the asymptotic variance of \( \hat{\alpha} \) will be estimated as

\[
\text{var} \; \hat{\alpha} = \left( \frac{\partial g}{\partial \theta} \right)_{\theta} \Sigma_\theta \left( \frac{\partial g}{\partial \theta} \right)_{\theta}^T
\]

where \( \frac{\partial g}{\partial \theta} \) is the \((1 \times 4)\) vector of partial derivatives of \( g \) with respect to \( \theta \) evaluated at the maximum likelihood estimates \( \theta \). The asymptotic covariance matrix of \((\lambda, \sigma_{11}, \sigma_{12}, \sigma_{22})\) is given by

\[
\Sigma_\theta = \frac{1}{T} T_{\sigma^2_{\lambda}} \times
\begin{bmatrix}
T_{\sigma^2_{\lambda}} & 0 & 0 & 0 \\
0 & 2\sigma_{11}^2 & 2\sigma_{11}\sigma_{12} & 2\sigma_{12}^2 \\
0 & 2\sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{22} + \sigma_{12}^2 & 2\sigma_{12}\sigma_{22} \\
0 & 2\sigma_{12}^2 & 2\sigma_{12}\sigma_{22} & 2\sigma_{22}^2
\end{bmatrix}
\]

where \( T_{\sigma^2_{\lambda}} \) is estimated by
21

\[
T_0^2 \lambda = \left[ - \frac{2 \log L}{\partial \lambda^2} \right]^{-1}
\]

and where \( \log L \) is the natural logarithm of the likelihood function (32). Notice that the maximum likelihood estimate of \( \lambda \) is asymptotically orthogonal to the estimates \( \sigma_{11}, \sigma_{12}, \sigma_{22} \). The preceding formula for \( L_0 \) can be derived by applying results of Wilson [10] and T. W. Anderson [1, pp. 159-61]. In the computations summarized below, the components \( \sigma_{11}, \sigma_{12}, \) and \( \sigma_{22} \) were estimated by

\[
D = \left( \begin{array}{cc} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_{22} \end{array} \right) = T^{-1} \sum_{t=1}^{T} \hat{a}_t \hat{a}_t',
\]

the maximum likelihood estimator. The term

\[
\left( - \frac{2 \log L}{\partial \lambda^2} \right) \hat{\lambda}
\]

was estimated numerically in the course of minimizing (33) to obtain the maximum likelihood estimates.

It bears emphasizing that \( \alpha \) is identifiable at all only on the basis of a restriction on \( \alpha_{e\eta} \), and that the estimator of \( \alpha \) obtained by imposing \( \alpha_{e\eta} = 0 \) depends sensitively on the covariance matrix of the errors in forecasting \( x_t \) and \( \mu_t \) from the past. The estimates of \( \alpha \) thereby obtained ought to be regarded as very delicate.

5. An Alternative Estimator

If it is assumed that \( \alpha_{e\eta} = 0 \), so that shocks to the demand for money and to the supply of money are uncorrelated, an instrumental variable estimator is available. From equations (30) and (29) we have that

\[
E_{t-1} x_t = E_{t-1} \mu_t = \frac{1 - \lambda}{1 - \lambda L} x_{t-1},
\]
and that

\[(42) \quad x_t - E_{t-1} x_t = a_{1t} = \frac{1}{\lambda + \alpha(1 - \lambda)} (\varepsilon_t - \eta_t) \]

\[(43) \quad \mu_t - E_{t-1} x_t = a_{2t} = \frac{1 - \lambda}{\lambda + \alpha(1 - \lambda)} (\varepsilon_t - \eta_t) + \varepsilon_t \]

Notice that

\[(44) \quad a_{2t} - (1 - \lambda)a_{1t} = \varepsilon_t. \]

This suggests the following procedure. Estimate by maximum likelihood the univariate first-order moving average process for $Ax_t$, i.e.,

\[(1 - L) x_t = (1 - \lambda L) a_{1t} \]

where $a_{1t} = \frac{1}{\lambda + \alpha(1 - \lambda)} (\varepsilon_t - \eta_t)$ is "white." This will yield consistent estimates of $\lambda$ and permit estimating the forecast errors. The forecasts $E_{t-1} x_t$ can be estimated from the above equation as

\[E_{t-1} x_t = x_{t-1} - \lambda a_{1t-1}. \]

Use of (44) shows that estimates of $\varepsilon_t$ can be extracted according to

\[(45) \quad \varepsilon_t = (\mu_t - E_{t-1} x_t) - (1 - \lambda)(x_t - E_{t-1} x_t). \]

On the assumption that $\varepsilon_t$ is uncorrelated with $\eta_t$, $\varepsilon_t$ is a valid instrument for estimating equation (1): it is correlated with the regressors but orthogonal to the disturbance. Letting $\tilde{\varepsilon}_t$ be the estimates of $\varepsilon_t$ obtained by applying (45), I propose fitting the first-stage regression
\[ \hat{x}_t = \sum_{i=0}^{n} w_i \hat{e}_{t-1} \]

where the hatted values denote least squares estimates. Then Cagan's equation (1) would be estimated by applying (nonlinear) least squares to the second-stage regression

\[ m_t - p_t = \alpha \left( \frac{1 - \lambda}{1 - \lambda L} \right) \hat{x}_t + u_t - \frac{\alpha(1 - \lambda)}{1 - \lambda L} (\hat{x}_t - x_t). \]

This procedure yields consistent estimates of \( \alpha \) and \( \lambda \) on the assumption that

\[ \text{plim} \ T^{-1} \sum_{t=1}^{T} \hat{e}_t \eta_t = 0, \]

a condition that the orthogonality of \( \hat{e}_t \) and \( \eta_t \) goes a long way toward delivering.

6. Testing the Model

Representation (27) shows that the model is a special case of the general vector first-order moving average, first order autoregressive process

\[ (27') \quad \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{1t-1} \\ a_{2t-1} \end{bmatrix} \]

where in (27) seven linear restrictions have been placed on the eight parameters \( (c_{11}, c_{12}, c_{21}, c_{22}, b_{11}, b_{12}, b_{21}, b_{22}) \) of (27') so that the systematic part of (27) only involves the single parameter \( \lambda \). The model (27) can be tested by relaxing some subset of the seven restrictions that were imposed on (27') to get (27), maximizing the likelihood function under the less restrictive parameterization, and calculating the pertinent
Let \( L(x_t, \mu_t; \theta_0) \) be the maximum of the likelihood function under parameterization (27), which is Cagan's model under rational expectations. Let \( L(x_t, \mu_t; \theta, q) \) be the maximum of the likelihood function under (27') with \( q \) of the seven restrictions in (27) being relaxed. Then

\[
-2 \log \left( \frac{L(x_t, \mu_t; \theta_0)}{L(x_t, \mu_t; \theta, q)} \right)
\]

is asymptotically distributed as \( \chi^2(q) \). High values of the test statistic lead to rejection of representation (27). Below this test is implemented under several alternative relaxations of the restrictions on (27).

7. Empirical Results

For Cagan's and Barro's data, respectively, Tables 2 and 3 report the estimates obtained using the maximum likelihood estimator and the assumption that \( \sigma_{\epsilon t} = 0 \). Asymptotic standard errors are in parentheses beneath each estimator. Cagan's and Barro's estimates are reported in Tables 4 and 5 for convenience. For Cagan's data, the maximum likelihood estimator recovers estimates of \( \alpha \) that are in most cases characterized by large standard errors. In particular, for the important German case, a case in which Cagan had apparently estimated \( \alpha \) with a tight confidence band, my estimate of \( \alpha \) has a big standard error, one nearly as big as the point estimate itself. Evidently, the estimate of \( \alpha \) is not statistically significantly different from zero even at modest confidence levels, at least if we are willing to use the asymptotic (normal) distribution of the estimates. For the Austrian and Russian cases, my estimate of \( \alpha \) is smaller than its standard error. Only in the case of Hungary I, and to a lesser extent in the case of Poland, is the standard error of \( \alpha \) small relative to the point estimate of \( \alpha \). Interestingly
enough, for Hungary I my estimate of $\alpha$ of -1.84 is much smaller in absolute value than Cagan's estimate of -8.70. The reciprocal of +1.84 is .54, while the average monthly rate of inflation in the Hungary I case was .46. In the case of Hungary I, my estimate of $\alpha$ suggests that the paradox with which I began this paper, the apparent tendency of creators of money to print money "too fast," was not present. For what it is worth, then, my estimate of $\alpha$ for Hungary I tends to explain away the paradox. For the other countries, the point estimates do not explain away the paradox. However, in each case, values of $\alpha$ that would cause the paradox to disappear do exist within confidence intervals of two standard errors on each side of the point estimate of $\alpha$. This suggests that perhaps the paradox ought not to be taken as having been seriously confirmed since the estimates of $\alpha$ on which it is based seem so shaky.

Notice that my estimates of $\lambda$ are always lower than Cagan's. That is an unexpected result, since according to the model, Cagan's estimate of $\lambda$ and my maximum likelihood estimator are each consistent. The systematic difference in estimates as between the two estimators may reflect the inadequacy of the model.

For Barro's data, the maximum likelihood estimates are reported in Table 3. For Austria and Germany, the estimated asymptotic standard errors of $\alpha$ are fairly large relative to the point estimates, while for Hungary I and Poland they are much smaller. As with Cagan's data, my estimate of $\alpha$ is much smaller than is Barro's for Hungary I. My estimate is somewhat smaller than Barro's for Poland. As with Cagan's data, my estimate of $\lambda$ is always smaller than Cagan's.

The main conclusion that I draw from these estimates is that even under the restriction $\sigma_{\epsilon \eta} = 0$, the slope parameter $\alpha$ is usually
poorly estimated. When to this is added the observations that $\alpha$ is not
even identifiable unless $\sigma_{\varepsilon \eta}$ is restricted and that economics does not
seem to restrict $\sigma_{\varepsilon \eta}$, the uncertainty about $\alpha$ only increases. It seems
correct to conclude that, with the possible exception of Hungary I, I
have not been able to estimate very well the slope of the portfolio
balance schedule.

This is not to say, however, that the model is necessarily
defective. It is certainly conceivable that the model approximated
reality quite well even though $\alpha$ cannot be estimated well or isn't even
identifiable. As pointed out in Section 6, the proper way to test the
model is to "overfit" the vector moving average, autoregressive repre-
sentation (27), and to test whether the restrictions imposed by (27) are
violated. For overfitting, I have estimated each of the six parameteri-
izations reported in Table 6. For each parameterization, the chi-square
statistic described in Section 6 was computed, and is reported in
Table 7 for Cagan's data and in Table 8 for Barro's data. High values
of the $\chi^2$ statistic lead to rejection of the null hypothesis that model
(27) is adequate.9/

For Cagan's data, at the .95 confidence level, the model is
rejected relative to parameterization 5 for Russia, relative to parame-
terizations 1, 2, 4, and 5 for Hungary I, and relative to parameterizations
2, 5, and 6 for Austria. For Germany, Greece, and Poland, the model is
not rejected relative to any of the six parameterizations at the .95
confidence level. For three of the hyperinflations, then, overfitting
representation (27) does turn up evidence that would prompt rejection of
the model. However, it surprised me just how adequately the model does
seem to perform relative to the six parameterizations in Table 6.
Representation (27) is a very stark, highly restricted parameterization; indeed, the systematic part of the vector autoregression has only one parameter. I had expected the model to be rather decisively rejected by these overfitting tests. It is remarkable that the model seems to survive those tests for even three of the hyperinflations.

For Barro's data, at the .95 confidence level the chi-square statistics call for rejecting representation (27) relative to parameterizations (1), (2), (4), and (5) for Hungary I. The statistics do not call for rejection of (27) for Germany, Poland, or Austria.

Notice that for both Cagan's and Barro's data, the overfitting tests reject representation (27) for the case of Hungary I, a case for which my estimator of \( \alpha \) obtained the tightest confidence band.

8. Conclusions

This paper has applied maximum likelihood techniques to derive a consistent estimator of a bivariate, rational expectations version of Cagan's model of hyperinflation. The estimator, in principle, eliminates the simultaneous-equations, asymptotic bias that characterizes Cagan's estimator. Application of the maximum likelihood estimator typically yields "loose" estimates of the slope parameter of the demand schedule for money. The estimates are so loose that confidence bands of two standard errors on each side of them include values that would imply that the creators of money were inflating at rates that maximized their command over real resources, thus maybe resolving the "paradox" with which I began this paper. While perhaps this resolves the paradox, it does so in a destructive way, by suggesting that the demand for money in hyperinflation has not been isolated as well as might have been thought. This is not a very satisfactory state of affairs in which to leave the
subject. In a subsequent paper, I intend to describe further efforts to isolate the demand schedule for money, using a technique which for special reasons cannot be applied to Cagan's model. Use of that technique will be shown to require abandoning the assumption of adaptive (geometric lag, unit-sum) expectations. The technique will be shown to break down under the singular circumstance that the model in the present paper is the correct one. However, the results of my "overfitting" tests, to the extent that they do not always emphatically reject the model in the present paper, suggest that the prospects for success are not great for using such a technique. It could just be true that the model in this paper is the "correct" one, so that even though the portfolio balance schedule was exactly the one Cagan assumed, the nature of the money supply regimes in effect during the hyperinflations makes difficult or impossible estimating the slope of that portfolio balance schedule.
Footnotes

1/Among such studies are some of those in Meiselman [5].

2/I assume that the information available consists (at least) of observations of current and past μ's and current and past x's. Thus
\[ E_{t} x_{t+1} = E[x_{t+1}|μ_{t}, μ_{t-1}, ..., x_{t}, x_{t-1}, ...]. \]
Similarly, where \( z_{t} \) is any arbitrary random variable, I will write \( E_{t} z_{t+1} \) for \( E[z_{t+1}|μ_{t}, μ_{t-1}, ..., x_{t}, x_{t-1}, ...]. \)

3/Substituting (3) into (1), first differencing, and shifting the time subscripts forward one period gives

\[ \mu_{t+1} - x_{t+1} = αE_{t} x_{t+2} - αx_{t+1} + (u_{t+1} - u_{t}). \]

Taking expectations conditional on information available at time \( t \) gives

\[ E_{t} x_{t+1} = \frac{1}{1-α} E_{t} μ_{t+1} - \frac{α}{1-α} E_{t} x_{t+2} - (E_{t} u_{t+1} - E_{t} u_{t}). \]

Recursion on the above difference equation shows that equation (4) is indeed a solution to that equation.

4/To see that process (9) satisfies (8), write (9) as

\[ \mu_{t+1} = (1-λ)x_{t+1} + \frac{(1-λ)λ}{1-λL} x_{t} + ε_{t+1}. \]

Taking expectations conditional on information available at \( t \), we have

\[ E_{t} μ_{t+1} = (1-λ)E_{t} x_{t+1} + \frac{(1-λ)λ}{1-λL} x_{t}. \]

But \( E_{t} x_{t+1} = \frac{1-λ}{1-λL} x_{t} \), so that we have

\[ E_{t} μ_{t+1} = ((1-λ)+λ) \left( \frac{1-λ}{1-λL} \right) x_{t} \]

\[ E_{t} μ_{t+1} = \frac{1-λ}{1-λL} x_{t}, \]

as required.

5/The most readily accessible reference in economics on the multivariate Wold representation is Sims [9], especially the appendix.

6/Sims [9] proved the equivalence of Granger causality with econometric exogeneity.
Wallace and I were mistaken when we asserted that "the system is one in which expectations of money creation could equally well be formed as a distributed lag of past rates of money creation," [8, p. 337]. It is true that

$$E[u_t | u_{t-1}, \ldots] = \frac{1 - T}{1 - TL} u_{t-1},$$

where $T$ is a parameter that depends on the ratio of the variance of $\varepsilon_t$ to the variance of $\eta_t$. However, in the model $E[u_t | u_{t-1}, \ldots] \neq E[u_t | u_{t-1}, \ldots, x_{t-1}, \ldots]$. Instead, $E[u_t | u_{t-1}, \ldots, x_{t-1}, \ldots] = E[u_t | x_{t-1}, \ldots]$, which, of course, has a smaller prediction error variance than $E[u_t | u_{t-1}, \ldots]$. The erroneous statements on page 337 of Sargent and Wallace [8] amount to an assertion that the Wold representation of the $x_t - \mu_t$ process contains only one noise, so that lagged values of either $x$ or $\mu$ exhaust all information in the past values of $x$ and $\mu$ useful for predicting either $x$ or $\mu$. That is wrong, as the triangular Wold representation derived in Section 3 of this paper verifies. The upshot of all this is that it was not necessary for Sargent and Wallace to posit measurement errors in the money supply to rationalize the empirical observation that $x$ causes $\mu$. That is already an implication of the system free of measurement errors.

Actually, the normality of the asymptotic distribution is conjectural. See Porter, [7].
References


Table 2

Estimates for Cagan's data (standard errors in parentheses)
(x and \( \mu \) are deviations from respective means)

<table>
<thead>
<tr>
<th>Country</th>
<th>( \lambda )</th>
<th>( \alpha )</th>
<th>( \sigma_{11} )</th>
<th>( \sigma_{12} )</th>
<th>( \sigma_{22} )</th>
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<td>(4.615)</td>
<td>(.0147)</td>
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<td>(1.5695)</td>
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<td>(.0109)</td>
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<td>.0524</td>
<td>.0138</td>
<td>.0205</td>
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<td>(10.742)</td>
<td>(.0145)</td>
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<tr>
<td>May '22-Nov '23</td>
<td>.5364</td>
<td>-2.529</td>
<td>.0566</td>
<td>.0149</td>
<td>.0089</td>
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<tr>
<td></td>
<td>(.0722)</td>
<td>(.8562)</td>
<td>(.0175)</td>
<td>(.0059)</td>
<td>(.0027)</td>
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</table>
Table 3

Estimates for Barro's data (standard errors in parentheses)  
(x and μ are deviations from respective means)

<table>
<thead>
<tr>
<th>County</th>
<th>( \hat{\lambda} )</th>
<th>( \hat{\sigma} )</th>
<th>( \hat{\sigma}_{11} )</th>
<th>( \hat{\sigma}_{12} )</th>
<th>( \hat{\sigma}_{22} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AUSTRIA</td>
<td>.6373</td>
<td>-3.979</td>
<td>.0584</td>
<td>.0161</td>
<td>.0081</td>
</tr>
<tr>
<td>Apr '21-Dec '22</td>
<td>(.0739)</td>
<td>(2.805)</td>
<td>(.0172)</td>
<td>(.0056)</td>
<td>(.0024)</td>
</tr>
<tr>
<td>GERMANY</td>
<td>.5921</td>
<td>-2.344</td>
<td>.1806</td>
<td>.0653</td>
<td>.0263</td>
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<tr>
<td>Feb '21-Aug '23</td>
<td>(.0510)</td>
<td>(1.223)</td>
<td>(.0445)</td>
<td>(.0165)</td>
<td>(.0065)</td>
</tr>
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<td>HUNGARY</td>
<td>.4323</td>
<td>-1.705</td>
<td>.0280</td>
<td>.0071</td>
<td>.0038</td>
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<td>Nov '21-Feb '24</td>
<td>(.0559)</td>
<td>(.2782)</td>
<td>(.0072)</td>
<td>(.0023)</td>
<td>(.0010)</td>
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<tr>
<td>POLAND</td>
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<td>.0319</td>
<td>.0063</td>
<td>.0040</td>
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<tr>
<td>Feb '22-Jan '24</td>
<td>(.0533)</td>
<td>(.3537)</td>
<td>(.0089)</td>
<td>(.0025)</td>
<td>(.0011)</td>
</tr>
</tbody>
</table>
Table 4

Cagan's estimates of $\alpha$, $\lambda$ together with confidence band for $\alpha$.

<table>
<thead>
<tr>
<th>Country</th>
<th>$\hat{\lambda}$</th>
<th>$\hat{\alpha}$</th>
<th>$(\alpha_e, \alpha_u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Austria</td>
<td>.95</td>
<td>-8.55</td>
<td>-(4.43, 30.0)</td>
</tr>
<tr>
<td>Jan '21-Aug '22</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Germany</td>
<td>.82</td>
<td>-5.46</td>
<td>-(5.05, 6.13)</td>
</tr>
<tr>
<td>Sept '20-July '23</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Greece</td>
<td>.86</td>
<td>-4.09</td>
<td>-(2.83, 32.5+)</td>
</tr>
<tr>
<td>Jan '43-Aug '44</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hungary</td>
<td>.90</td>
<td>-8.70</td>
<td>-(6.36, 42.2+)</td>
</tr>
<tr>
<td>July '22-Feb '24</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hungary</td>
<td>.86</td>
<td>-3.63</td>
<td>-(2.55, 4.73)</td>
</tr>
<tr>
<td>July '45-Feb '46</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poland</td>
<td>.74</td>
<td>-2.30</td>
<td>-(1.74, 3.94)</td>
</tr>
<tr>
<td>Apr '22-Nov '23</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Russia</td>
<td>.70</td>
<td>-3.06</td>
<td>-(2.66, 3.76)</td>
</tr>
<tr>
<td>Dec '21-Jan '24</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$(\alpha_e, \alpha_u) = 90$ percent confidence band calculated by Cagan using likelihood ratio method.

*$\alpha_u$ actually exceeds right-hand figure in parentheses.

Source: Cagan's Table 3, p. 43.
Table 5
Barro's Estimates of λ and α

<table>
<thead>
<tr>
<th>Country</th>
<th>λ</th>
<th>α*</th>
</tr>
</thead>
<tbody>
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<td>Austria</td>
<td>.829</td>
<td>-4.09</td>
</tr>
<tr>
<td>Jan '21-Dec '22</td>
<td></td>
<td>(-3.6 , -4.5)</td>
</tr>
<tr>
<td>Germany</td>
<td>.824</td>
<td>-3.79</td>
</tr>
<tr>
<td>Jan '21-Aug '23</td>
<td></td>
<td>(-3.3 , -4.3)</td>
</tr>
<tr>
<td>Hungary</td>
<td>.861</td>
<td>-5.53</td>
</tr>
<tr>
<td>Oct '21-Feb '24</td>
<td></td>
<td>(-4.6 , -6.9)</td>
</tr>
<tr>
<td>Poland</td>
<td>.709</td>
<td>-2.56</td>
</tr>
<tr>
<td>Jan '22-Jan '24</td>
<td></td>
<td>(-2.1 , -3.3)</td>
</tr>
</tbody>
</table>

*95 percent confidence intervals in parentheses beneath each estimate.

Source: Barro, Table 3.
Table 6

Parameterizations for Overfitting

\[
\begin{bmatrix}
  x_t \\
  \mu_t
\end{bmatrix} = C \begin{bmatrix}
  x_{t-1} \\
  \mu_{t-1}
\end{bmatrix} + \begin{bmatrix}
  a_{1t} \\
  a_{2t}
\end{bmatrix} + B \begin{bmatrix}
  a_{1t-1} \\
  a_{2t-1}
\end{bmatrix}
\]

1. \( C = \begin{bmatrix}
  c & 0 \\
  (1-\lambda) & \lambda
\end{bmatrix}, \quad B = \begin{bmatrix}
  -\lambda & 0 \\
  0 & -\lambda
\end{bmatrix} \)

2. \( C = \begin{bmatrix}
  1 & c \\
  1-\lambda & \lambda
\end{bmatrix}, \quad B = \begin{bmatrix}
  -\lambda & 0 \\
  0 & -\lambda
\end{bmatrix} \)

3. \( C = \begin{bmatrix}
  1 & 0 \\
  1-\lambda & \lambda
\end{bmatrix}, \quad B = \begin{bmatrix}
  -\lambda & b \\
  0 & -\lambda
\end{bmatrix} \)

4. \( C = \begin{bmatrix}
  c_1 & c_1 \\
  1-\lambda & \lambda
\end{bmatrix}, \quad B = \begin{bmatrix}
  -\lambda & 0 \\
  0 & -\lambda
\end{bmatrix} \)

5. \( C = \begin{bmatrix}
  1 & c \\
  (1-\lambda) & \lambda
\end{bmatrix}, \quad B = \begin{bmatrix}
  -\lambda & b \\
  0 & -\lambda
\end{bmatrix} \)

6. \( C = \begin{bmatrix}
  1 & 0 \\
  c_1 & c_2
\end{bmatrix}, \quad B = \begin{bmatrix}
  -\lambda & 0 \\
  0 & -\lambda
\end{bmatrix} \)
<table>
<thead>
<tr>
<th>Country</th>
<th>Parameterization</th>
<th>Number</th>
<th>$\chi^2$ (1)</th>
<th>$\chi^2$ (1)</th>
<th>$\chi^2$ (2)</th>
<th>$\chi^2$ (2)</th>
<th>$\chi^2$ (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GERMANY</td>
<td>Oct '20-July '23</td>
<td>1</td>
<td>.52</td>
<td>1.12</td>
<td>2.06</td>
<td>.95</td>
<td>3.37</td>
</tr>
<tr>
<td>RUSSIA</td>
<td>Feb '22-Jan '24</td>
<td>2</td>
<td>.21</td>
<td>3.05</td>
<td>2.84</td>
<td>3.90</td>
<td>7.79</td>
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<tr>
<td>GREECE</td>
<td>Feb '43-Aug '44</td>
<td>3</td>
<td>1.04</td>
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<td>.25</td>
<td>4.14</td>
<td>1.87</td>
</tr>
<tr>
<td>POLAND</td>
<td>May '22-Nov '23</td>
<td>5</td>
<td>.19</td>
<td>.04</td>
<td>.22</td>
<td>.31</td>
<td>.56</td>
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<tr>
<td>AUSTRIA</td>
<td>Feb '21-Aug '22</td>
<td>6</td>
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<td>4.97</td>
<td>.63</td>
<td>4.97</td>
<td>10.05</td>
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</tbody>
</table>

Significance Levels: $\chi^2(1)_{.05} = 3.84$, $\chi^2(2)_{.05} = 5.99$, $\chi^2(1)_{.01} = 6.63$, $\chi^2(2)_{.01} = 9.21$
### Table 8

Barro's Data
Results of Overfitting—Chi Square Statistics

<table>
<thead>
<tr>
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<th>$\chi^2(1)$</th>
<th>$\chi^2(2)$</th>
<th>$\chi^2(2)$</th>
<th>$\chi^2(2)$</th>
</tr>
</thead>
<tbody>
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<td>1.272</td>
<td>.382</td>
<td>.3</td>
<td>3.5</td>
<td>.33</td>
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<td></td>
</tr>
<tr>
<td>HUNGARY I</td>
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<td>5.424</td>
<td>7.6</td>
<td>1.232</td>
<td>7.63</td>
<td>8.49</td>
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<td>Nov '21-Feb '24</td>
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</tr>
<tr>
<td>POLAND</td>
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<td>1.58</td>
<td>.528</td>
<td>.184</td>
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</tr>
</tbody>
</table>

Significance Levels:
- $\chi^2(1).05 = 3.84$
- $\chi^2(2).05 = 5.99$
- $\chi^2(1).025 = 5.02$
- $\chi^2(2).025 = 7.37$
- $\chi^2(1).01 = 6.63$
- $\chi^2(2).01 = 9.21$