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A MODEL OF COMMODITY MONEY

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A Model of Commodity Money

This paper describes and partially analyzes a set of models designed to help answer the following questions about commodity money systems. Are commodity money systems generally inefficient? Does the degree of inefficiency depend on what commodity ends up being the commodity money? Under what circumstances do quantity-theory-of-money like observations turn up under commodity systems? Are interest rates determined by the return on the commodity money, or can it happen that commodity money is driven out because it cannot compete with interest-bearing inside money? Does one commodity emerge as a natural commodity money because of its depreciation rate and technological conditions of production? Can a government choose the commodity or combination of commodities that is to serve as the commodity money? Does a commodity money system limit seignorage, thereby disciplining fiscal policy? Finally, what happens if restrictions are placed on privately-issued inside money, as has often been the case historically?

The models we use to approach these questions are essentially growth models with a single consumption good (at each date) and with one or several capital goods (at each date), and with technological features of production designed to make the capital goods resemble commodity monies. People in our model consist of members of two-period lived overlapping generations who care only about their own consumption. Finally, aside from explicit legal restrictions, we put nothing into the models to distinguish a priori the demands for assets that one may wish to identify as monies from the demands for other assets.

The rest of the paper proceeds as follows. In Section 1, we describe the physical environment of our model. In Section 2, we describe conditions for a perfect foresight competitive equilibrium--henceforth,

called, simply, an equilibrium--under laissez-faire. We also describe the equilibrium in some simple cases. In Section 3, we use those descriptions to answer some of the questions listed above. In Section 4, we consider governmental choice of the commodity standard. Then, in Section 5, we examine the role of legal restrictions that limit the degree to which private debts can compete with the commodity money and the resulting potential intermediary role of a central bank. Finally, in the concluding section, we comment on some issues we have not addressed, including differences between our modelling approach and that of others.

1. The Model

In this section, we describe a one consumption good, one capital good version of the model. We call the consumption good "bread"--bread at t , bread at $t + 1$, and so on--and we call the capital good "gold"--gold at t , gold at $t + 1$, and so on. The model uses discrete time and the current or initial date is labeled $t = 1$. We now describe, in turn, total resources, technology, and people and their preferences.

Total Resources

At each date, there is exogenously available to the economy $Y(t)$ units of time t bread. We also assume that the economy starts at $t = 1$ with $Z(1)$ units of time 1 gold. This is the only exogenously available gold. Thus, while the economy gets a new endowment of bread each period, it does not get a new endowment of gold.

Technology

There are two aspects of the technology; a static aspect which involves possibilities for converting time t bread into time t gold and vice versa, and an intertemporal aspect which involves possibilities for converting time t gold into time $t + 1$ gold. We assume that there is no direct technology for converting time t bread into time $t + 1$ bread.

Figure 1 describes the static features of the technology.

[Insert Figure 1]

In Figure 1, $Z(t)$ represents the amount of gold the economy starts with at date t . This amount is determined by actions taken at $t - 1$ and earlier in a way we will describe below. The segment of the frontier to the right of B represents the possibility of devoting some of the bread resource, $Y(t)$, to the production of additional gold. The parameter λ_{bg} (the subscripts are meant to connote "bread into gold") has the units, ounces of time t gold (output) per ounce of time t bread (input). The segment of the frontier to the left of B represents the possibility of devoting some of the gold resource, $Z(t)$, to the production of additional bread. This is meant to represent an "industrial use" for gold. Given that we assume that only bread appears as an argument of utility functions (gold is not consumed directly), the existence of such a use for gold is critical and is what justifies calling this object a commodity or capital good. The parameter λ_{gb} ("gold into bread") has the units ounces of time t bread (output) per ounce of time t gold (input). We assume $0 < \lambda_{gb} \leq 1/\lambda_{bg}$.

We assume a very simple intertemporal technology. As already noted, we assume that time t bread cannot be directly converted into time $t + 1$

bread. As for gold, for any $k \geq 0$, k units of time t gold can be converted into $(1-\delta)k$ units of time $t + 1$ gold. In other words, gold depreciates geometrically at the rate δ per period. Thus, letting $ZP(t)$ be net gold production at t , the amount of gold produced at t minus the amount of gold used as input into the production of bread at t , the starting stock at t and net gold production at t are related to the starting stock at $t + 1$ according to $Z(t+1) = (1-\delta)[Z(t) + ZP(t)]$.

Population and Preferences

The population consists of two-period lived overlapping generations. Each member h of generation t (young at t , old at $t + 1$), $t \geq 1$, has a utility function $u_t^h(c_t^h(t), c_t^h(t+1))$, where $c_t^h(s)$ is the consumption of time s bread by h in generation t . We assume that u_t^h is twice continuously differentiable, increasing, and strictly quasi-concave. Later on, we consider additional assumptions imposed implicitly by way of assumptions on aggregate saving. Each member of generation 0, the old at the first date, wants to maximize his or her own consumption of time 1 bread.

2. Equilibrium Under Laissez-Faire

As regards individual endowments, we distinguish between the members of generation 0 and everyone else. We assume that the members of generation 0 own among them $Z(1)$, the starting stock of gold. They also have an endowment of time 1 bread. Everyone else has endowments of bread only; member h of generation t , $t \geq 1$, has an endowment consisting of some time t bread, $w_t^h(t) > 0$, and some time $t + 1$ bread, $w_t^h(t+1) > 0$. We assume that the individual endowments exhaust $Y(t)$.

Roughly speaking, an equilibrium consists of sequences of quantities and prices with the property that when individuals take the relevant (current

and future) prices parametrically, the quantities are utility maximizing and clear markets. We now proceed to derive equilibrium conditions.

First, letting $p(t)$ be the price of time t gold in terms of time t bread, we obviously have (see Figure 1)

Proposition 1: In any equilibrium, $\lambda_{gb} \leq p(t) \leq 1/\lambda_{bg}$. Moreover, if $ZP(t) > 0$, then $p(t) = 1/\lambda_{bg}$; while if $ZP(t) < 0$, then $p(t) = \lambda_{gb}$. (We omit the proofs of this and other obvious propositions. Other proofs, as noted, appear in the Appendix.)

We now describe the choice problem of a member of generation t , $t > 1$, but only for prices that satisfy Proposition 1. This allows us to ignore, for the moment, static production possibilities, since at such prices they do not enhance consumption opportunities. We do, however, explicitly introduce the possibility of storing gold and two markets in loans: loans stated in terms of bread at t , $l^h(t)$, with gross payoff in bread at $t + 1$, denoted $r(t)l^h(t)$; and loans stated in terms of time t gold, $l_g^h(t)$, with gross payoff in gold at $t + 1$, denoted $i(t)l_g^h(t)$. (Think of $r(t)$ as the gross real rate of interest and of $i(t)$ as the gross nominal rate of interest.) Then consumption of bread at t and $t + 1$ for h is related to loans granted and gold stored by

$$(1) \quad c_t^h(t) \leq w_t^h(t) - l^h(t) - p(t)l_g^h(t) - p(t)z^h(t+1)/(1-\delta)$$

$$(2) \quad c_t^h(t+1) \leq w_t^h(t+1) + r(t)l^h(t) + p(t+1)i(t)l_g^h(t) + p(t+1)z^h(t+1)$$

Here, $z^h(t+1)$ is the output from the gold h stores at t . For each such unit, he or she must have stored $1/(1-\delta)$ units at t , where δ is the physical depreciation rate. The aggregate $Z(t+1)$ is simply the sum of the $z^h(t+1)$'s, the sum being taken over the members of generation t .

Since $\ell^h(t)$ can be positive (lending by h) or negative (borrowing by h), (1) and (2) are equivalent in terms of consumption bundles to the following single constraint obtained by solving (1) for $\ell^h(t)$ and substituting the result into (2):

$$(3) \quad c_t^h(t) + c_t^h(t+1)/r(t) \leq w_t^h(t) + w_t^h(t+1)/r(t) \\ + \ell_g^h(t) [p(t+1)i(t)/r(t) - p(t)] + z^h(t+1) [p(t+1)/r(t) - p(t)/(1-\delta)]$$

Since utility is increasing in consumption and since consumption is bounded by resources in equilibrium, it follows that in equilibrium, the right-hand side of (3) must be bounded. Therefore, in any equilibrium, $i(t) = r(t)p(t)/p(t+1)$, $r(t) > (1-\delta)p(t+1)/p(t)$ and with equality if $z^h(t+1) > 0$, and

$$(4) \quad c_t^h(t) + c_t^h(t+1)/r(t) \leq w_t^h(t) + w_t^h(t+1)/r(t).$$

In other words, utility maximizing consumption choices in equilibrium are constrained by (4).

Let $s_t^h[r(t)]$ (s for saving) be the solution for $w_t^h(t) - c_t^h(t)$ to the problem, maximize u_t^h subject to (4). It follows that s is a bounded (above), continuous function which has a unique zero. We will describe equilibrium conditions in terms of the aggregate saving function for generation t , $S_t[r(t)] = \sum s_t^h[r(t)]$, the sum being taken over the members of generation t . From now on, we make the convenient, but strong assumption that S_t is strictly increasing.

The above results allow us to work with the following definition of an equilibrium.

Definition: An equilibrium consists of nonnegative sequences for $r(t)$, $p(t)$ and $Z(t+1)$ and a sequence for $ZP(t)$ that for all $t > 1$ satisfy

(i) $S_t[r(t)] = p(t)Z(t+1)/(1-\delta);$

(ii) $r(t) \geq p(t+1)(1-\delta)/p(t)$ and with equality if $Z(t+1) > 0;$

(iii) $Z(t+1) = (1-\delta)[Z(t) + ZP(t)];$

(iv) $\lambda_{gb} \leq p(t) \leq 1/\lambda_{bg}$ with

$$p(t) = \begin{cases} \lambda_{gb} & \text{if } ZP(t) < 0 \\ 1/\lambda_{bg} & \text{if } ZP(t) > 0 \end{cases}$$

We now describe the equilibrium for some special cases.

An Unchanging Monotone Saving Function and No Depreciation

Here we assume $S_t[r(t)] = S[r(t)]$ for all t and $\delta = 0$. Let r^* be such that $S(r^*) = 0$.

Proposition 2. If $r^* > 1$, then there exists an equilibrium with $r(t) = r^*$, $p(t) = \lambda_{gb}$, and $Z(t+1) = 0$ for all $t \geq 1$. If $r^* < 1$, then there exists an equilibrium with $r(t) = 1$, $p(t) = p$ and $Z(t+1) = S(1)/p$ for all $t \geq 1$ where

$$p = \begin{cases} \lambda_{gb} & \text{if } Z(1) \geq S(1)/\lambda_{gb} \\ S(1)/Z(1) & \text{if } S(1)/\lambda_{gb} > Z(1) > S(1)\lambda_{bg} \\ 1/\lambda_{bg} & \text{if } Z(1) \leq S(1)\lambda_{bg} \end{cases}$$

That this is an equilibrium is immediate from the definition given above. Note that all of the endogenous variables are constant over time and that in the case $r^* < 1$, the size of the starting stock of gold, $Z(1)$, determines $p(t)$.

It turns out, with monotonicity of S , that the Proposition 2 equilibrium is unique in all significant respects. We state this as

Proposition 3: Under the assumptions of Proposition 2, there is a unique equilibrium $r(t)$ sequence. (For a proof, see the Appendix.)

Only as regards $p(t)$ is the Proposition 2 equilibrium not unique and, then, only in the case $r^* > 1$. In that case, if $Z(1) > 0$, then $p(1) = \lambda_{gb}$ in order that $ZP(1) = -Z(1)$. However, other terms of the $p(t)$ sequence are somewhat free. While (ii) and (iv) must be satisfied, these do not determine a unique $p(t)$ sequence when $r^* > 1$.

An Unchanging Saving Function and Depreciation

Depreciation changes matters in several respects. First, if the amount of gold held is not to approach zero, then gold must be produced. That, in turn, requires that its price be at the upper bound, $1/\lambda_{bg}$. At this upper bound price, the rate of return on gold is $1 - \delta$. Hence, with depreciation, the crucial value of r^* is $1 - \delta$. Finally, if $r^* < 1 - \delta$, then although $p(t)$ must eventually be such as to permit gold to be produced, it need not start out that high. If $Z(1)$ is sufficiently large, then the price starts out below $1/\lambda_{bg}$. The description of the equilibrium in this case is somewhat complicated, and we leave the details to the Appendix. There we give a more or less constructive proof of the existence of equilibrium, a proof of the following proposition.

Proposition 4. If $r^* > 1 - \delta$, then there exists an equilibrium with $r(t) = r^*$, $p(t) = \lambda_{gb}$, and $Z(t+1) = 0$ for all $t > 1$. If $r^* < 1 - \delta$, then there exists an equilibrium with $r(t) = 1 - \delta$, $p(t) = 1/\lambda_{bg}$, and $Z(t+1)/(1 - \delta) = \lambda_{bg}S(1 - \delta)$ for all $t > T$; where $1 < T < \bar{T}$, \bar{T} is the smallest positive integer j that satisfies

$$(5) \quad (1 - \delta)^{j-1}Z(1) < \lambda_{bg}S(1 - \delta),$$

$1 > r(1) > r(2) > \dots > r(T-1) > r(T)$, and $p(1) < p(2) < \dots < p(T-1) < p(T)$.

Note that if $r^* < 1 - \delta$, then desired saving at $r(t) = 1 - \delta$ and $p(t) = 1/\lambda_{bg}$ is $\lambda_{bg}S(1-\delta)$ in units of gold. If $Z(1)$ does not exceed this quantity, then $\bar{T} = 1$ (see the definition of \bar{T} and equation (5)) and, hence, $T = 1$. If, however, $Z(1)$ exceeds this quantity, then the price starts out lower and increases in such a way that $r(t)$ decreases. The monotonicity of S plays a role in our proof that there exists an equilibrium with these properties. It also gives us uniqueness subject only to the kind of proviso about $p(t)$ noted above.

Proposition 5: Under the assumptions of Proposition 4, there is a unique equilibrium $r(t)$ sequence. (For a proof, see the Appendix.)

Note, of course, that uniqueness of the $r(t)$ sequence implies uniqueness of the $Z(t+1)$ sequence and of the consumption allocation.

Growth and Two Limiting Cases

Here we discuss briefly simple proportional growth and two special static production cases: $\lambda_{bg} = 0$ (gold cannot be produced) and $\lambda_{gb} = 1/\lambda_{bg}$ (constant costs).

As regards growth, suppose that for all $t > 1$ the aggregate saving function for generation $t + 1$ is a positive constant, n , times the aggregate saving function of generation t . (This happens if the number of type i members of generation $t + 1$ is n times the number of type i members of generation t for all types i and for all $t > 1$, where a type is a description of preferences and endowments.) Then with all other assumptions intact, there are obvious analogues of Propositions 2 and 4.

If $r^* > 1 - \delta$, then the claims made in Proposition 4 hold. If $r^* < 1 - \delta$, then a comparison between n and $1 - \delta$ is relevant. If $n = 1 - \delta$, then the description of Proposition 2 applies; if $n > 1 - \delta$, then that of Proposi-

tion 4 applies; finally, if $n < 1 - \delta$, then the price of gold must eventually be at its lower bound.

If gold cannot be produced, $\lambda_{bg} = 0$, then Propositions 2 and 4 do not apply because the price can rise indefinitely. This permits a very different equilibrium to occur.

Proposition 6: If $n > 1 - \delta$, $S(n) > 0$, and $\lambda_{bg} = 0$; then there exists a unique equilibrium with $r(t) = n$ for all t .

In such an equilibrium, the depreciation rate plays no role.

The special case $\lambda_{gb} = 1/\lambda_{bg}$ is covered by Propositions 2 and 4. It merits comment only as being particularly simple. It is also, however, the case discussed by many writers, some of whom assert that it prevails only in the "long run." One way to capture what was meant by "long run" is to assume a production technology with adjustment costs in a model with uncertainty. We do not study such setups here.

3. Features of the Laissez-Faire Equilibrium

In this section, we use the above results to discuss some of the questions listed at the outset of this paper.

Inefficiency of Commodity Money

Since bread is the only consumption good in our model, the standard definition of inefficiency applied to our model takes the following simple form: a sequence $C(t)$, $t \geq 1$, where $C(t)$ is total consumption of time t bread, is inefficient if there exists a feasible alternative, $\hat{C}(t)$, with $\hat{C}(t) \geq C(t)$ for all $t \geq 1$ and with strict inequality for some $t \geq 1$. It is immediate from this definition that many of the equilibria described above are inefficient; the inefficient ones are those in which a commodity money is held and $n \geq 1 - \delta$.

Thus, for example, consider the Proposition 2 equilibrium in the case $r^* < 1$. This equilibrium satisfies $C(1) < Y(1) + \lambda_{gb} Z(1)$, $C(t) = Y(t)$ for all $t \geq 2$. It can be dominated by the feasible path, $\hat{C}(1) = Y(1) + \lambda_{gb} Z(1)$, $\hat{C}(t) = Y(t)$, $t \geq 2$.

Not surprisingly, the inefficiency is "worse" with depreciation. Thus, a Proposition 4 equilibrium with $r^* < 1 - \delta$ satisfies $C(1) < Y(1) + \lambda_{gb} Z(1)$; $C(t) \leq Y(t)$; $t = 2, 3, \dots, T$; and $C(t) < Y(t)$, $t > T$. This can be dominated by the same $\hat{C}(t)$ path used to dominate the Proposition 2 equilibrium.

In addition to these inefficiency results, there is a kind of paradox relating the degree of inefficiency to the productivity of the bread-into-gold production technology. In general, the more productive is that technology--i.e., the larger is λ_{bg} --the "worse" the equilibrium. Thus, consider two economies, A and B, identical in all respects except for λ_{bg} ; $\lambda_{bg}^A > \lambda_{bg}^B$. It follows that the corresponding laissez-faire equilibria satisfy $C^A(t) \leq C^B(t)$ for all $t \geq 1$. Moreover, it is easy to construct examples, with or without depreciation, in which some of the inequalities are strict. Of course, this ceases to be paradoxical once one recognizes that any production of the commodity money is inefficient and that less tends to be produced the more costly it is in terms of the consumption good required to produce it.

The above inefficiencies are "capital theoretic;" they are instances of capital overaccumulation. It is well-known that the no-last-period feature of our model is necessary for producing such outcomes. If there were a last period, a last generation, then under our assumptions, any competitive equilibrium is both efficient and Pareto optimal. Thus, while our model provides an interpretation of the usual claim that commodity money is inefficient--why use resources to mine gold in order to bury it under Fort Knox?--one may wonder whether there are not more plausible interpretations.

An alternative interpretation must rely, somehow, on the notion that the commodity money is a relatively low return asset--relative, that is, to other available assets. However, if one is discussing a laissez-faire equilibrium, then in order to use that notion, one must explain why one of the higher return assets does not take over as the commodity money. We do not think it is easy to provide such an explanation.

The Quantity Theory of Money

Is there any sense in which our model implies proportionality between the price level and the quantity of money and the neutrality of money? One way to consider this is to compare the equilibria for two economies that are physically identical except as regards starting stocks of gold, $Z(1)$'s.

One case in which we get a quantity theory-like outcome from this comparison is in the Proposition 2 setup (or, slightly more generally, $n = 1 - \delta$) with $r^* < 1$ and with $Z(1)$ for both economies in the intermediate range, i.e., in the interval $[S(1)\lambda_{bg}, S(1)/\lambda_{gb}]$. Then the model is compatible with the following remarks of Samuelson:

Given physical amounts of tobacco, food, ballet, etc., have significance in terms of the want pattern of the consumer, but it is not possible to attach similar significance to a given number of physical units of money, say to a number of ounces of gold. It would be otherwise in the case of gold which was to be used to fill teeth, but such uses of gold in the industrial arts we purposely neglect. The amount of money which is needed depends upon the work that is to be done, which in turn depends upon the prices of all goods in terms of gold. (Foundations of Economic Analysis, pp. 118-119.)

Obviously, though, the quantity theory result is not robust in models like ours. It disappears, for example, if our static production assumptions are replaced by those that would produce a smooth production possibilities curve in Figure 1. That being so, either our model is defective or

one should not be concerned if one's model of commodity money does not give quantity theory-like results. Which attitude one adopts depends, in part, on how pervasive are quantity theory-like observations. The most often cited such observations concern large discoveries of the precious metals. However, to us it is not clear that these have been accompanied by anything other than ordinary price theory-like outcomes according to which it is not surprising if a large discovery of "x" is accompanied by a large price decrease in x, whether x is gold, oil, cotton, or Picassos. In order to argue that a special theory is required to explain the observations, one would have to show that they regularly come "close" to satisfying proportionality; we agree that outcomes satisfying proportionality almost exactly should be "rare" according to ordinary price theory models.

Interest Rates and Outside and Inside Money

As regards interest rates and portfolios in equilibrium, our models give rise to two general possibilities. Before describing these, we want first to note that our equilibria very generally contain private securities, loans issued by members of generation t who are sufficiently--relative to other members of generation t --heavily endowed with time $t + 1$ bread relative to time t bread and/or have a sufficiently strong preference for time t bread. At time t , these loans are safe titles to bread or gold at $t + 1$ and, hence, are "as good" as gold. If, therefore, we regard gold holdings as commodity money, and, hence, as outside money, then we ought to regard the loans as inside money, if only because they and gold are perfect substitutes from the point of view of potential holders when both are held.

One possible pattern of equilibrium portfolios is that only private loans are held; that is, there is only inside money. This occurs when $r^* > 1 - \delta$. The other general possibility is that both loans and gold are held. In

the first case, the gross real interest rate, $r(t)$, can be higher than any possible equilibrium return on the commodity money. In the second case, it is equal to that on the commodity money. These possibilities match almost completely two possibilities Samuelson described in the following passage:

It is true that in a world involving no transaction friction and no uncertainty, there would be no reason for a spread between the yield on any two assets, and hence there would be no difference in the yield on money and on securities. Hicks concludes, therefore, that securities will not bear interest but will accommodate themselves to the yield on money. It is equally possible and more illuminating to suppose that under these conditions money adjusts itself to the yield of securities. In fact, in such a world securities themselves would circulate as money and be acceptable in transactions; demand bank deposits would bear interest, just as they often did in this country in the period of the twenties. And if money could not make the adjustment, as in the case of metal counters which Aristotle tells us are barren, it would pass out of use, wither away and die, become a free good. (Foundations of Economic Analysis, p. 123.)

In the case $r^* > 1 - \delta$, our gold does not become a free good, but it does pass out of use.

It seems clear from this passage and a subsequent one that Samuelson thought that the relevant case empirically is $r^* > 1 - \delta$, the case of a pure credit economy. But even so, it is not clear why he goes on to say

Of course, the above [disappearance of outside money] does not happen in real life, precisely because uncertainty, contingency needs, non-synchronization of revenues and outlay, transaction frictions, etc., etc., all are with us. But the abstract special case [of disappearance] analyzed above should warn us against the facile assumption that the average levels of the structure of interest rates are determined solely or primarily by these differential factors. (Op cit., p. 124.)

If interest rates in this context mean the spread between the yield on securities and the yield on currency and if eliminating "these differential factors"

eliminates the spread (either by eliminating outside money or by eliminating the spread), then it seems that an observed spread should be attributed to these factors.

We can also use the outside-inside money interpretation of gold and loans to comment on Friedman's estimate of the resource costs of a commodity standard. He writes:

. . . The amount is by no means negligible--for example, under a pure commodity standard, the United States would at present be devoting about 2 1/2% of its national product or about \$8 billion a year to produce directly or indirectly through foreign trade additional amounts of the monetary commodity to add to the amounts already in circulation or in warehouses. (A Program for Monetary Stability, p. 5.)

As described in the accompanying footnote, this estimate is the amount required to provide for growth in currency plus demand and time deposits at commercial banks (M_2) assuming that this total is replaced by holdings of a commodity, that is, by holdings of outside money. Such a replacement is what Friedman means by a "pure" commodity standard. But "pure" must be quite different than laissez-faire. Under a laissez-faire commodity standard, it is likely that at least as much of the total money supply would be inside money as under the current standard. This would lead to a much lower estimate of the cost of providing for growth under a commodity standard.

Multiple Commodity Monies

We now describe a class of economies which are identical with the previous ones, except that an additional storable good called silver is also potentially available. The economy starts out with a stock $Z_s(1) \geq 0$ of silver at $t = 1$, all in the hands of those who are old at $t = 1$. Silver is characterized by a technology analogous to that governing gold, one that is

described by two parameters λ_{bs} and λ_{sb} , respectively, which give the terms on which bread can be converted into silver, and on which silver can be converted into bread. In particular, bread can be converted at constant returns to scale into silver at the rate of λ_{bs} units of silver per unit of bread; silver can be converted into bread at constant returns to scale at the rate of λ_{sb} units of bread per unit of silver. We assume that $\lambda_{sb} \leq 1/\lambda_{bs}$. Silver and gold cannot be converted directly into one another. Silver depreciates geometrically at the rate δ_s per period, $0 \leq \delta_s < 1$, while gold depreciates at the rate δ_g per period, $0 \leq \delta_g < 1$. We now let $Z_g(t)$ be the stock of gold held at the beginning of period t ; $Z_s(t)$ the stock of silver at the beginning of period t ; $ZP_g(t)$ and $ZP_s(t)$ the production of gold and silver, respectively, during period t ; and $p_g(t)$ and $p_s(t)$ the prices of gold and silver, respectively, in terms of bread.

We define an equilibrium for this economy as follows.

Definition: An equilibrium consists of nonnegative sequences for $r(t)$, $p_g(t)$, $p_s(t)$, $Z_g(t+1)$, and $Z_s(t+1)$ and sequences for $ZP_s(t)$ and $ZP_g(t)$ for $t \geq 1$ that satisfy:

- (i) $S_t[r(t)] = p_g(t)Z_g(t+1)/(1-\delta_g) + p_s(t)Z_s(t+1)/(1-\delta_s)$;
- (ii) $r(t) \geq p_g(t+1)(1-\delta_g)/p_g(t)$ with equality if $Z_g(t+1) > 0$;
- (iii) $r(t) \geq p_s(t+1)(1-\delta_s)/p_s(t)$ with equality if $Z_s(t+1) > 0$;
- (iv) $Z_g(t+1) = (1-\delta_g)[Z_g(t)+ZP_g(t)]$;
- (v) $Z_s(t+1) = (1-\delta_s)[Z_s(t)+ZP_s(t)]$;
- (vi) $\lambda_{gb} \leq p_g(t) \leq 1/\lambda_{bg}$ with

$$p_g(t) = \begin{cases} \lambda_{gb} & \text{if } ZP_g(t) < 0 \\ 1/\lambda_{bg} & \text{if } ZP_g(t) > 0 \end{cases}$$

(vii) $\lambda_{sb} \leq p_s(t) \leq 1/\lambda_{bs}$ with

$$p_s(t) = \begin{cases} \lambda_{sb} & \text{if } ZP_s(t) < 0 \\ 1/\lambda_{bs} & \text{if } ZP_s(t) > 0 \end{cases}$$

We shall consider the case of proportional growth in which the saving function satisfies $S_t(r) = n^{t-1}S_1(r)$, $t \geq 1$, and $n > 1$. As above we let r^* be the value that satisfies $S_1(r^*) = 0$.

Proposition 7 (Unequal Depreciation Rates): Assume that $\delta_g < \delta_s$. There exists a unique equilibrium. If $r^* > 1 - \delta_g$, then $Z_g(t+1) = Z_s(t+1) = 0$ for all $t \geq 1$. If $r^* < 1 - \delta_g$, then $r(t) = 1 - \delta_g$, $p_g(t) = 1/\lambda_{bg}$, and $Z_g(t+1)/(1 - \delta_g) = \lambda_{bg}S_1(1 - \delta_g)n^{t-1}$ for all $t > T$, where T is a finite integer (determined in a manner analogous to that described in Proposition 4), and $p_s(t) = \lambda_{sb}$, $Z_s(t+1) = 0$ for all $t \geq 1$. For $t \leq T$, returns and prices of gold obey $n > r(1) > r(2) > \dots > r(T-1) > r(T)$ and $p_g(1) < p_g(2) < \dots < p_g(T-1) < p_g(T)$.

Proposition 7 states that the commodity with the lower rate of depreciation is stored. Thus, it provides one example in which "the market" selects one commodity to act as outside money. Proposition 10 will provide another example, one not based on different depreciation rates.

Preparatory to discussing government interventions designed to influence the choice of standard, from now on we assume equal depreciation rates. To obtain the following proposition, we assume $0 < \lambda_{gb} < 1/\lambda_{bg}$ and $0 < \lambda_{sb} < 1/\lambda_{bs}$. We also assume that $Z_s(1)/\lambda_{bs} = Z_g(1)/\lambda_{bg}$ and $\lambda_{gb}\lambda_{bg} = \lambda_{sb}\lambda_{bs}$, although these latter conditions can be weakened, as will be described below.

Proposition 8 (Equal Depreciation Rates): Assume that $\delta_g = \delta_s = \delta$. If $r^* > 1 - \delta$, then there exists a unique equilibrium with $r(t) = r^*$, $p_g(t) = \lambda_{gb}$, $p_s(t) = \lambda_{sb}$, $Z_g(t+1) = Z_s(t+1) = 0$ for all $t > 1$. If $r^* < 1 - \delta$, then there exist multiple equilibria, each one having $r(t) = 1 - \delta$ for $t > T$, where T is a finite number that is in each case determined in a manner analogous to that described in Proposition 4. We can categorize these equilibria as follows:

(a) An equilibrium with $p_g(t) = 1/\lambda_{bg}$ and $Z_g(t+1)/(1-\delta) = \lambda_{bg}S_1(1-\delta)n^{t-1}$ for $t > T$; $p_s(t) = \lambda_{sb}$ and $Z_s(t+1) = 0$ for all $t > 1$; and $n > r(1) > \dots > r(T)$ and $p_g(1) < p_g(2) < \dots < p_g(T)$.

(b) An equilibrium symmetric to equilibrium (a) with the roles of gold and silver reversed.

(c) A class of equilibria with $p_s(t) = 1/\lambda_{bs}$, $p_g(t) = 1/\lambda_{bg}$ with $Z_s(t+1) > 0$ and $Z_g(t+1) > 0$ satisfying $S_1(1-\delta)n^{t-1} = Z_g(t+1)/(1-\delta)\lambda_{bg} + Z_s(t+1)/(1-\delta)\lambda_{bs}$ for $t > T$, with $n > r(1) > \dots > r(T)$, $p_g(1) < p_g(2) < \dots < p_g(T)$ and $p_s(t)/p_g(t) = \lambda_{bg}/\lambda_{bs}$ for all $t > 1$. Notice that $Z_s(t+1)$ and $Z_g(t+1)$ are not uniquely determined for $t > T$.

(d) A class of equilibria with $p_s(t) = 1/\lambda_{bs}$, $p_g(t) = p_g$, $t > T$, where p_g is a constant that satisfies $\lambda_{gb} < p_g < 1/\lambda_{bg}$ with $Z_g(t+1) = (1-\delta_g)Z_g(t)$ for all $t > 1$, $n > r(1) > \dots > r(T)$, $p_s(1) < p_s(2) < \dots < p_s(T)$, and $p_s(t)/p_g(t) = 1/p_g\lambda_{bs}$ for all $t > 1$, with $Z_g(t+1)$ satisfying (a) with the above values of $Z_g(t+1)$, $p_g(t)$, and $p_s(t)$ for $t > 1$.

(e) A class of equilibria symmetric to class (d), with the roles of silver and gold reversed.

If $r^* < 1 - \delta$, equilibria of types (a) and (b) both exist, even if we abandon the conditions $Z_s(1)/\lambda_{bs} = Z_g(1)/\lambda_{bg}$ and $\lambda_{gb}\lambda_{bg} = \lambda_{sb}\lambda_{bs}$. These conditions are sufficient, although not necessary, to permit equilibria of types (c), (d), and (e) to exist. The role of these conditions is to permit the bread prices of both metals to rise in proportion in each period to T , so that each metal can bear the same rate of return exceeding $1 - \delta$ prior to T . To illustrate how such conditions are needed, suppose that $Z_g(1) > (1-\delta)S_1(1-\delta)$, that $\lambda_{gb}\lambda_{bg} < 1$, and that $\lambda_{sb}\lambda_{bs} = 1$. In this case, equilibria of types (c), (d), and (e) do not exist because the technology does not permit the price of silver to move over time, as is required in such equilibria for some initial period.

Any two equilibria from among types (a)-(e), with $T > 1$ for one of them, differ with respect to their prices of gold and silver, the rates of return, and the consumption allocations assigned to generations born before dates at which the steady state has commenced for each of the equilibria. However, the consumption allocations assigned to generations born at dates after both of any two such equilibria have attained their steady states are identical.

The range of these indeterminacies in the Proposition 8 economy gets much smaller when we restrict the technologies for gold and silver to be perfectly reversible. We state this in the following proposition, which is really about an economy that is a special limiting case of a Proposition 8 economy.

Proposition 9 (Reversible Technologies): Suppose that $\lambda_{bs}\lambda_{sb} = \lambda_{bg}\lambda_{gb} = 1$.

If $r^* < 1 - \delta$, then there exists a continuum of equilibria, each one having $r(t) = 1 - \delta$, $p_g(t) = \lambda_{gb}$, $p_s(t) = \lambda_{sb}$ for $t > 1$; and with any nonnegative sequences $Z_g(t+1)$, $Z_s(t+1)$ that solve

$$S_1(1-\delta)n^{t-1} = \lambda_{gb} Z_g(t+1)/(1-\delta) + \lambda_{sb} Z_s(t+1)/(1-\delta) \text{ for } t > 1.$$

In the Proposition 9 equilibrium, the prices and rates of return are determinate, only the quantities of gold and silver stored being indeterminate when $r^* < 1 - \delta$. The consumption allocations are identical in all Proposition 9 equilibria with $r^* < 1 - \delta$, including the consumption of the current old at $t = 1$. The Proposition 9 economy thus fits Friedman's description:

"The commodity in question might be gold or silver or copper or bricks or some combination of these or of other goods in fixed proportions, as under any of the variety of symmetrical or commodity reserve standards that have been proposed The maintenance of a commodity standard requires the use of real resources to produce additional amounts of the monetary commodity--of men and other resources to dig gold or silver or copper out of the ground or to produce whatever other commodities constitute the standard. In a stationary economy, production is needed solely to make good losses through wear and tear; in a growing economy, also to provide for an increase in the stock of money. Interestingly enough, the amount of resources required to provide for growth does not depend on the commodity or commodities used as the standard, but only on the cash balance preferences of the public and on the rate of growth of the economy." (A Program for Monetary Stability, pp. 4, 5.)

We now describe a variation on the above economy in which one of the metals is getting cheaper to produce through time. We assume that $\lambda_{gb} = 1/\lambda_{bg}$ for all $t > 1$, but that the technology for silver is changing over time in the following way. We assume that at time t , bread can be converted into silver at constant returns to scale at a rate of $\lambda_{bs}(t)$ units of silver per unit of bread, and that the technology is perfectly reversible so that $\lambda_{sb}(t) = 1/\lambda_{bs}(t)$. We also assume that silver gets cheaper to produce over time so that $\lambda_{bs}(t+1) > \lambda_{bs}(t)$ for all $t > 1$. For this economy, we have:

Proposition 10 (A Time Varying Technology for Silver): Suppose that $r^* < 1 - \delta$. Then there exists a unique equilibrium with $p_g(t) = 1/\lambda_{bg}$, $p_s(t) = 1/\lambda_{bs}(t)$, $Z_s(t+1) = 0$ and $Z_g(t+1)/(1-\delta) = \lambda_{bg}S_1(1-\delta)n^{t-1}$ for $t > 1$.

In the Proposition 10 economy, the metal that is getting cheaper to produce over time (silver) is not produced or stored because it is dominated in rate of return by the metal that is not getting cheaper to produce (gold).

We now return to the case in which the technology is constant through time. The next proposition considers an economy that is another limiting case of a Proposition 8 economy.

Proposition 11 (Gold Cannot be Produced): Suppose that $0 < \lambda_{bs} = 1/\lambda_{sb}$, but that $0 = \lambda_{bg} < 1/\lambda_{gb}$. For convenience, we also assume that $Z_g(1) < S_1(1-\delta)/\lambda_{gb}$ so that the initial gold stock is not too big. Suppose that $r^* < 1 - \delta$. Then there exist the following three classes of equilibria:

(a) (Gold-and-silver equilibria) A continuum of equilibria with $r(t) = 1 - \delta$, $p_s(t) = 1/\lambda_{bs}$, and $p_g(t) = p_g$ where p_g is a constant that obeys $\lambda_{gb} < p_g \leq S_1(1-\delta)/Z_g(1)$, for $t > 1$; $Z_g(t+1) = (1-\delta)^t Z_g(1)$ for $t > 1$; $Z_s(t+1) = \lambda_{bs}[(1-\delta)n^{t-1}S_1(1-\delta) - p_g Z_g(1)(1-\delta)^t]$

(b) (A silver-only equilibrium) A unique equilibrium in which $p_g(t) = \lambda_{gb}$, $p_s(t) = 1/\lambda_{bs}$, $Z_g(t+1) = 0$ and $Z_s(t+1) = \lambda_{bs}S_1(1-\delta)n^{t-1}$ for all $t > 1$.

(c) (A gold-only equilibrium) A unique equilibrium with $r(t) = n$, $p_s(t) = 1/\lambda_{bs}$, $Z_s(t+1) = 0$, $Z_g(t+1) = (1-\delta)^t Z_g(1)$ for $t > 1$, and $p_g(t) = S_1(n)n^{t-1}/Z_g(1)(1-\delta)^{t-1}$.

Equilibria among classes (a) and (b) can differ with respect to the consumption allocated to the old at $t = 1$, since this can be affected by the initial prices of gold and silver. Otherwise, equilibria of types (a) and (b)

lead to identical consumption allocations for all generations born from time 1 onward. The type (c) equilibrium results in different allocations for all generations, as compared to both type (a) and (b) equilibria. In it, the rate of return on holding gold equals n .

4. Government Choice of a Commodity Standard

The class of models that we are using is shot through with propositions of a Modigliani-Miller kind, in which agents see through various institutions and, having the technologies to do so, costlessly undo their effects. This property of the models poses problems when it comes to considering senses in which the government can be said to choose or to influence the choice of a commodity money. The nature of these problems can be illustrated by considering two possible government measures which might be thought to influence the choice of commodity: administering a government mint, and choosing the commodity in which to denominate government debt. Setting up a government mint to stamp coins costlessly, even assigning the government mint a monopoly, in itself has no effects in our models, since private agents can costlessly assay and melt down coins. Private agents can easily be imagined to render innocuous government decisions to stamp differing amounts of gold, silver, and other metals as "one dollar." For example, if the government freely coins gold and silver dollars at a ratio of 16:1, and requires that all stored metals be stamped "dollars," the equilibria of the models of Propositions 7-11 are unaffected, since there is nothing to prevent gold dollars from exchanging for silver at a discount or a premium. Similarly, if fiscal policy is held constant in a natural sense, the equilibria of the Propositions 8-11 economies, suitably modified to include a role for government finance, would be unaffected by a decision to denominate the government debt in silver as opposed to gold.

These properties of our model reveal the need to impose supplementary legal restrictions in order to render the government capable of influencing the choice of commodity standard. We have found that without imposing such legal restrictions, it is difficult either to imagine a role for the government in choosing a standard commodity, or to make sense of "Gresham's Law," which seems to hinge on imputing to the government a significant role in influencing the choice of standard.

Partly to illustrate how stringent such restrictions must be in our economies, this section describes two hypothetical systems of government interventions that are directed at influencing the choice of the commodity to be used as the standard. The systems of intervention that we shall analyze are particular abstract versions of bimetallism and symmetallism. The plan of this section is to describe in turn each of these systems of intervention, together with its effects on some of the different economies that were described in Propositions 8-11. Throughout this section, we deal with identical depreciation rates, and furthermore, for simplicity, with depreciation rates equal zero.

Bimetallism

We suppose that the government imposes a legal restriction that states that while evidences of private indebtedness (consumption loans) can be held, neither gold nor silver can be stored directly. Instead, the government requires that only (intrinsically useless) paper certificates called dollars that it alone issues and that are "backed" in a particular way by gold or silver can be stored. The government sets up a printing press, which operates costlessly, and which issues dollars according to the following rules. It offers freely to issue dollars on the basis of silver that is brought to it at the price of d_s dollars per unit of silver, and freely to issue dollars for

gold at the price of d_g dollars per unit of gold. The government stores the gold and silver that are brought to it. Furthermore, the government offers to redeem dollar certificates for a metal of the government's own choosing, giving $1/d_s$ units of silver per dollar or $1/d_g$ units of gold per dollar. The government's choice is to pay in gold if $p_g(t)/p_s(t) < d_g/d_s$ and in silver otherwise.

The preceding set of government regulations is economically equivalent with one in which the government simultaneously sets up a mint, offers freely to coin or stamp silver dollars at a price of d_s and freely to coin gold dollars at a price of d_g , and costlessly enforces a legal restriction that in all private trades a "dollar is a dollar," so that gold and silver dollars are always required to circulate at par.^{1/} Despite their economic equivalence, we prefer to think of the scheme in the terms described in the preceding paragraph. Under both versions of the scheme, it will be seen that in some of the economies of Propositions 8-11, but not in others, there are strong incentives for avoiding the restriction that "a dollar is a dollar."

Bimetallism in an Equal-Depreciation-Rate (Proposition 8) Economy

We consider the Proposition 8 economy with $\delta_g = \delta_s = 0$ and $r^* < 1$. As we have seen, this economy has a continuum of equilibria, one that can be indexed by price ratios p_g/p_s in the closed interval $[\lambda_{gb}\lambda_{bs}, 1/\lambda_{bg}\lambda_{sb}]$. Any constant ratio in this interval is an equilibrium price ratio, and corresponds to one of the equilibria described in Proposition 8. In the context of this economy, the institution of bimetallism works as follows. If the government

^{1/}Milton Friedman and Anna Schwartz (1963) [fn, p. 27] point to the need for some device to fix the market rate of exchange between two monies if Gresham's law is to apply.

sets the "mint ratio" d_g/d_s in the interval $[\lambda_{gb}\lambda_{bs}, 1/\lambda_{bg}\lambda_{sb}]$, the effect is to select one from among the continuum of equilibria, namely the one that satisfies $d_g/d_s = p_g/p_s$. The particular choice of d_g/d_s affects consumption allocations of those born before T, and in some versions T itself, in the manner described above. In equilibria in which p_g/p_s is set in the open interval $(\lambda_{gb}\lambda_{bs}, 1/\lambda_{bg}\lambda_{sb})$, both gold and silver are used as "backing," so there is a sense in which bimetallism works.

If the government sets $d_g/d_s > 1/\lambda_{bg}\lambda_{sb}$, the effect is to select equilibrium (a), while if the government sets $d_g/d_s < \lambda_{gb}\lambda_{bs}$, the effect is to select equilibrium (b). There is a sense in which these cases exhibit a version of Gresham's law. In the former case, gold is "overvalued at the mint" causing it alone to be taken to the mint. In the latter case, silver is overvalued at the mint, and it alone is taken to the mint.

In this economy, in general, the government has at least some latitude for setting the price of gold relative to silver, p_g/p_s . There is no uniquely determined "market price ratio" from which the government can depart only at the cost of driving one metal or the other out of use as dollars. However, a special case of the Proposition 8 economy evidently does have this property, namely the Proposition 9 economy in which both gold and silver technologies are perfectly reversible. In this economy, if $d_g/d_s > \lambda_{gb}\lambda_{bs}$, then only gold is brought to the mint, while if $d_g/d_s < \lambda_{gb}\lambda_{bs}$, then only silver is brought to the mint. In this way, the choice of d_g/d_s resolves the indeterminacy about which metals are stored in the Proposition 9 economy. However, the institution of bimetallism as we have specified it leaves all of the real aspects of the economy unaffected by the choice of d_g/d_s . The choice of d_g/d_s only affects whether the "dollar" represents gold or silver "certificates."

Bimetallism in an Economy Where Silver Gets Cheaper to Produce (Proposition 10)

In the Proposition 10 economy, we assume that d_g and d_s are constant over time and set in the following way. We assume that $d_g/d_s > p_g(1)/p_s(1) = \lambda_{gb}\lambda_{bs}(1)$ so that initially silver is undervalued at the mint, implying that only gold is taken to the mint and stored. We assume that there is a finite $T > 1$ such that $d_g/d_s > \lambda_{gb}\lambda_{bs}(t)$ for $t < T$ and $d_g/d_s < \lambda_{gb}\lambda_{bs}(t)$ for $t > T$. For $t < T$, only gold is taken to the mint and stored there. For such t , the institution of bimetallism is innocuous since agents would freely choose to store gold anyway since it dominates silver in rate of return. For $t > T$, only silver is brought to the mint, since from T on it becomes overvalued at the mint. For $t > T$, the institution of bimetallism makes a difference, since some lenders would like to store gold, but are prevented from doing so by the legal restriction requiring that only minted dollar certificates be stored. The gross rate of returns on loans and dollars for $t > T$ becomes $r(t) = \lambda_{bs}(t)/\lambda_{bs}(t+1)$. The equilibrium condition determining the amount of silver minted and stored for $t > T$ is $n^{t-1}S_1(\lambda_{bs}(t)/\lambda_{bs}(t+1)) = \lambda_{bs}(t)[Z_s(t+1) + Z_g(T)]$, where $Z_g(T)$ is the amount of gold stored by the government from $(T-1)$ to T .^{2/}

We note that borrowers are better off after silver displaces gold, since both loans and dollars bear a lower rate of return than when gold is the standard. There is an inflation after "silver drives out gold." Thus, after $\lambda_{bs}(t)$ has risen enough by date T to make silver cheap enough to function as dollars, there is a force which previously had not been present for lenders

^{2/}This assumes that $n^{t-1}S_1(\lambda_{bs}(T)/\lambda_{bs}(T+1)) > \lambda_{bs}(T)Z_g(T)$ and that $n^{t-1}S_1(\lambda_{bs}(t)/\lambda_{bs}(t+1)) > n^{t-2}S_1(\lambda_{bs}(t-1)/\lambda_{bs}(t))$ for $t > T + 1$, so that there is a motive to coin more dollars from T onward.

and holders of dollars to evade or agitate against the institutions that operate to depress the rate of return on assets.

Bimetallism in an Economy Where Gold Cannot Be Produced (Proposition 11)

In the economy of Proposition 11, the government is free to set p_g/p_s at any value in the interval $[\lambda_{gb}\lambda_{bs}, S_1(1)/Z_g(1)]$. This will determine a unique equilibrium from among the continuum described in Proposition 11. In those equilibria for which $\lambda_{gb}\lambda_{bs} < d_g/d_s < \lambda_{bs}S_1(1)/Z_g(1)$ all of the initial gold stock is used as dollars, and additional silver is taken to the mint at a rate determined by the equation $S_t(1) = p_g Z_g(1) + \lambda_{sb} Z_s(t+1)$. If the government sets $d_g/d_s > \lambda_{bs} S_1(1)/Z_g(1)$, then the effect is to select the "gold only" (c) equilibrium of Proposition 11.

This economy shares with the Proposition 8 economy that the government is free to preserve bimetallism while setting d_g/d_s within some interval governed by the range of indeterminacy inherited from the technology. Also, a version of Gresham's law obtains in the sense that an attempt to set d_g/d_s too high moves the economy to a "gold only" equilibrium. However, in this economy, the "gold only" equilibrium differs from the other equilibria in allocations, rates of return, and economic efficiency.

Symmetallism

Under symmetallism, the government again has a monopoly on issuing dollars. Besides evidences of private indebtedness, only dollars can be stored. The government defines a dollar as consisting of γ units of gold and σ units of silver. The government stands ready to issue freely a dollar certificate to anyone who brings it γ units of gold and σ units of silver. The government costlessly stores the gold and silver. The government is willing to redeem dollar certificates for $1/\gamma$ units of gold plus $1/\sigma$ units of silver.

We let $p_d(t)$ be the bread price of a dollar at time t , measured in units of bread per dollar. In equilibrium, we must have

$$(6) \quad p_d(t) = \gamma p_g(t) + \sigma p_s(t).$$

We let $D(t)$ be the total stock of dollars at the beginning of t , which obeys

$$(7) \quad D(t) = Z_g(t)/\gamma = Z_s(t)/\sigma.$$

An equilibrium under symmetallism with dollars being stored satisfies the following version of (i),

$$(8) \quad S_t(p_d(t+1)/p_d(t)) = p_d(t)D(t+1)$$

Symmetallism in an Equal-Depreciation Rate (Proposition 8) Economy

For the Proposition 8 economy, with $n > 1$, there is a unique equilibrium of type (c). In this case, there is a unique price ratio $p_s(t)/p_g(t) = \lambda_{bg}/\lambda_{bs}$ which is determined by the technology and the need eventually to accommodate growth.

Symmetallism in an Economy (Proposition 10) Where Silver Gets Cheaper to Produce

In the Proposition 10 economy, there is a unique equilibrium, in which the equilibrium bread price of dollars obeys $p_d(t) = \gamma/\lambda_{bg} + \sigma/\lambda_{bs}(t)$ for all $t \geq 1$. The equilibrium quantity of dollars is determined by

$$S_t[(\gamma/\lambda_{bg} + \sigma/\lambda_{bs}(t+1))/(\gamma/\lambda_{bg} + \sigma/\lambda_{bs}(t))] = D(t+1)/(\gamma/\lambda_{bg} + \sigma/\lambda_{bs}(t))$$

with $Z_g(t) = \gamma D(t)$ and $Z_s(t) = \sigma D(t)$. The equilibrium rate of return in this economy is between that which emerges in the gold-only equilibrium under laissez-faire and that which emerges eventually under bimetallism in the silver-only equilibrium.

Symmetallism in an Economy (Proposition 11) Where Gold Cannot Be Produced

An equilibrium of the Proposition 11 economy solves (6), (7), and (8) with $Z_g(t) = Z_g(2)$ for all $t > 2$. We will first seek a solution with $Z_g(t+1) = Z_g(1)$ for $t > 1$, which amounts to assuming that the initial stock $Z_g(1)$ is sufficiently small. Under this condition, equation (8) becomes

$$(9) \quad S_t [p_d(t+1)/p_d(t)] = p_d(t)Z_g(1)/\gamma.$$

In the special case $S_t(r) = n^{t-1}S_1(r)$ with $n > 1$, equation (9) has a solution

$$(10) \quad n^{t-1}S_1(n) = p_d(t)Z_g(1)/\gamma.$$

This gives the unique solution of the model, provided that the implied value of $p_g(1) > \lambda_{gb}$. The implied value of $p_g(t)$ is $p_g(t) = p_d(t)/\gamma - \sigma/\gamma\lambda_{bs}$. If $Z_g(1) < S_1(n)/(\lambda_{gb} + \sigma/\gamma\lambda_{bs})$, then $p_g(1) > \lambda_{gb}$ and the solution is, indeed, given by (10). If not, then the solution is $p_d(t)n^{t-1}S_1(n)\gamma/Z_g(2)$, where $Z_g(2) = S_1(n)/(\lambda_{gb} + \sigma/\gamma\lambda_{bs})$. The old at $t = 1$ convert $Z_g(1) - Z_g(2)$ into bread.

We note that in the Proposition 11 model, symmetallism has the effect of selecting an equilibrium with a rate of return sequence that is identical to that associated with the gold-only equilibrium. Symmetallism thus eliminates the $r(t) = 1$ equilibria in which gold and silver coexist as backing for dollars. Therefore, in these economies, borrowers generally prefer bimetallism to symmetallism.

Seignorage

In the context of steady states in which per capita real government debt held by the private sector is constant, we regard the government as earning seignorage if there accrues government revenue from maintenance of the stock of its indebtedness. This happens if the real rate of return on its

debt is less than the growth rate n . Under laissez-faire, the return on government debt in a steady state cannot be less than the maximum of $1 - \delta$ and r^* . The kind of legal restriction discussed above, under which outside money must be in the form of government-issued "dollars," removes $1 - \delta$ as a lower bound on the return on government debt. Hence, if $r^* < 1 - \delta < n$, the existence of such a restriction enhances seignorage possibilities. In the next section, we discuss legal restrictions on private borrowing and lending which can remove r^* as a lower bound on the return on government debt. Here, in order to discuss the sense in which adopting a commodity standard limits seignorage, we want briefly to describe seignorage through fiat-money issue under laissez-faire.

Under laissez-faire, if the growth rate n exceeds $1 - \delta$ and exceeds r^* , then there is "room" in the economy for seignorage earning government debt. One form that this debt can take is a stock of fiat money which the market prices. If the stock of fiat money is kept constant over time, an equilibrium rate of return on fiat money and private securities is n . Now imagine that the government lets the stock of fiat money evolve according to $M(t+1) = \beta M(t)$, where $M(t)$ is the stock of fiat money at time t , and β is a constant greater than one. Imagine, also, that the government spends the newly created money on "bread," which it consumes. Provided that n/β exceeds $1 - \delta$ and exceeds r^* , there is an equilibrium in which fiat money is valued and has a rate of return equal to n/β .

While adoption of a commodity standard would seem to rule out the issue of fiat money, it is misleading to say that it limits seignorage. The seignorage possibility just described can be achieved without having the government issue fiat money. It can be achieved by having the government properly choose and keep outstanding a stock of government debt, denominated,

say, in gold or, better yet, in bread. This conclusion is one of those Modigliani-Miller like results referred to above and is the sense in which it is misleading to say that adoption of a commodity standard limits seignorage and imposes fiscal discipline.

There is, however, a sense in which adoption of a commodity standard does limit seignorage. Given a rule like that assumed above under which outside money is limited to government-issued dollars and given $r^* < 1 - \delta$, adoption of a commodity standard in any of the variants described above makes $1 - \delta$ a lower bound on the return on government debt. However, in these circumstances, other standards could be adopted, some of which would permit a lower return on government debt, and, perhaps make possible greater seignorage. In this sense, adoption of a commodity standard limits seignorage.

5. Restrictions on the Issue of Private Inside Money and the Role of an Intermediating Central Bank

By an intermediating central bank, we mean one that exchanges assets, say, through open market operations or some sort of discount window. Such asset exchanges have no effects unless the central bank has effective monopoly power--for example, over note issue. If it does not have such power, then a Modigliani-Miller type of irrelevance result applies to its intermediation activities. If it has effective monopoly power, then its choice of an intermediation strategy matters, affecting interest rates, the price level, and the equilibrium allocation.

In our view, whether a central bank has effective monopoly power does not depend on the commodity standard in effect, or on whether there is a commodity or fiat standard in effect. Instead, it depends on whether there are restrictions on privately-issued liabilities and privately-held assets--for example, restrictions on issues of notes by private banks, reserve requirements against bank deposits, and so on.

Since, by definition, there are no such restrictions in effect under the laissez-faire regime examined above, there is no role for an intermediating central bank under that regime. In describing laissez-faire, we have assumed that private borrowers can costlessly issue loans in a form that competes perfectly with any potential commodity money. However, the irrelevance of a central bank's intermediation holds in more general set-ups. If the costs of intermediating private loans are the same for private intermediaries and the central bank, then the existence of such costs does not overturn the irrelevance result. Indeed, it is accurate to say that intermediation by a central bank is relevant only if there are potential profits to such intermediation. When these exist, we attribute them to legal restrictions on private intermediation. Such profits, of course, constitute one of the forms of seignorage mentioned in the last section.

Since this point of view and many of its implications have been spelled out elsewhere (see Bryant-Wallace, (1980), Wallace (1981), and Sargent-Wallace (1982)), we will devote the rest of this section to using it to interpret some of the well-known debates concerning the Bank Charter Act of 1845 (Peel's Act), which limited bank note issue in England.

This Act required a 100 percent gold backing behind all bank note issues over a certain inherited amount. Some proponents of the Act wanted the stock of "currency" to behave exactly as would a currency that consisted entirely of gold. Bank notes consisting of warehouse receipts to gold were, on this view, the only acceptable kind of "paper currency." The Act accomplished this; it prevented private borrowers--directly or through banks--from issuing liabilities in the form of small denomination bearer notes.

Peel's Act did not seem to give the Bank of England an effective monopoly because it was subject to the 100 percent marginal reserve require-

ment on note issue. We regard the Bank of England as having had and as having exercised monopoly power only to the extent that the terms of the Act limiting its note issue were subject to suspension. In fact, they were suspended on several occasions during the second half of the 19th century.

Jevons, a defender of the Act, described the views of some of its opponents:

"The objectors to the Bank Charter Act urge that we want more currency, but they cannot really mean more metallic currency. We must not look to changes in the law to increase the amount of specie in the country, and, as I have remarked, any one can get sovereigns if he has the needful gold. . . . What the currency theorists want, then, is not more gold, but more promises to pay gold. The Free-Banking School especially argue that it is among the elementary rights of an individual to make promises, and that each banker should be allowed to issue as many notes as he can get his customers to take, keeping such a reserve of metallic money, as he thinks in his own private discretion, sufficient to enable him to redeem his promises. But this free issue of paper representative money does not at all meet the difficulty of the money market, which is a want of gold, not of paper; on the contrary, an unlimited issue of paper would tend to reduce the already narrow margin of gold upon which we erect an enormous system of trade. (Money and the Mechanism of Exchange, pp. 307-308.)

Jevons seems dubious of the notion that more promises to pay gold could meet the needs of the money market. However, that notion is understandable if we interpret the Act as limiting the form that private debts can take and, therefore, as making it unnecessarily expensive to conclude private agreements to borrow and lend. By doing that, the Act made effective interest rates to borrowers higher than they would otherwise be. The money market could have used more promises to pay gold in forms convenient for most lenders in the sense that relieving the restrictions would have led to a different outcome, one in which borrowers' demands would have been met at lower interest rates.

6. Concluding Remarks

The propositions and interpretations which we have advanced in this paper are evidently products of the models that we have chosen to use, in particular, products of the finance-theory like approach to the pricing of all assets which is inherent in our use of what are essentially growth models. The principle of finance theory to which we refer is the notion that assets are priced according to the consumption streams that they support. Some features of our results also depend quite sensitively on particular features of the technology which we have specified, and are not robust. For example, as indicated above, results of a quantity-theory kind are very special. Other results, such as those regarding differential depreciation rates for different metals and the inefficiency of commodity money systems, are much more robust within the class of growth theories like ours.

We are aware that while growth models and what we have called the associated "finance-theory like" property are widely thought to be promising for building theories of asset pricing for privately-issued securities, they are much more controversial as models of "money" and maybe government debts in general. Thus, our model and our selection of issues to study differ from those in recent writings on commodity standards including those of Barro (1979), Whitaker (1979), and Flood and Garber (1981). These differences partly reflect the fact that we have chosen to explore the consequences of a different set of first principles for pricing assets than they did.

These differences having been noted, we would like to assert the virtues of models like ours for contributing to understanding and clarifying a number of ideas and issues. We find these models particularly useful for distinguishing between inside and outside money, and for bringing into focus distinctions such as that between a "pure" gold standard, and a "gold reserve

standard." In our models, these systems differ either simply because more private borrowing and lending happens to emerge in the latter--say because of differing endowment patterns and preferences across agents--or because laissez-faire prevails in the latter, while legal restrictions are imagined to inhibit intermediation in the former. Because our models are rigged to handle heterogeneous agents in a tractable way, they are manageable vehicles for studying these and other questions involving private borrowing and lending. (Indeed, it is this manageability in the two-period lived overlapping generations model, not an attraction to its demographics, which prompts our use of this model.) Other issues on which we find our models shedding useful insights include the issues of economic inefficiency of commodity standards, the delicacy of quantity theory predictions, and the nature of the restrictions that must be imposed to get a version of Gresham's law to operate.

Finally, we should like to add one question to the list in our introduction, which is in the same spirit as those questions, but a full answer to which we have not worked out. The question is: How does adopting a commodity standard as opposed to a fiat standard affect the "Phillips curve?" Lucas's (1972) analysis of the Phillips curve was conducted using a growth model like the ones that we use, but assumed a fiat money standard that was administered with a special structure of monetary transfers. Since the nature of the Phillips curve correlation in Lucas's set-up depends on how the money is handed out, there is a presumption that moving to a commodity standard would alter the Phillips curve in interesting ways in such a model. Completing such an analysis would be of substantial interest, since interesting sets of observations on the Phillips curve are drawn from periods and countries on some sort of commodity standard.

Appendix: Proofs

This appendix contains proofs of Propositions 3, 4, and 5. The proofs of our other propositions are either straightforward or follow closely one of the proofs given below.

Proof of Proposition 3.

We prove this by contradiction, letting $\bar{\cdot}$'s denote an alternative equilibrium and $\hat{\cdot}$'s denote the Proposition 2 equilibrium. We suppose that for some date, t , $\bar{r}(t) \neq \hat{r}(t)$ and show that this gives rise to a contradiction.

Suppose $\bar{r}(t) < \hat{r}(t)$. This is possible only if $\hat{r}(t) = 1$ and $S(1) > 0$. Therefore, $\bar{r}(t) < 1$. But, then, since $\bar{p}(t+1) \leq \bar{p}(t)\bar{r}(t)$, $ZP(t+1) \leq 0$ and hence, $S[\bar{r}(t+1)] \leq S[\bar{r}(t)]$. This implies $\bar{r}(t+1) \leq \bar{r}(t)$ and, hence, $\bar{p}(t+2) \leq \bar{p}(t+1)\bar{r}(t) \leq \bar{p}(t)[\bar{r}(t)]^2$. Repeating these steps, we conclude that $\bar{p}(t+k) \leq \bar{p}(t)[\bar{r}(t)]^k$ for all $k > 0$. Since $\bar{r}(t) < 1$, this violates Proposition 1.

Suppose $\bar{r}(t) > \hat{r}(t)$. It follows that $\bar{r}(t) > 1$ and that $S[\bar{r}(t)] > 0$, which implies $\bar{p}(t+1) = \bar{r}(t)\bar{p}(t)$ and $ZP(t+1) > 0$. Hence, we have $S[\bar{r}(t+1)] > S[\bar{r}(t)]\bar{r}(t)$. It follows that $\bar{r}(t+1) > \bar{r}(t)$ and that $\bar{p}(t+2) = \bar{p}(t+1)\bar{r}(t+1) > \bar{p}(t)[\bar{r}(t)]^2$. Proceeding in this way, we get $S[\bar{r}(t+k)] > S[\bar{r}(t)][\bar{r}(t)]^k$ for all $k > 0$. This violates the upper bound on $S(r)$.

Proof of Proposition 4.

What has to be proved are the assertions about T and $r(t)$ and $p(t)$ for $t \leq T$ in the case $r^* < 1 - \delta$.

We first consider the possibility that $T = \bar{T}$ and that $p(t) > \lambda_{gb}$ for all t . We denote the implied rates of return and prices by $\bar{r}(t)$ and $\bar{p}(t)$, respectively.

If $p(t) > \lambda_{gb}$ for all t , then no gold is used to produce bread. It follows that the return on saving at $\bar{T} - 1$, and the price of gold at $\bar{T} - 1$ satisfy

$$(A.1) \quad S[\bar{r}(\bar{T}-1)] = \bar{p}(\bar{T}-1)Z(1)(1-\delta)^{\bar{T}-2}$$

Using $\bar{r}(\bar{T}-1) = (1/\lambda_{gb})(1-\delta)/\bar{p}(\bar{T}-1)$ to eliminate $\bar{p}(\bar{T}-1)$ from (A.1), we get

$$(A.2) \quad \bar{r}(\bar{T}-1)S[\bar{r}(\bar{T}-1)] = (1/\lambda_{bg})Z(1)(1-\delta)^{\bar{T}-1}.$$

The monotonicity of S implies that (A.2) has a unique solution for $\bar{r}(\bar{T}-1)$. We now show that this solution is the interval $(1-\delta, 1)$.

From the definition of \bar{T} as the smallest integer j that satisfies (4), it follows that $(1-\delta)S(1-\delta) < (1/\lambda_{bg})Z(1)(1-\delta)^{\bar{T}-1}$. This implies that $\bar{r}(\bar{T}-1) > 1 - \delta$.

If $r > 1$, then $rS(r) > S(r) > S(1-\delta) > (1/\lambda_{bg})Z(1)(1-\delta)^{\bar{T}-1}$, where the last inequality follows from the definition of \bar{T} . This implies that the solution to (A.2) is less than unity.

Now we proceed, working backwards, to find $\bar{r}(\bar{T}-2)$, then $\bar{r}(\bar{T}-3)$, and finally $\bar{r}(1)$.

For any k such that $1 < k \leq \bar{T} - 1$,

$$(A.3) \quad S[\bar{r}(\bar{T}-k)] = \bar{p}(\bar{T}-k)Z(1)(1-\delta)^{\bar{T}-k-1}$$

Using $\bar{r}(\bar{T}-k) = \bar{p}(\bar{T}-k+1)(1-\delta)/\bar{p}(\bar{T}-k)$ to eliminate $\bar{p}(\bar{T}-k)$ from (A.3) and using (A.3) for $S[\bar{r}(\bar{T}-k+1)]$, we see that (A.3) is equivalent to

$$(A.4) \quad \bar{r}(\bar{T}-k)S[\bar{r}(\bar{T}-k)] = S[\bar{r}(\bar{T}-k+1)].$$

With $\bar{r}(\bar{T}-k+1)$ given, this is an equation of the form $rS(r) = S(x)$ with x given. From the properties of S it follows that if $x \in (1-\delta, 1)$, then this equation has a unique solution for r and that that solution is in the interval $(x, 1)$. Since we have established that $\bar{r}(\bar{T}-1) \in (1-\delta, 1)$, it follows that solving (A.4) repeatedly--for $k = 2$, then $k = 3$, and finally $k = \bar{T} - 1$ --gives

us a unique $\bar{r}(1), \bar{r}(2), \dots, \bar{r}(\bar{T}-1)$ which satisfies $1 > \bar{r}(1) > \bar{r}(2) > \dots > \bar{r}(\bar{T}-1) > (1-\delta)$. This, in turn, implies a unique $\bar{p}(1), \bar{p}(2), \dots, \bar{p}(\bar{T}-1)$ using equilibrium condition (ii) at equality and $\bar{p}(\bar{T}) = 1/\lambda_{bg}$. This is our Proposition 4 equilibrium if $\bar{p}(t) > \lambda_{gb}$ for all t .

We now use the above sequences to describe the Proposition 4 equilibrium if it happens that $\bar{p}(t) < \lambda_{gb}$ for some t . Let $\bar{p}(\bar{T}-K)$ be the largest of the $\bar{p}(t)$ that is less than λ_{gb} . (Since $\bar{p}(\bar{T}-1) > \bar{p}(\bar{T}-2) > \dots > \bar{p}(1)$, if we work backwards from $\bar{p}(\bar{T}-1)$, then $\bar{p}(\bar{T}-K)$ is the first $\bar{p}(t)$ that is less than λ_{gb} .) In this case, we let $T = K + 1$, we let $p(1) = \lambda_{gb}$ and we let $p(2), p(3), \dots, p(K)$ and $Z(2)$ be the solution to the following K equations:

$$\begin{aligned}
 S[p(2)(1-\delta)/\lambda_{gb}] &= \lambda_{gb}Z(2)/(1-\delta) \\
 S[p(3)(1-\delta)/p(2)] &= p(2)Z(2) \\
 &\vdots \\
 S[p(t+1)(1-\delta)/p(t)] &= p(t)Z(2)(1-\delta)^{t-2} \\
 &\vdots \\
 S[(1/\lambda_{bg})(1-\delta)/p(K)] &= p(K)Z(2)(1-\delta)^{K-2}.
 \end{aligned}
 \tag{A.5}$$

We now show that equations (A.5) have a unique solution and that that solution satisfies the claims made in Proposition 4.

We deal only with the case $K > 2$. If $K = 1$, the existence and uniqueness is trivial. We begin by rewriting equations (A.5) in terms of rates-of-return; $r(t) = p(t+1)(1-\delta)/p(t)$. In particular, we consider the $K + 1$ equations

$$\begin{aligned}
 S[r(1)] &= \lambda_{gb} Z(2)/(1-\delta) \\
 S[r(2)] &= r(1)S[r(1)] \\
 &\vdots \\
 S[r(K)] &= r(K-1)S[r(K-1)] \\
 \\
 r(K)S[r(K)] &= (1/\lambda_{bg})Z(2)(1-\delta)^{K-1}
 \end{aligned}
 \tag{A.6}$$

The last two equations of (A.6) come from the last equation of (A.5). It is easy to see that there is a one-to-one correspondence between solutions to (A.6) and solutions to (A.5). (Notice that any $r(t); t = 1, 2, \dots, K$ that solves (A.6) satisfies the condition that the product of these $r(t)$'s equals $(1-\delta)^K(1/\lambda_{bg})/\lambda_{gb}$.)

We define a mapping from values of $Z(2)$ on the right-hand side (RHS) of the last equation of (A.6) to values of $Z(2)$ on the RHS of the first equation of (A.6). We will establish enough facts about this mapping to imply the existence of a fixed point (and certain facts about the fixed point).

We let the domain for this mapping be the interval $D = [d_1, d_2]$ where

$$d_1 = S(1-\delta)\lambda_{bg}/(1-\delta)^{K-2}$$

$$d_2 = S(1)\lambda_{bg}/(1-\delta)^{K-1}$$

The mapping is defined by proceeding equation by equation through (A.6), from the last to the first. That is, for any element of D inserted in place of $Z(2)$ on the RHS of the last equation of (A.6), one solves that equation for $r(K)$. Then, using that solution for $r(K)$ in the LHS of the next-to-last equation of (A.6), one solves for $r(K-1)$. One proceeds backward in this fashion, finally solving the first equation of (A.6) for $Z(2)$, which is the value of the mapping. We denote the mapping by Γ .

It is obvious from this definition that Γ is a continuous, increasing function on D . We now show that $\Gamma(d_1) > d_1$ and that $\Gamma(d_2) < d_2$.

As regards $\Gamma(d_1)$, note that when $Z(2) = d_1$ then $r(K) = 1 - \delta$. Thus our task is to find the corresponding $S[r(1)]$. But $S[r(K)] = S(1-\delta)$ and equations (A.6) are exactly those equations we solve when $\bar{T} = K$ and $Z(1)$ is such that no gold gets produced at \bar{T} . Thus, by (A.3), the corresponding $S[r(1)]$ satisfies

$$(A.7) \quad S[r(1)] = \bar{p}(\bar{T}-K+1)Z(1) = \bar{p}(\bar{T}-K+1)S(1-\delta)\lambda_{bg}/(1-\delta)^{K-1}$$

where the second equality uses the condition that no gold is produced at $t = K$. Upon inserting the second equality of (A.7) on the LHS of the first equation of (A.6), we find (as a solution for $Z(2)$)

$$\Gamma(d_1) = \bar{p}(\bar{T}-K+1)S(1-\delta)\lambda_{bg}/(1-\delta)^{K-2}\lambda_{gb}$$

Since $\bar{p}(\bar{T}-K+1) > \lambda_{gb}$, we conclude $\Gamma(d_1) > d_1$.

As regards $\Gamma(d_2)$, note that when d_2 is inserted for $Z(2)$ on the RHS of the last equation of (A.6), we get $r(K) = 1$. It follows immediately that the corresponding $S[r(1)] = S(1)$. Thus

$$\Gamma(d_2) = (1-\delta)S(1)/\lambda_{gb}$$

Therefore, $\Gamma(d_2) < d_2$ if $(1-\delta)/\lambda_{gb} < \lambda_{bg}/(1-\delta)^{K-1}$ or if $\lambda_{gb} > (1-\delta)^K/\lambda_{bg}$. This follows from the fact that $\lambda_{gb} > \bar{p}(\bar{T}-K)$ and the fact that $\bar{r}(t) < 1$ for all t . In other words, $\bar{p}(\bar{T}-K) > (1-\delta)^K/\lambda_{bg}$ follows from the fact that the $\bar{p}(t)$ prices increase at a rate less than $1/(1-\delta)$.

The above facts about Γ imply that there is a d^* in the interval (d_1, d_2) such that $\Gamma(d^*) = d^*$. This d^* and the associated $r(1), r(2), \dots, r(K)$ satisfy (A.6). Moreover, because d^* is not an endpoint of D , the associated r 's satisfy the claim made in Proposition 4.

We now show that d^* is unique. Suppose instead that there are two fixed points of Γ , $d^{(1)}$ and $d^{(2)}$, with $d^{(1)} < d^{(2)}$. It follows from (A.6) that $r(t)^{(1)} < r(t)^{(2)}$ for $t = 1, 2, \dots, K$. But this violates the condition on the product of the r 's that any solution satisfies. Hence, d^* is unique.

Proof of Proposition 5

We proceed by contradiction, letting $\bar{\cdot}$'s denote an alternative equilibrium and $\hat{\cdot}$'s the Proposition 4 equilibrium.

First, suppose that $\bar{r}(t) < \hat{r}(t)$ for some t . This implies that the equilibrium satisfies $S[\hat{r}(t)] > 0$ for all t and hence $\hat{r}(t) = 1 - \delta$ for all $t > T$. Since the proof of Proposition 4 shows that positive saving equilibria with $r(t) = 1 - \delta$ for $t > T$ have a unique T and unique $r(t)$ for $t < T$, we can, without loss of generality, take $\bar{r}(t) < 1 - \delta$. This possibility is ruled out by proceeding exactly as in the first part of the proof of Proposition 3; essentially, by repeatedly using $\bar{p}(t+k) \leq \bar{p}(t+k-1)\bar{r}(t+k-1)/(1-\delta)$ to get $\bar{p}(t+k) \leq \bar{p}(t)[\bar{r}(t)/(1-\delta)]^k$, which violates the lower bound on the price of gold.

Now suppose that $\bar{r}(t) > \hat{r}(t)$ for some t . It follows that $\bar{r}(t) > 1 - \delta$ and that $S[\bar{r}(t)] > 0$ and, therefore, that $\bar{p}(t+1) = \bar{r}(t)\bar{p}(t)/(1-\delta)$. This and $\bar{r}(t) > (1-\delta)$ imply $\bar{p}(t+1) > \bar{p}(t)$ and, hence, $\bar{Z}(t+1) > 0$ and $\bar{Z}(t+2) > (1-\delta)\bar{Z}(t+1)$. Multiplying both sides of this last inequality by $\bar{p}(t+1)/(1-\delta)$ we get

$$(A.8) \quad S[\bar{r}(t+1)] > \bar{p}(t+1)\bar{Z}(t+1) = \bar{r}(t)\bar{p}(t)\bar{Z}(t+1)/(1-\delta) = \bar{r}(t)S[\bar{r}(t)].$$

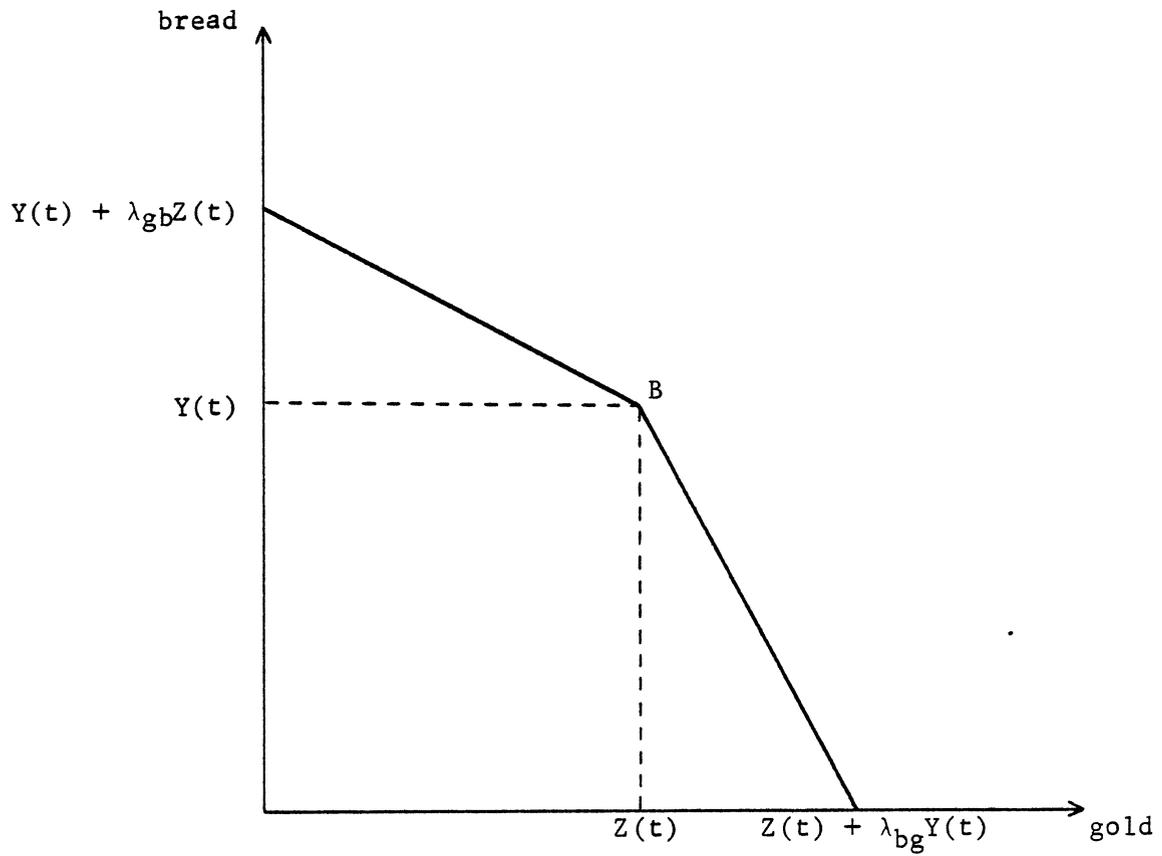
We now consider separately the possibilities $\bar{p}(t+1) = 1/\lambda_{bg}$ and $\bar{p}(t+1) < 1/\lambda_{bg}$. In the former case, it follows that $\bar{p}(t+2) \leq \bar{p}(t+1)$, so $\bar{r}(t+1) \leq 1 - \delta$. The strict inequality was ruled out above. The equality, $\bar{r}(t+1) = 1 - \delta$, implies $\bar{r}(t+k) = 1 - \delta$ for all $k > 1$, which, as noted above, cannot be other than a Proposition 4 equilibrium.

If $\bar{p}(t+1) < 1/\lambda_{bg}$, then we get equality in (A.8) and, hence, $\bar{r}(t+1) = \bar{r}(t) = 1$ and $\bar{p}(t+2) = \bar{p}(t+1)/(1-\delta)$.

But the latter implies that saving at $t + 2$ is at least as large saving at $t + 1$. Thus, $\bar{r}(t+2) > 1$ which implies $\bar{p}(t+3) > \bar{p}(t+2)/(1-\delta)$. Proceeding in this manner, we violate the upper bound on the price of gold.

Figure 1

The Static Production Possibility Frontier



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