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A LAW OF LARGE NUMBERS FOR LARGE ECONOMIES

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ABSTRACT

If [0,1] is a measure space of agents and $(X_1)_{1\in[0,1]}$ a collection of pairwise uncorrelated random variables with common finite mean μ and variance σ^2, one would like to establish a law of large numbers (*) $\int X_1 dl = \mu$. In this paper we propose to interpret (*) as a Pettis integral. Using the corresponding Riemann–type version of this integral, we establish (*) and interpret it as an L_2–law of large numbers. Intuitively, the main idea is to integrate before drawing an ω, thus avoiding well–know measurability problems. We discuss distributional properties of i.i.d. random shocks across the population. We given examples for the economic interpretability of our definition. Finally, we establish a vector–valued version of the law of large numbers for economies.

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<u>A law of large numbers for large economies</u>

I. Introduction

In the analysis of economies with a continuum of agents, the following problem often arises. Suppose, each agent has to bear a certain risk. Each agents risk is independent from the identical risk, any other agent faces. Does the risk dissappear upon aggregation?

More concretely, imagine the following insurance arrangement. There is only one good. There is a continuum of identical agents i, each of which will independently receive an endowment $e_1 > 0$ of that good with probability p or an endowment $e_2 > e_1$ with probability 1–p. Their endowment can be publicly observed. Suppose, agents are risk–averse and have utility functions E[ln(c)], say, where E denotes the expectation operator. These agents might decide ex ante to enter a contract of mutual insurance, in which all endowments are collected and each agent is given the expected endowment $c \equiv \bar{e} = pe_1 + (1-p)e_2$. It is clear that agents prefer this arrangement to autarky, but the question is: is this a meaningful and feasible contract? The intuition is, that a law of large numbers should guarantee that we can pay $\bar{e}$ to everybody almost surely. After all, a continuum is "even more" than a sequence, and for sequences, the validity of the law of large numbers is well known.

Examples, in which such a law of large numbers is implicitely or explicitely assumed or used, include Bewley [1], Diamond and Dybvig [4], Green [6], Lucas [10], Marimon [11] and Prescott and Townsend [13].

Stochastic continuum economies with a law of large numbers allow us to lift insurance–type arrangements , where some agent faces a risk–neutral insurance agency, from the microeconomic, partial equilibrium perspective to a macroeconomic, general equilibrium framework: it all adds up. Furthermore, while it might be quite hard to keep track of the rich world of contingent contracts in a finite random economy, many of these contingencies disappear or become trivial in the limit [0,1]–economy: there is no need anymore to write contracts contingent on certain random aggregates. Random [0,1]–economies are often easier to analyze, yet help in understanding the "true" object: a given finite economy. Therefore one needs to know wether a reasonable law of large numbers holds.

Formally the problem can be formulated as follows. Let there be a random variable X_l for each agent $l \in [0,1]$. X_l is defined on a probability space (Ω, Σ, P) and represents e.g. endowment shocks. Suppose, all X_l are independent and identically distributed with finite mean μ and variance σ. One likes to have a theorem (the law of large numbers), which essentially states

(*) $\int X_l dl = \mu.$

Observe, that this is a meaningless expression, as long as we do not have a definition of the integral on the left hand side. One would like such an integral to have the following properties in addition to (*), if possible:

(i) it blends with economic interpretations of stochastic continuum–economies.

(ii) the law of large numbers holds "automatically", i.e. it is not just built into the mathematical construction.

(iii) the law of large numbers also holds on a large class of reasonable subgroups of [0,1].

(iv) the integral is easy to use and known as a mathematical concept.

Justifying the intuition, however, turned out to be more problematic than initially thought. Judd [9] pointed out that severe measurability problems arise when attempting to generalize the strong law of large numbers in a straight–forward manner. Different remedies have been considered. Judd himself suggests extending the measure space in such a way as to make the law of large numbers hold. Thereby he shows that the strong law is not inconsistent with mathematical theory. Unfortunately, it does not "automatically" follow either: it is possible to extend the measure space in such a way that the law of large numbers never holds[1]. Feldman and Gilles [5] consider relaxing the independence condition. For sequences they solve the problem using finitely additive measures or Banach limits. Bewley [1] proposes to use a continuum of agents to draw randomly a sequence of ever–increasing finite economies. Almost surely then, such a sequence will satisfy the sequential law of large numbers. Stutzer [16] uses nonstandard analysis and establishes the law of large numbers for a hyperfinite set of agents. It seems, however, that his result cannot be carried over to models with a continuum of agents.[2]

The main difficulty that these authors encountered arise from interpreting (*) as a strong law of large numbers, i.e. as

$$\int X_l(\omega)dl = \mu \quad P.\text{–a.e..}$$

But if this should hold not only on [0,1], but also on every nondegenerate subinterval, say, one can easily prove (using Radon–Nikodyms theorem) that $X_1(\omega)$ has to be almost surely constant for almost all ω!

For these reasons I propose to interpret (*) as a version of Khinchines law of large numbers, which is weaker than the strong law, but stronger than the weak law of large numbers (see Appendix). My proposed "L_2–law of large numbers" will satisfy the properties (ii) to (iv) stated above, interpreting the integral in (*) as a Pettis–integral. For the simple case described above and to deliver the intuition of an "L_2–law", we can use a Riemann–type version of the Pettis integral.

As for (i), there doesn't seem to be an agreement yet among economists, what we actually think of when analyzing a stochastic continuum economy. Lacking such an agreement, I give three examples to explore the meaning of a law of large numbers. The first example examines the insurance arrangement described above. The important feature is that the difference between the utility of an agent living in some finite economy to the utility of the agent in the continuum economy is small if the finite economy is sufficiently large. Hence, we are not too far off in terms of the welfare of people if we consider a continuum economy instead of a given finite economy.

The paper is organized as follows: in section II, we introduce the Pettis integral and a Riemann–type version of the Pettis integral and prove the law of large numbers. In section III, we discuss distributional properties of the individual shocks, i.e. answer the question, what fraction of the population receives a shock in a given set A.

Section IV gives a vector–valued version of the law of large numbers when the random variables take on values in some Banach space rather than the real line[3]. Section V contains two examples for the application and interpretation of the law of large numbers and one counterexample. Section VI contains some concluding remarks. In the appendix I, we prove relationships between the three sequential laws of large numbers. In appendix II, we examine the relationship between the Riemann–type integral and the Pettis integral and discuss further the relationships between vector–valued integrals, a continuum of random variables and the size of the underlying probability space.

II. The law of large numbers.

We propose to interpret the integral in (*) as a Pettis integral. Recall the following definitions.

Definition 1:

Let X be a Banach space, X' its dual space, (L,Λ,λ) a a finite measure space.

A function $f: L \to X$ is called weakly λ–measurable, if for each $x' \in X'$, the function $x'f$ is λ–measurable.

(Diestel–Uhl [2], Def. II.1.1)

A function $f: L \to X$ is called **Pettis–integrable**, if f is weakly λ–measurable, if $x'f \in L_1(\lambda)$ for all $x' \in X'$ and if for all $E \in \Lambda$, there exists a vector $x_E \in X$, such that

$$x' x_E = \int_E x'f \, d\lambda \text{ for all } x' \in X'.$$

In this case, we define the **Pettis–integral**

$$(P)-\int_E f \, d\lambda := x_E.$$

(Diestel–Uhl [2], Definition II.3.2 or Rudin [15], Def. 3.26)

In what follows, the Banach space X will be the space $L_2(\Omega,\Sigma,P)$ of square integrable random variables over some probability space (Ω,Σ,P). (L,Λ,λ) will be the unit interval [0,1] with its Borel sets and the Lebesgue measure. We call a collection of random variables $(X_l)_{l\in[0,1]}$ pairwise uncorrelated, if $\mathrm{Cov}(X_l,X_s) = 0$ for all $s\neq t$.

Theorem 1: The law of large numbers for a large economy

Let $(X_l)_{l\in[0,1]}$ be a collection of pairwise uncorrelated random variables with common finite mean μ and variance σ^2, defined over some probability space (Ω,Σ,P). Then (X_l) is Pettis–integrable in $L_2(\Omega,\Sigma,P)$ and we have

$$\mu = (P)-\int X_l \, d\lambda.$$

(Observe, that we use somewhat imprecisely the same symbol μ to denote the mean as well as the random variable Y defined by $Y(\omega) = \mu$ for all $\omega \in \Omega$. The integral of (X_l) is really a random variable.).

Proof:

The dual space of L_2 is (naturally isomorphic to) L_2. Thus, let $Z \in L_2$. Z operates on $X \in L_2$ via $Z(X) = E[ZX]$. Thus, we have to show that the function $g(l) = E[Z(X_l)]$ is measurable with respect to λ and that

$$0 = \int E[Z(X_l - \mu)]\, d\lambda.$$

But this is trivial: observe that $E[Z(X_l - \mu)] = 0$ for almost every $l \in L$ since

$$\sum_{j=0}^{\infty} (E[Z(X_{l_j} - \mu)])^2 \leq \mathrm{Var}(Z)\sigma^2$$

for any countable selection of different l_j's by Bessel's inequality.

The Pettis integral is a Lebesgue–type integral for vector valued functions. It has the usual properties that changes on a null set do not matter etc. However, familiar theorems like the dominated convergence theorem are not easily available for the Pettis integral. The other, "nicer" integral for vector valued functions – the Bochner integral – is unfortunately not available in our situation, see Appendix II. Thus, the Pettis integral is the natural choice here.

For most applications, it suffices to use the Riemann–type version of the Pettis–integral. Furthermore the Riemann–type version offers useful insights and provides a powerful tool for computing Pettis–integrals or proving theorems.

As in Calculus, let

$$\Gamma = \{ (n, l_0, l_1, .., l_n, \psi_1, .., \psi_n) \mid n \in \{1,2,..\},$$
$$0 = l_0 < l_1 .. < l_n = 1,\ l_{j-1} \leq \psi_j \leq l_j,\ j=1,..,n \}$$

be the set of all partitions T of the interval [0,1]. For $T \in \Gamma$, we define the mesh

$$\zeta(T) := \max \{ l_j - l_{j-1} \mid j \in \{1,..,n\} \}.$$

In order to define the integral, we need a convergence concept for random variables.

We chose the mean square as a measure of distance. What we want to define is the Riemann–type integral of a vector–valued function:

Definition 2:

Let $(X_l)_{l\in[0,1]}$ be a collection of random variables, defined on the probability space (Ω,Σ,P). If there is a random variable Y, such that

$$\lim_{\zeta(T)\to 0} E[(Y-\sum_{j=1}^{n} X_{\psi_j}(l_j-l_{j-1}))^2] = 0,$$

we write[4]

$$Y = \int X_l \, dl$$

and call Y the integral of $(X_l)_{l\in[0,1]}$.

We call (X_l) **Riemann–type integrable.**[5]

It is possible to prove the following result:

Theorem 2:

Every integrable collection $(X_l)_{l\in[0,1]}$ (in the sense of Definition 2) with $X_l \in L_2(\Omega,\Sigma,P)$ is Pettis–integrable and we have

$$\int X_l dl = (P)\text{–}\int X_l \, d\lambda.$$

The proof is in Appendix II.

Thus the following Corollary comes at no surprise. We provide an independent proof, since it is this proof which gives the intuition that we are dealing with an "L_2–law of large numbers" and since the bounds in the proof are useful in comparing finite economies with continuum economies.

Corollary 1: The law of large numbers for a large economy

Let $(X_l)_{l\in[0,1]}$ be a collection of pairwise uncorrelated random variables with common finite mean μ and variance σ^2. Then (X_l) is Riemann–type integrable and we have

$$\mu = \int X_l \, dl.$$

Proof:

Calculate

$$\begin{aligned}
E[\,(\mu - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1}))^2\,] &= \sum_{j=1}^{n} E[\,((X_{\psi_j} - \mu)(l_j - l_{j-1}))^2\,] \\
&= \sum_{j=1}^{n} (l_j - l_{j-1})^2 \sigma^2 \\
&\le \zeta(T)\sigma^2 \sum_{j=1}^{n} (l_j - l_{j-1}) \\
&= \zeta(T)\sigma^2
\end{aligned}$$

converging to zero as $\zeta(T)$ converges to zero. This completes the proof.

We note for later purposes, that the rate of convergence of the Riemann–sums to the integral is given by $\zeta(T)\sigma^2$.

It is clear, that the proof of Corollary 1 is not "tight", i.e. that we can prove the following improved version:

Proposition 1:

Let $(X_l)_{l\in[0,1]}$ be a stochastic process of pairwise uncorrelated random variables with finite mean μ_l and variance σ_l^2, such that the function $f:[0,1]\to\mathbb{R}$, $f(l) := \mu_l$ is L_1 and the variances are bounded above by some constant M, say.

Then $(X_l)_{l\in[0,1]}$ is integrable with

$$\int \mu_l \, d\lambda = \int X_l \, d\lambda.$$

If f is (bounded and) Riemann integrable, then (X_l) is Riemann–type integrable.

Proof:

The proof for the Pettis integration is completely analoguous to the proof in Theorem I:given $Z \in L_2$, we have $E[ZX_l] = \mu_l E[Z]$ (argue with X_l/σ_l).

For the Riemann–type integrability, let $\mu = \int \mu_l \, dl$ Let $\epsilon > 0$, $\epsilon < 1/2$ be arbitrary and choose $\delta > 0$ small enough, so that for all Partitions T with $\zeta(T) \leq \delta$, we have

$$|\mu - \sum_{j=1}^{n} \mu_{\psi_j}(l_j - l_{j-1})| < \epsilon$$

and $\zeta(T)M < \epsilon/2$.

For these partitions, we now calculate:

$$E[\,(\mu - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1}))^2\,]$$

$$= E[\,(\sum_{j=1}^{n}(X_{\psi_j}-\mu_{\psi_j})(l_j-l_{j-1}) - (\mu-\sum_{j=1}^{n}\mu_{\psi_j}(l_j-l_{j-1})))^2\,]$$

$$= \sum_{j=1}^{n}(l_j-t_{t-1})^2\sigma^2_{\psi_j} + (\mu-\sum_{j=1}^{n}\mu_{\psi_j}(l_j-l_{j-1}))^2$$

$$\leq \zeta(T)M + \epsilon^2$$

$$< \epsilon.$$

Since $0 < \epsilon$ was arbitrary, our claim follows.

It is often useful to apply Theorem 1 via the following, trivial "functional principle": suppose S is some abstract set and $f: S \to \mathbb{R}$ a function. Let $X_l: \Omega \to S$ be mappings associated with the agents $l \in [0,1]$. If $(f(X_l))$ constitutes a collection of measurable, pairwise uncorrelated random variables with common finite mean μ_f and variance σ_f^2, then $\mu_f = \int f(X_l)\, dl$ and $|\mu_f| \leq \sup_{s \in S} |f(s)|$.

III. Distributional properties of the individual shocks.

In our large economy, it is desirable to have a result, in which the fraction of the population experiencing a certain type of shock is equal to the probability of this shock. This statement is as vage as the statement (*) in the beginning, that the average shock equals the expectation of these shocks. We can make this statement precise with the help of our integral from Definition 1. With a finite population $I=\{1,..,n\}$ and a function $f: I \to \mathbb{R}$, the fraction of the population receiving a certain "shock" $f(j) \in A \subset \mathbb{R}$ is $\frac{1}{n}\sum_{j=1}^{n} \chi_A(\, f(j)\,)$, where χ_A is the characteristic function. Thus, the fraction of the population [0,1] experiencing a certain type of shock $A \subset \mathbb{R}$ should be

$$\int \chi_A(X_l)\, dl,$$

where now this integral is an integral in the sense of Definition 1:

Corollary 2:

Let $(X_l)_{l\in[0,1]}$ be a collection of pairwise independent and identically distributed random variables X_l. Let A be a Borel–measurable subset of $\mathbb{R}$. Then

$$\int \chi_A(X_l)\, dl \;=\; P(\, X \in A\,),$$

where X is some random variable with the same distribution function as any X_l.

Proof:

Obviously, $(\chi_A(X_l))$ is a collection of pairwise uncorrelated random variables with mean $\mu = P(\, X \in A\,)$ and finite, constant variance. Hence, the claim follows immediately from Theorem 1 resp. the functional principle with f being the indicator–variable χ_A.

It is easy to find similar conclusions for more general situations, applying e.g. Proposition 1.

IV. A vector–valued version of the law of large numbers.

It is often desirable to let X_l take values in a (possibly infinite dimensional) vector space V rather than just in the real line. The most desirable setup would be as follows. Let V be a Banach space, we denote the norm by $\|\cdot\|$. . We assume that $X_1 \in L_2(P,V)$, the Lebesgue–Bochner space of P–Bochner integrable functions $X:\Omega \to V$ such that $\sigma_X^2 := \| X \|_2 := (\int \|X(\omega)\|^2 dP(\omega))^{1/2} < \infty$ (observe the different meaning of the norms). For Definitions, see appendix II of this paper resp. Diestel–Uhl [2]. Definition 2 now has to be altered appropriately. The most obvious way would be to require that

$$\lim_{\zeta(T)\to 0} E(\|Y - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1})\|^2) = 0.$$

The question then is: is there a version of Corollary 1 or Theorem 1 in this context? This is as yet an unsolved problem.

However, there is a version for vector–valued random variables using weak convergence instead of strong convergence, exploiting the functional principle mentioned in section II. We need the concept of Pettis integration on two levels: we need it to reduce our vector–valued random variables to of real–valued random variables and we need it to find the law of large numbers via integrating a collection of these real–valued random–variables (obtained from a collection of vector–valued random variables). To reduce notation and introduce a few new concepts, we need to following Definitions. These concepts also seem to be useful, when describing limiting concepts via limits of distributions of distributions for finite random economies to continuum random economies.

Definition 3:

Let (Ω,Σ,P) be a probability space. Let V be a locally convex topological vector space and V' its dual space. Let $X : \Omega \to V'$ be a mapping.

a) If for every $v \in V$, $Xv : \Omega \to \mathbb{R}$ is measurable and $Xv \in L_2(\Omega,\Sigma,P)$, we call X a **Pettis*–square–integrable random–vector**. We write

$$\mu_v := E[\, Xv \,]$$

and

$$\sigma_v^2 := \mathrm{Var}[\, Xv \,].$$

(Compare to Definition 1 in section II of this paper).

b) Let X and Y be two Pettis*–square–integrable random vectors. Let $v \in V$. We write

$$\mathrm{Cov}_v(\, X \,,\, Y \,) := \mathrm{Cov}(\, Xv \,,\, Yv \,)$$

and call this the **Pettis* covariance of X and Y in direction v.**

We call X and Y **weak* uncorrelated**, if $\mathrm{Cov}_v(\, X \,,\, Y \,) = 0$ for all $v \in V$.

Definition 4:

Let (Ω,Σ,P) be a probability space. Let V be a locally convex topological vector space and V' its dual space. Let $(X_l)_{l\in[0,1]}$ be a collection of Pettis*–square integrable random vectors $X_l : \Omega \to V'$.

a) (X_l) is called **uncorrelated collection**, if X_l and X_s are weak* uncorrelated for all $s \neq t$.

b) If there is a weak*–measurable mapping $Y : \Omega \to V'$ (i.e., if Yv is measurable for all $v \in V$) such that

$$Yv = (P)\text{–}\int X_l v \, dl \quad \text{for all } v \in V,$$

where the integral is a Pettis–integral in the sense of Definition 3, then we write

$$Y = \int X_l \, dl$$

and call Y the **Pettis*–Pettis–integral of (X_l).**

Theorem 3: A vector–valued law of large numbers for a large economy

Let V be a Banach space and $(X_l)_{l\in[0,1]}$ be an uncorrelated collection $X_l : \Omega \to V'$ with common finite mean μ_v and variance σ_v for all $v \in V$. Assume that

$$|\mu_v| \leq M \cdot \|v\|$$

for some constant M, independent of v.

Then the Pettis*–Pettis–integral of (X_l) exists and is constant φ for some $\varphi\in V'$. We have

$$\varphi = \int X_l \, dl \, (\omega) \quad \text{for P.a.e. } \omega \in \Omega.$$

It also follows that

$$\varphi v = (\int X_l dl)(v) = \int X_l v \, dl = \mu_v = E(X_r v)$$

for all $v\in V$ and $r\in[0,1]$

Proof:

Apply the functional principle to find

$$\int X_l v \, dl = \mu_v = E(X_r v).$$

Define $\varphi(v):=\mu_v$, it remains to show, that $\varphi\in V'$ (the weak*–measurability is then trivial, since our function Y is constant). Observe, that φ is linear by the linearity of the expectation–operator. φ is continuous, since by assumption $|\varphi(v)| \leq M \|v\|$. Thus, our claim is proved.

Of course, in many cases we have to get X artificially into a dual space: if X maps into a Banach space V, apply the theorem to the mapping $\tilde{X}$ which maps into the bidual V'' via the natural embedding of V into V''.

For particular applications, we need the full power of Theorem 3: like normally distributed random variables with real values, random vectors might take values anywhere in the Banach space V' and there is no way around checking the condition, that the means μ_v are uniformly bounded across v. However, in some cases, the random vectors are already bounded themselves, in which case this condition is easy to check. Since we consider this an important special case, we formulate this as Corollary 3.

Corollary 3:

In Theorem 3, replace the condition $|\mu_v| \leq M \cdot \|v\|$ by the condition that for some constant K,

$$\| X_l(\omega) \| \leq K$$

for all $l \in [0,1]$, and all $\omega \in \Omega$. Then the conclusions of Theorem 3 hold, i.e. for some $\varphi \in V'$, we have

$$\varphi = \int X_l \, dl\,(\omega) \quad \text{for P.a.e. } \omega \in \Omega.$$

Proof:

This follows from the inequality mentioned with the functional principle in section I and Theorem 3, since then

$$E(X_1 v) \leq K \|v\|.$$

V. Interpreting the law of large numbers: the "large economy" as an approximation for a "large" but finite economy.

By now, the relationship of nonrandom continuum economies with finite economies is fairly well understood (see e.g Hildenbrand [8] or Mas–Colell [12]). This, however, is not true for random continuum economies of the type we are dealing with in this paper. The biggest obstacle is, that we cannot approximate a continuum of independent and identically distributed random variables (with nonzero, finite variance) through a sequence of finite sets of such random variables in a sensible way: random variables that are independent of each other are just too far apart"[6]. Nonetheless, random continuum economies are appealing because they match certain aspects of a possibly large, but finite economy that we are ultimately interested in analyzing. As of now, there is just not a precise agreement on what this analogy between the finite case and the continuum case is all about. Lacking such an agreement, I provide two examples in which I show how to derive useful insights about some given finite economy, using the "approximate" continuum economy and the proposed integral instead. I also show a counterexample to the claim that results of the finite economies always carry over (compare to similar results in limiting game theory). The examples are meant to be simple and instructive.

Example 1: An insurance scheme

We imagine an economy in which agents engage in risk sharing. More precisely suppose, that I denotes a set of agents ,where I is a finite set or the continuum. Each agent $i \in I$ is endowed with X_i, but X_i depends on the state of nature ω. X_i could e.g. represent net income after a possible car accident, medical costs or other insurable risks. $X_i(\omega)$ can be publicly observed. For simplicity, let X_i be one–dimensional and positive, i.e. $X_i \in \mathbb{R}_+$. We imagine all X_i to be independent and identically distributed with finite mean μ and variance σ^2 according to a distribution function F. Also, we imagine all agents to have the same utility function, i.e. agents maximize expected utility $U(c) = E[\, u(c)\,]$, where consumption c is a random variable taking values in $\mathbb{R}_+$ and u is a utility function, bounded from below, monotone increasing, continuous and concave[7]. Recall $L_2^+(\Omega,\Sigma,P) = \{ f \in L_2(\Omega,\Sigma,P) \mid f(\omega) \geq 0 \text{ P.a.e.} \}$. We need the following

Lemma 1:

a) U is a uniformly continuous function on $L_2^+(\Omega,\Sigma,P)$.

b) Let $c \in L_2^+(\Omega,\Sigma,P)$ and $\mu = E[c]$. Suppose one of the following conditions:

(i) u is differentiable on the range of c with $\sup u'(c(\omega)) \leq M$.

(ii) $\inf c(\omega) = \underline{c} > 0$. Set $M := (u(\underline{c}) - u(0))/\underline{c}$.

Then

$$|U(c) - u(\mu)| \leq M(\mathrm{Var}(c))^{1/2}.$$

Proof:

Assume w.l.o.g. $u(0)=0$. Choose some $\epsilon>0$. Fix some $x>0$ and let $M=u(x)/x$. Fix α at $\alpha:=\epsilon/(2M(x+1))$. Note that u must be equicontinuous, so choose a $\delta>0$ to $\epsilon/2$. Finally note that we can find $\nu>0$, $\nu<1$ so that $c,d \in L_2^+(\Omega,\Sigma,P)$ and $E[(c-d)^2] < \nu$ implies $P(|c-d|\geq\delta) < \alpha$. Distinguish the cases where $|c-d|<\delta$ or otherwise $|c|<x$ resp. $|d|<x$.Use $|u(a)-u(b)|\leq M|a-b|$ for $a,b\geq x$ and the Cauchy–Schwartz inequality to find immediately

$$|U(c) - U(d)| \leq (1-\alpha)\epsilon/2 + \alpha Mx + \alpha M\nu^{1/2} \leq \epsilon.$$

The second claim is an immediate implication of the mean value theorem and the Cauchy–Schwartz inequality, q.e.d..

Now, if the utility fuction u is strictly concave, agents are risk–averse and it is clear that they prefer ex ante to share all risks rather than consuming whatever endowments they get, i.e. they prefer mutual insurance over autarky. Suppose then, that they sign a contract before anybody knows the realization of his or her random endowment, in which they agree to split the total endowment in equal parts. In the case of the finite economy, each agent gets the average

$$c_l = \overline{X} = \frac{1}{n} \sum_{l=1}^{n} X_l$$

of all individual endowments. Observe, that $\overline{X}$ and hence c_l is a random variable.

To relate this with the continuum economy and our integral, we restrict our attention to partitions T where all l_j are equidistant, i.e $l_j=j/n$, $j=0,..n$. We imagine the finite economy as being drawn from the continuum economy in the sense that we select, for each j, some $\psi_j \in (l_{j-1} , l_j]$ and set $X_j := X'_{\psi_j}$, where the latter represents the individual risks X_l for agents $l \in [0,1]$ in the continuum economy.

Now, $\overline{X}$ is nothing but the Riemann sum corresponding to the partition T and we proved in Theorem 1, that

$$E[(\int X_l^{\prime} dl - \overline{X})^2] \leq \sigma^2 \zeta(T) = \sigma^2/n.$$

With the Lemma above, it now follows that for any such finite economy which is sufficiently large, the difference between the expected utility $U(\overline{X})$ in the finite economy and the expected utility $u(\mu) = U(\int X_l^{\prime} dl)$ in the continuum economy is smaller than some given $\epsilon>0$.

This is the kind of result we are interested in: ultimately, we have to deal with one large, but finite economy, and not with a continuum economy or with some sequence of finite economies. Given certain characteristics of this finite economy, we want to make sure that we are not too far off in terms of the welfare of the people if we consider a continuum–model instead. It is an important step in any particular application of the law of large numbers to establish this.[8]

How much then does an agent receive as a result of the insurance in the continuum economy? Regardless of the state of nature ω, the agent receives μ (with probability 1). This contract is feasible, because the Riemann–sums $\overline{X}$ of the finite economies are computed "ω by ω". $\overline{X}$ is close to the random variable $\int X_l^{\prime} dl$ in the L_2–sense if the finite economy is large enough – and that is all we need to guarantee closeness of the utilities.

Observe furthermore, that the results above did not depend on mutual independence of the X_l: as long as the integral $Z = \int X_l' \, dl$ exists in $L_2^+(\Omega,\Sigma,P)$, $\overline{X}$ will approach Z in L_2 (and it will do so at the rate σ^2/n under reasonably weak assumptions about mutual cross–correlations) and thus $U(\overline{X})$ will approach $U(Z)$ at a corresponding rate due to the uniform continuity of U. This can be especially important if there is a high amount of correlation between the individual risks (as an extreme example, consider $X_l' \equiv X$, i.e. every agent faces exactly the same risk!). Our construction of the integral is robust against such variations.

One might consider decentralized versions of this model, in which agents trade in contingency claims. The insurance contracts above are then the outcome of a symmetric equilibrium. Observe that we need claims contingent on the aggregate outcome in the finite economies. There is no need for that in the continuum case: the only contingency relevant for an agent is the variation in his own endowment. The analysis becomes simpler.

Example 2: Infering the distribution of the risk.

The following problem arises e.g. in the analysis of bank–runs (see Drees [3]). Suppose agents line up at a bank counter. In the "good" equilibrium (which is the only one we want to consider here), each agent withdraws money according to their preferences of allocating consumption goods across time without taking into account the possibility of a bank run. Let us assume that the withdrawal X_1 of each agent i is random and realized according to some distribution F with finite variance. However, F is unknown to the agents in the line and for this example, we want to assume that agents want to infere wether $F=F_1$ or $F=F_2$, both of which seem equally probable a priori. Let us assume that F_1 is actually the true distribution. The action of the agents then might depend on their inference and we want to assume for simplicity, that agents get consumption w if they make the correct inference but face an extra loss $d < w$ and thus consume only $w-d$ if they come up with the wrong conclusion.

How do agents arrive at their decision about which distribution to choose? One very simple method (although not the best one for our agent but good enough for our example) works as follows: let b be a real number so that (w.l.o.g.) $F_1(b) < F_2(b)$. Let $a = (F_1(b) + F_2(b))/2$. Now, if k people are before you in the line and exactly j of them withdraw no more than b, calculate the ratio $R = j/k$. If $R \geq a$, settle on F_2 and if $R < a$, choose F_1. Suppose that p is the probability of making the wrong inference.

As in example 1, agents ultimately care about $U(c) = E[\ u(c)\ ]$, where consumption c is now a random variable, taking value w with probability $1-p$ and $w-d$ with probability p. We imagine the line in front of the bank counter to be normalized to unit length. We suppose that for finite economies there will be one agent per 1/nth of the line. The distribution of agents in the continuum case is of course uniform.

Suppose now, an agent is located at position s of the line, say. The ratio R that he calculates is a random variable and it will converge in the L_2-sense to the corresponding "ratio" in the continuum case, i.e. to the constant random variable

$$Q := \frac{1}{s}\int_0^s \chi_{(-\infty,b]}(X_l)\, dl$$
$$\equiv r = F_1(b),$$

using Corollary 1. In the continuum case, our agent is bound to make the correct decision (with probability 1). Moreover, we can find an upper bound for the probability p by calculating

$$p = P(R\geq a) \leq P(\ |R-r|\geq a-r\ )$$
$$\leq Var(R-r)/(a-r)^2$$
$$\leq r(1-r)/((a-r)^2 sn),$$

which converges to zero as n tends to infinity. As p converges to zero at a certain speed, so does the utility of the agent converge to u(w) at the same rate. This is what we ultimately care about. The rate of convergence is the slower, the smaller s is, reflecting the fact that agents early on in the line have to base their inference on less data than agents that have a position which is closer to the end.[9]

What then does an agent in our continuum economy observe, given a state of nature ω? He observes the "ratio"

$$Q(\omega) = \frac{1}{s}\int_0^s \chi_{(-\infty,b]}(X_l)\, dl(\omega) \ = \ r,$$

and a casual look might suggest, that he has to know the outcome of X_l's across all states of nature to compute that. This however is not true: he truly observes the realization of the random variable Q which is the limit of ratios R observed in the finite economies – and these ratios are calculated ω by ω. The limit–concept that we choose is the L_2–convergence which is enough to ensure convergence of the welfare of our agents. It is this link that makes the use of our integral and stochastic continuum economies economically meaningful.

A counterexample

Here we want to take up a variation of example 1 that convergence of the outcome of the finite economies to the outcome of the continuum economy is not always guaranteed, i.e. that the correspondence of allocations achievable by contracts is not upper hemicontinuous.

Suppose we have the same economy as in example 1, but we assume now that the individual realizations of the random variable X_i is private information. Let us assume however, that the aggregate outcome $n\overline{X} = \sum_{i=1}^{n} X_i$ is public information. The transfer payment $c(m_i, \frac{1}{n}\sum_{j \neq i} m_j, \overline{X})$ to (resp. transfer tax on) an individual shall now depend on messages m_i, in which the agent i announces his realization X_i. A contract consists of this function c and (invoking the relevation principle) the truth–telling clause that agents truthfully announce $m_i = X_i$. Such a contract has to obey the incentive compatibility constraint that

$$(IC)\ c(X_i, \frac{1}{n}\sum_{j \neq i} X_j, \overline{X}) > c(\tilde{m}_i, \frac{1}{n}\sum_{j \neq i} X_j, \overline{X}),$$

$$\text{for all } \omega \in \Omega \text{ and all } \tilde{m}_i \in \mathbb{R}.$$

Let us assume that $\min_{\omega} X_j(\omega) > EX_j - \max_{\omega} X_j(\omega) + 2\epsilon$ for all j and some $\epsilon > 0$.

Observe that the contract with c specified by

$$c(x,y,z) = \begin{cases} -\max X_j - \epsilon & \text{if } x/n+y \neq z \\ z-x & \text{if } x/n+y=z \end{cases}$$

works and complete insurance is possible in the truth–telling Nash equilibrium simply since we can always deduce the true realization of X_i from the truthful messages of the other agents.

This contract however will no longer satisfy the incentive compatibility constraint in the continuum economy since the individual outcome is negligible: it always pays for the individual to claim poverty. Autarcy will be the only solution here.

Obviously, the aggregate $\overline{X}$ only reveals information about the individual agents in the finite economy.

VI. Concluding remarks

Our Definitions 1 and 2 give an uncomplicated and straightforward way of interpreting (*). We can easily prove a version of the law of large numbers, using Pettis integration or mean square convergence. It seems upon first sight , however, that this interpretation is not the best one, one would like to have. A strong law of large numbers, interpreting (*) as pathwise integration, would be nicer. However, it is clear (due to the measurability problems mentioned in the introduction), that one cannot hope for a sensible strong law which e.g. also works on subintervals. The examples in section V indicate that this might not be a big loss, especially since the use of the mean square metric enables us to compare results in large finite economies to continuum economies. We find e.g. that expected utility changes continuously with the mean square of the random consumption, so that we do not make a big error in terms of welfare of the people, if we consider the continuum econmy and use our law of large numbers instead of analyzing a (sufficiently big) finite economy. Even if a strong law were available, it probably would not add any improvement here.

Further research has to be done on a better understanding of the properties of the integral proposed in Definition 1 and the economic relevance of the law of large numbers derived from it. It is clear, that the link between large, but finite stochastic economies and our models of stochastic economies with a continuum of agents is not very well understood at this point (see Section V). The question of the "validity" of a certain interpretation of the integral (*) for a specific problem crucially depends on this link resp. on the interpretation of large, stochastic economies for the "real world".

Appendix I

Let $(X_n)_{n=1}^{\infty}$ be a stochastic process of i.i.d random variables on the probability space (Ω,Σ,P) with finite mean μ and variance σ^2. The well known strong law of large numbers (Kolmogorov's Theorem) states, that

$$\frac{1}{N}\sum_{n=1}^{N} X_n \to \mu \quad P\text{–a.e.},$$

i.e. except on a set of measure zero, the sample average converges to the mean. The weak law of large numbers (Chebyshev's Theorem) states, that

$$\text{for every } \epsilon > 0,\ P(\{\omega \mid |(\frac{1}{N}\sum_{n=1}^{N} X_n)(\omega) - \mu| > \epsilon\} \to 0,$$

i.e. the sample average converges to the mean in probability. In between lie the ℓ_p–laws of large numbers (Khinchine's Theorem) $(1 \leq p < \infty)$:

$$E[\ |\frac{1}{N}\sum_{n=1}^{N} X_n - \mu|^p\ ] \to 0.$$

Of special importance is the ℓ_2–law of large numbers, stating that the variance of the sample averages converges to zero. It is easy to prove, that these ℓ_p–laws imply the weak law. They are implied by the strong law, using Lebesgues dominated convergence theorem, as the following proposition shows:

Proposition:

Let $1 \leq p < \infty$. Let (X_n) be a sequence of random variables with finite means μ_n . Suppose , that for some $r > p$ and all positive integers n, $E[|X_n - \mu_n|^r]$ exists and that for some M and all positive integers n,

$$E[|X_n - \mu_n|^r] \leq M.$$

Let

$$S_N := \frac{1}{N} \sum_{n=1}^{N} (X_n - \mu_n).$$

Suppose, that the "strong law of large numbers" holds, i.e. that

$$S_N \to 0 \text{ P-a.e.}.$$

Then the ℓ_p-law of large numbers holds, i.e.

$$E[|S_N|^p] \to 0.$$

Proof:

Recall that for any set E, the characteristic function χ_E is defined as 1 for all $x \in E$ and as 0 for all $x \notin E$. Let $\epsilon > 0$ be arbitrary. Let q be such that $1/q + p/r = 1$. First calculate, that by Minkovsky's inequality,

$$E[|S_N|^r] \leq \left(\frac{1}{N} \sum_{n=1}^{N} (E[|X_n - \mu_n|^r])^{1/r} \right)^r$$

$$\leq M.$$

Hence by the general Chebyshev-inequality, we can find a (large) positive number K, such that

$$P(\{|S_N| > K\}) \leq (\epsilon \cdot M^{-p/r})^q \text{ for all N.}$$

Let $Q_N := S_N \cdot \chi_{\{|S_N| \leq K\}}$ and $R_N := S_N - Q_N = S_N \cdot \chi_{\{|S_N| > K\}}$. Now observe, that

$$E[|S_N|^p] = E[|R_N|^p] + E[|Q_N|^p],$$

for the sets $\{|S_N| \leq K\}$ and $\{|S_N| \leq K\}$ are disjoint. Since $Q_N \to 0$ P.a.e. by assumption and since $|Q_N|^p \leq K^p$, it follows immediately from the theorem of Lebesgue on dominated convergence, that

$$E[|Q_N|^p] \to 0.$$

From Hoelders inequality, it follows, that

$$\begin{aligned} E[|R_N|^p] &= (E[(|S_N| \cdot \chi_{\{|S_N|>K\}})^p]) \\ &\leq E[|S_N|^r]^{p/r} \cdot E[\chi_{\{|S_N|>K\}}^{pq}]^{1/q} \\ &= E[|S_N|^r]^{p/r} \cdot P(\{|S_N|>K\})^{1/q} \leq \epsilon. \end{aligned}$$

Since $\epsilon>0$ was arbitrary, our claim follows.

Appendix II

Relationships between the Pettis integral, the Riemann type integral and the Bochner–integral.

We want to prove Theorem 2 from above, that the Riemann–type integral corresponds in a measure–theoretic setting to a (vector–valued) Pettis integral. Recall the following Definitions (see Diestel–Uhl [2])

Definition 5:

Let X be a Banach space, X' its dual space, (L,Λ,λ) a finite measure space.

A function $f:L\to X$ is called **simple**, if there exist $x_1,x_2,\dots,x_n \in X$ and $E_1,E_2,\dots,E_n \in \Lambda$, such that $f=\sum_{t=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}(t) = 1$, if $l \in E_i$ and $\chi_{E_i}(t) = 0$, if $t \notin E_i$ (χ_{E_i} is called the **characteristic function** of E_i).

A function $f:L\to X$ is called weakly λ–measurable, if for each $x' \in X'$, the function $x'f$ is λ–measurable.

A function $f:L\to X$ is called λ–measurable, if there exists a sequence (f_n) of simple functions with $\lim_{n\to\infty} \|f_n - f\| = 0$ λ–almost everywhere.

(Diestel–Uhl [2], Def. II.1.1)

Definition 6:

Let X be a Banach space and (L,Λ,λ) a finite measure space.

A function $f:L\to X$ is called **Pettis–integrable**, if f is weakly λ–measurable, if $x'f \in L_1(\lambda)$ for all $x' \in X'$ and if for all $E \in \Lambda$, there exists a vector $x_E \in X$, such that

$$x'x_E = \int_E x'f \, d\lambda \text{ for all } x' \in X'.$$

In this case, we define the **Pettis–integral**

$$(P)\!-\!\int_E f \, d\lambda := x_E.$$

(Diestel–Uhl [2], Definition II.3.2 or Rudin [15], Def. 3.26)

A function $f:L\to X$ is called **Bochner–integrable**, if f is λ–measurable and if there exists a sequence of simple functions (f_n), such that

$$\lim_n \int_T \| f_n - f \| \, d\lambda = 0.$$

In this case, the **Bochner–integral** $\int_E f \, d\lambda$ is defined for each $E \in \Sigma$ by

$$\int_E f \, d\lambda = \lim_n \int_E f_n \, d\lambda,$$

where $\int_E f_n \, d\lambda$ is defined in the obvious way.

(Diestel–Uhl [2], Definition II.2.1).

In our context, X is the space $L_2(\Omega,\Sigma,P)$ of random variables with finite variance, where (Ω,Σ,P) is a probability space (different from (L,Λ,λ), in general). In the sequel we shall refer to Riemann integrals of real valued functions only when they are computed "directly", i.e. without taking limits for certain points within the interval over which we integrate. This implies, that we only Riemann–integrate bounded functions.

Lemma 2:

An integrable collection $(X_l)_{l\in[0,1]}$ (i.e. in the sense of Definition 1) with $X_l \in L_2(\Omega,\Sigma,P)$ is weakly λ–measurable, where λ is the Lebesgue–measure on the Borel–sets of [0,1]. For every $Z\in L_2(\Omega,\Sigma,P)$, the function $f_Z:[0,1]\to\mathbb{R}$, $f_Z(t):= E(ZX_l)$ is Riemann–integrable and we have

$$\int f_Z(t)\,dl = E(ZY),$$

where $Y = \int X_l dl$.

More suggestively, we can write the formula above as

$$\int E(ZX_l)dl = E(Z\int X_l dl).$$

Observe the different meanings of the integrals.

Proof:

The dual space of $L_2(\Omega,\Sigma,P)$ is (canonically isomorphic to) $L_2(\Omega,\Sigma,P)$ itself. Therefore, let $Z\in L_2(\Omega,\Sigma,P)\cong L_2(\Omega,\Sigma,P)'$. The duality is given by $Z(X) = E[ZX]$ for $X\in L_2(\Omega,\Sigma,P)$. Consider a partition $T\in\Gamma$. Then

$$|E[ZY] - \sum_{j=1}^{n} f_Z(\psi_j)(l_j-l_{j-1})| = |E(Z(Y - \sum_{j=1}^{n} X_{\psi_j}(l_j-l_{j-1})))|$$

$$\leq E(Z^2)^{1/2} E((Y - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1}))^2)^{1/2}$$

(with Cauchy–Schwarz), which converges to zero, as $\zeta(T) \to 0$ by definition of our integral. This proves, that f_Z is Riemann–integrable with $\int f(l)\, dl = E(ZY)$. Since Riemann–integrable functions are measurable, we have proved the weak λ–measurability of (X_l).

Theorem 2:

Every Riemann–integrable collection $(X_l)_{l \in [0,1]}$ (in the sense of Definition 2) with $X_l \in L_2(\Omega, \Sigma, P)$ is Pettis–integrable and we have

$$\int X_l dl = (P) - \int f\, d\lambda,$$

where $f:[0,1] \to L_2(\Omega, \Sigma, P)$ is defined by $f(l) := X_l$.

Proof:

It is clear, that $\int X_l dl \in L_2(\Omega, \Sigma, P)$ (argue with an approximating partition).

Recall, that every Riemann–integrable function on $[0,1]$ must be Lebesgue–integrable, since it is bounded and is the a.e.–pointwise limit of step–functions. Hence, for every $Z \in L_2(\Omega, \Sigma, P)'$, we have $f_Z \in L_1(\lambda)$, using Lemma 2. Since furthermore by Lemma 2, our function f is weakly λ–measurable, an application of Lemma II.3.1 in Diestel–Uhl [2] together with the reflexivity of $L_2(\Omega, \Sigma, P)$ proves the claim. The equality of the Pettis–integral with our integral is clear with Lemma 2, q.e.d..

The Pettis–integral is not a very "powerful" (but a rather general) vector–integral.

Some of the facts known about the Pettis–integrals are:

- If f is Pettis–integrable, then $\rho(E) := (P)\text{–}\int_E f\,d\lambda$ is a countably additive λ–continuous vector measure on Λ. (Diestel–Uhl [2], Theorem II.3.5).
- Furthermore, the set $\{\rho(E) \mid E \in \Lambda\}$ is a relatively weakly compact subset of the bidual space X''(Diestel–Uhl [2], Corollary II.3.9).

It has not been possible to uncover much additional information about the Pettis–integral beyond the results established in the original paper by Pettis (Diestel–Uhl [2], II.5).

The Bochner–integral however, possesses all the nice properties, we are accustomed to from Real Analysis (e.g. the Theorem on dominated convergence holds, see Diestel–Uhl [2], II.2). It would therefore be nice, if one could Bochner–integrate our integrable stochastic processes[10]. This is possible for a certain class of stochastic processes, as the next Theorem shows.

Theorem 4:

Let $(X_l)_{l\in[0,1]}$ be a collection of random variables $X_l \in L_2(\Omega,\Sigma,P)$ such that the function $f : [0,1] \to L_2(\Omega,\Sigma,P)$, $f(l) := X_l$ is continuous.

Then (X_l) is integrable, f is Bochner–integrable and we have

$$\int X_l dl = \int f\,d\lambda.$$

Proof:

We show the Bochner–integrability first. The λ–measurability of f follows immediately from Lemma 2 and Pettis's measurability theorem, Theorem II.1.2 in

Diestel–Uhl [2], observing that f has a separable range, since it is a continuous function on [0,1]. As sequence of simple functions f_n approximating our function f according to the definition of the Bochner–integral, consider

$$f_n := X_{1/n}\chi_{[0,1/n]} + \sum_{j=2}^{n} X_{j/n}\chi_{((j-1)/n,j/n]},$$

i.e. f_n belongs to the partion

$$T_n=(n,0,1/n,2/n,\dots,1,1/n,2/n,\dots,1).$$

For any $\delta > 0$, define

$$w_\delta := \sup\{ E(X_t-X_s)^2)) \mid |s-t|\le \delta \}.$$

We find

$$\int\|f-f_n\|\, d\lambda = \sum_{j=1}^{n} \int_{[(j-1)/n,\, j/n]} E((X_t-X_{j/n})^2)^{1/2} d\lambda(t)$$

$$\le (w_{1/n})^{1/2},$$

converging to 0, as $n\to\infty$. Hence, f is Bochner–integrable. Let $Y := \lim_{n\to\infty} \int f_n\, d\lambda$.

For the integrability in the sense of our Definition 2, we repeat the same exercise, but now for functions $g_T = \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1})$, T a partition. We find $\int\|f-g_T\|\, d\lambda \le (w_{\zeta(T)})^{1/2}$ and it is clear, that for every $\epsilon>0$, we can find n sufficiently large and $\delta>0$ sufficiently small, so that for all $\zeta(T)<\delta$, we have

$$\begin{aligned}(E((Y-g_T)^2))^{1/2} &= \|Y-g_T\| \\ &\le \|Y-\int f_n d\lambda\| + \int\|f_n-f\|d\lambda + \int\|f-g_T\|d\lambda \\ &\le \epsilon.\end{aligned}$$

This proves the integrability (in the sense of Def. 2) and the equality of this integral to Y, the Bochner–integral.

The Theorem above seems promising at first. However, it is not possible to generalize it to the case of pairwise uncorrelated random variables, with which we started out: (The following Proposition is essentially a reminder of well–known facts)

Proposition 2:

Let $(X_l)_{l\in[0,1]}$ be a collection of pairwise uncorrelated random variables $X_l \in L_2(\Omega,\Sigma,P)$. Let there be a $\delta>0$, such that $Var(X_l)\geq\delta$ for all $l\in[0,1]$.

Then the function $f:[0,1]\to L_2(\Omega,\Sigma,P)$ is not λ–measurable and hence not Bochner–integrable.

Ω is not a separable metric space.

Proof:

Calculate, that for $l\neq s$,

$$\begin{aligned} \|X_l - X_s\| &\geq (Var(X_l - X_s))^{1/2} \\ &= (\ Var(X_l) + Var(X_s)\ )^{1/2} \\ &\geq (2\delta)^{1/2} =: d, \end{aligned}$$

i.e. two different random variables of our stochastic process have at least distance d. But then for any uncountable subset A of [0,1], the set $\{X_l \mid l\in A\}$ cannot be separable in $L_2(\Omega,\Sigma,P)$. Hence, using Pettis's measurability theorem (Theorem II.1.2 in Diestel–Uhl [2]), f cannot be λ–measurable.

Furthermore, the same theorem implies, that $L_2(\Omega,\Sigma,P)$ cannot be separable.

Hence Ω cannot be a separable metric space.

The Proposition above might shed some further light on the difficulty of proving laws of large numbers in large economies: quite in contrast to probability spaces "sufficient" for a sequence of independent and identically distributed random variables, we have to choose much "bigger" probability spaces in the case of large economies!

Footnotes:

(1) The other problem with this approach is, that the law still doesn't hold on a large class of subgroups of agents. Green(1987), however, showed how to get around that problem to a certain extent.

(2) On a more technical level, observe that not every hyperfinite sequence of random variables F_j, j in the hyperfinite unit interval grid T, gives rise to a collection of random variables X_l, $l \in [0,1]$ by the rule $X_l = F_{o_j}$ in a well-defined way (consider the example, where F_j is identically 1 for every nonstandard odd number $j*L$ and identically 0 else. This is an internal object and therefore "limit" of standard, "real" objects). But then the hyperfinite sequences, that do carry over, might be an ill-behaved set and we are back in our original problem.

(3) As a byproduct, covariances for infinite dimensional random variables are defined.

(4) Observe that we could have chosen convergence P–a.e. instead of L_2–convergence. We would then essentially be back at a formulation for the strong law and encounter all the problems mentioned in the introduction.

(5) Notice the similarity to the Ito–integral which is also defined by using the L_2–distance. Let W_l be the Brownian motion on [0,1] and X_l its "derivative", "white noise", which we understand as independent and standard normally distributed. Let Z be a random variable with standard normal distribution. Then the Ito–integral and some heuristics yields $\int X_l \, dl = \int dW_l = Z$. The reason that the result differs from ours is that other weights are used in the definition of the Ito–integral. We have

$$\lim_{\zeta(T)\to 0} E[\ (Z - \sum_{j=1}^{n} X_{\psi_j}(l_j - l_{j-1})^{1/2}\ )^2\ ] = 0,$$

i.e in the Ito–integral we use $(l_j - l_{j-1})^{1/2}$ instead of $(l_j - l_{j-1})$. We conjecture that this result relates to our Theorem 1 like the central limit theorem to the law of large numbers for sequences.

(6) Bewley [1] thus suggested drawing such sequences at random from the continuum economy and analyzing the continuum via these sequences. This

approach then leads him to his version of the law of large numbers mentioned in the introduction.

(7) This of course excludes the favorite $u(c) = \ln(c)$. But it is clear that one can obtain a version similar to Lemma 1 even for this utility function in most applications: usually, it is possible to restrict the relevant set of random variables in $L_2^+(\Omega,\Sigma,P)$ to e.g. the set $\{ f \in L_2^+(\Omega,\Sigma,P) \mid f \geq c_{min} \}$, where $c_{min} > 0$ is given a priori or to a bigger set of random variables that don't put too much mass close to zero. If a restriction like that is not possible, one might want to be cautious to use the continuum model!

(8) This is actually a problem in Bewleys approach [1]. While for any randomly selected sequence of finite economies, the randomness disappears in the limit, we have no idea how fast this happens. Furthermore, the convergence speed depends on the (random) choice of the sequence.

(9) Again, I don't see, how Bewley's approach [1] can deliver this type of analysis.

(10) It is an interesting question, how the Bochner integral relates to pathwise measurability of the process (X_l).

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