An Application of Schauder's Fixed Point Theorem to an Overlapping Generations Model

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The economy under consideration in this paper is the standard two-sector model populated by two-period lived people. The commodities produced are a capital good and a consumption good. People care only about the amount of consumption they may enjoy in each period of their life.

A young person is endowed with labor which he sells for a wage. Next period, when he is old, he can no longer work. The consumption good is nonstorable, and therefore the young person has to save some other asset. The only asset available is the capital good. Next period the capital saved will be used in production and will yield a rental with which consumption may be purchased.

A supply for next period's capital is derived from the production side of the economy. Demand is derived from the young person's utility maximization problem. An equilibrium is defined, and an existence proof concludes this paper.

Supply of Next Period's Capital

Production in this model is described by two functions which are homogeneous of degree 1 and strictly quasi concave. Let commodity 1 be the consumption good and commodity 2 the capital good. Production of each commodity takes place using two factors: factor 1--capital and factor 2--labor. Factor intensities are different in the production of the two goods, and factor intensity reversal is ruled out.

The two factors are supplied inelastically and are perfectly mobile between industries. In a competitive economy these assumptions yield a supply for capital given by

(1) $k(t+1) = S(p(t),k(t))$

where

$k(t)$ is the capital labor ratio at time t and

$p(t)$ is the price of capital at time t using the consumption good as numeraire.

The assumptions made mean that the economy will be on the production possibilities frontier. The supply may therefore be read off this frontier with the price being represented by the slope (or its reciprocal).

I assume that the supply function is continuously differentiable. The concavity means that $S_{p(t)} > 0$ for all prices in which the economy produces both goods.

$S_{k(t)}$ is less (greater) than zero if the consumption good is capital (labor) intensive.

Let $\underline{p}(k)$ be the price at which a country is on the verge of specializing in the consumption good and $\overline{p}(k)$ be the price at which the country is on the verge of specializing in the capital good. The functions $\underline{p}(k)$ and $\overline{p}(k)$ are continuous, monotone increasing if consumption is capital intensive, and monotone decreasing otherwise. For every positive k, $\underline{p}(k)$ and $\overline{p}(k)$ are positive numbers.

Before turning to the demand side, I shall add another assumption on the capital production function.

Assumption 1: There exists a $\tilde{k} > 0$ such that $\tilde{k} = f_2(\tilde{k})$.

This assumption means that at some input level, capital production can no longer reproduce the input.

Demand for Next Period's Capital

Every member of the young generation at time t wants to maximize his life-time utility derived from consuming the consumption good at each period of his life.

Let $c_1(t)$ denote first-period consumption of a person born at t and $c_2(t)$ be his second-period consumption. Then the life-time utility is given by $u(c_1(t),c_2(t))$. I assume u to be twice differentiable with $u_1 > 0$, $u_2 > 0$, $u_{11} < 0$, $u_{22} < 0$. Also, $\lim_{c_1 \to 0} u_1 = \lim_{c_2 \to 0} u_2 = \infty$.

Further, $c_1(t)$ and $c_2(t)$ are gross substitutes.

All people in the economy have the same utility function. A representative member of the young generation at time t faces the following problem:

$$(2) \qquad \max_{c_1(t),c_2(t),k(t+1)} \{u(c_1(t),c_2(t))\}$$

subject to

$$c_1(t) + p(t)k(t+1) \leq w(t)$$

$$c_2(t) \leq r(t+1)k(t+1)$$

where

$p(t)$ is the price of capital in terms of the consumption good at time t,

$w(t)$ is the real wage at time t,

$r(t+1)$ is the real rental at time $t+1$, and

$k(t+1)$ is the amount of capital the young person buys at time t to be used at time $t+1$.

All prices are taken as parameters, and the first-order conditions yield

$$(3) \qquad v(c_1(t),c_2(t)) \overset{\text{def}}{=} \frac{u_2(c_1(t),c_2(t))}{u_1(c_1(t),c_2(t))} = \frac{p(t)}{r(t+1)}.$$

Using the constraints in (2), define

$$(3') \qquad F(p(t),w(t),r(t+1),k(t+1)) = v(w(t)-p(t)k(t+1),r(t+1)k(t+1)) - \frac{p(t)}{r(t+1)}.$$

<u>Claim</u>: Given any positive $p(t)$, $w(t)$, $r(t+1)$, there exists a unique $k(t+1)$ such that $F(p(t),w(t),r(t+1),k(t+1)) = 0$.

<u>Proof</u>: For $k(t+1) = \frac{w(t)}{p(t)}$, $v \to 0$. For $k(t+1) = 0$, $v \to \infty$. $F_{k(t+1)} = -v_1 p(t) + v_2 r(t+1) < 0$, so the existence and uniqueness of $k(t+1)$ is established.

Using the implicit function theorem one may write

$$k(t+1) = D(p(t),w(t),r(t+1)).$$

It is a well-known result that for an economy which is producing both commodities, the wage and rental are functions of the prices of the two commodities alone. These functions are homogeneous of degree 1, and therefore the price of the consumption good may be used as numeraire for all t with $w(t)$ and $r(t+1)$ being functions of the price ratios $p(t)$ and $p(t+1)$. Using these facts the demand function for next period's capital is

$$(4) \qquad k(t+1) = D(p(t),p(t+1)).$$

The functions $r(p)$ and $w(p)$ are monotone and continuously differentiable under the assumptions already made on the production side.

The derivatives of the demand functions are of interest.

$$D_{p(t)} = - \frac{F_{p(t)}}{F_{k(t+1)}} = - \frac{v_1(w'(t)-k(t+1)) - \frac{1}{r(t+1)}}{-v_1 p(t) + v_2 r(t+1)}.$$

The numerator can be written as

$$v_1(w'(t)-k(t+1)) - \frac{1}{r(t+1)} = v_1(w'(t) - \frac{w(t)}{p(t)} + \frac{w(t)}{p(t)} - k(t+1))$$

$$- \frac{v}{p(t)} = \frac{1}{p(t)}(v_1 c_1 - v) + v_1(w' - \frac{w(t)}{p(t)}).$$

The first expression is nonpositive due to the gross substitution assumption. The second is positive if the consumption good is capital intensive, negative otherwise, by the Stolper-Samuelson theorem.

$$D_{p(t+1)} = - \frac{F_{p(t+1)}}{F_{k(t+1)}} = - \frac{\frac{r'(t+1)}{r(t+1)}(v_2 c_2 + v)}{-v_1 p(t) + v_2 r(t+1)}.$$

Gross substitution yields $v_2 c_2 + v \geq 0$. Therefore, $\text{Sign}(D_{p(t+1)}) = \text{Sign}(r'(t+1))$. If the consumption good is capital intensive, $D_{p(t+1)} \leq 0$. If the capital good is capital intensive, $D_{p(t+1)} \geq 0$.

Market Equilibrium

I assume that all capital existing at time t is completely destroyed. Therefore, demand for next period's capital can be met only by the new capital produced each period. The market clearing condition is then

(5) $\quad D(p(t),p(t+1)) - S(p(t),k(t)) = 0.$

Claim: For any $k(t) > 0$ and $p(t+1) > 0$, there exists a positive price $p(t)$ which solves (5).

Proof: Given any $k > 0$, there exist $\underline{p}(k)$ and $\overline{p}(k)$ in which the country is on the verge of specializing in the production of the consumption good and the capital good, respectively (see Figure 1). At $p(t) = \underline{p}(k(t))$, $\frac{p(t)}{r(t+1)} > 0$ for all $p(t+1) > 0$, and clearly there is a positive demand for $k(t+1)$, since $\lim_{\substack{c_2 \to 0 \\ c_1 \neq 0}} v = 0$. At $p(t) = \overline{p}(k(t))$ there is a positive demand for $c_1(t)$. So $c_1(t) = w(t) - p(t)k(t+1) > 0$. But at this point the country is producing only capital, and hence $w(t) = p(t)(f_2(k(t))-k(t)f_2'(k(t)))$. Therefore, we get $p(t)[f_2(k(t))-k(t+1)-k(t)f_2'(k(t))] > 0$, which yields $f_2(k(t)) - k(t+1) > 0$. Hence, the quantity of $k(t+1)$ demanded at this point is less than the quantity of capital produced. Figure (2) depicts the situation, and clearly there is a $p(t)$ which solves (5).

Figure 1

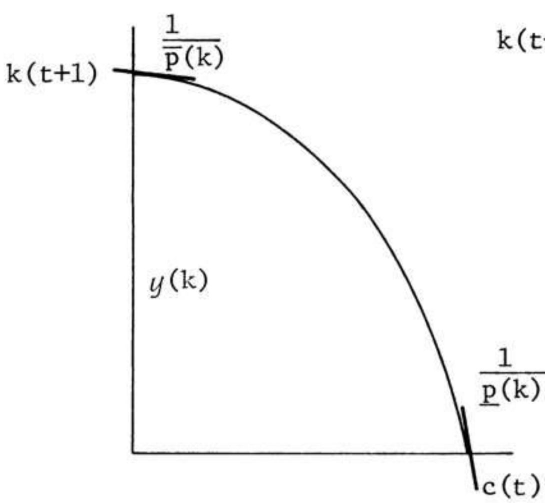
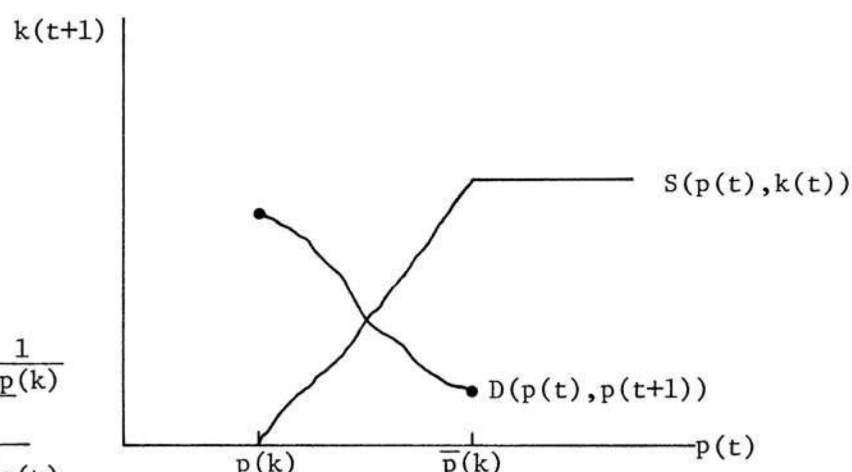


Figure 2



It is important for subsequent argumentation to assume that $D_{p(t)} - S_{p(t)} \neq 0$ at any solution. I also want the solution to be unique. It seems natural to assume the following on the excess demand function:

<u>Assumption 2</u>: At $p(t)$ which solves (5), $D_{p(t)} - S_{p(t)} < 0$ for all positive $k(t)$, $p(t+1)$.

We shall make one further assumption bounding demand away from 0. To state the assumption we shall first define a set K and a related set P.

Let $K = [\underline{k}, \overline{k}]$ such that $0 < \underline{k} < \tilde{k} < \overline{k} < \infty$. The set P is related to K in the following manner:

Let

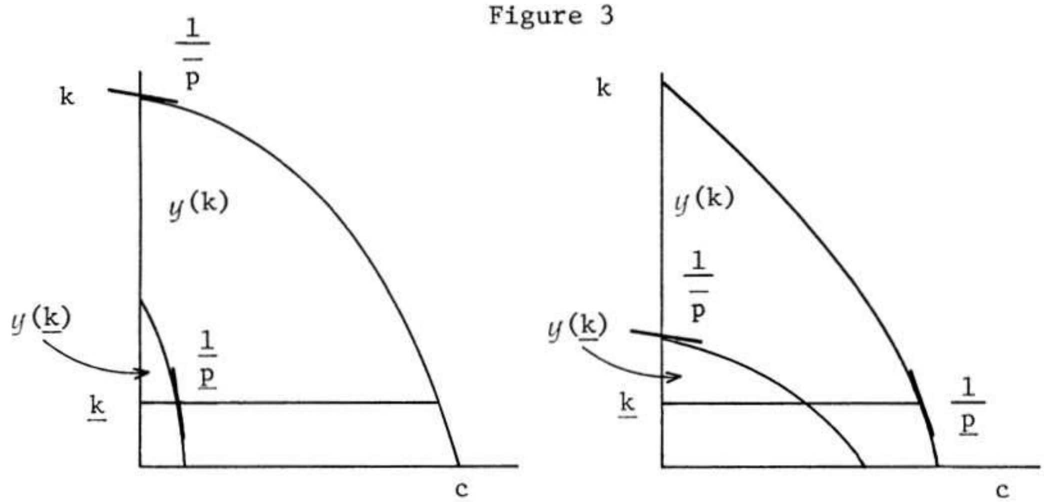
$$\overline{p} = \begin{cases} \overline{p}(\overline{k}) & \text{if the consumption good is capital intensive} \\ \overline{p}(\underline{k}) & \text{otherwise} \end{cases}$$

and let

$$\underline{p} = \begin{cases} \text{solution to } \underline{k} = S(p, \underline{k}) & \text{if the consumption good is k-intensive} \\ \text{solution to } \underline{k} = S(p, \overline{k}) & \text{otherwise.} \end{cases}$$

Figure 3 shows $\underline{p}$, $\overline{p}$ in each case.

Figure 3



c is k-intensive   k is k-intensive

Assumption 3 enables us to choose $\underline{k}$ in a particular manner.

Assumption 3: There exists a $\underline{k} > 0$ such that for some $\overline{k} > \tilde{k}$, $D(p(t),p(t+1)) \geq \underline{k}$ for all $p(t)$, $p(t+1) \varepsilon P$ where P is related to $K = [\underline{k},\overline{k}]$ as described above.

This assumption restricts in a complicated way the class of utility and production functions with which we may operate. A sufficient condition for assumption 5 to hold may be stated as follows:

The function v has to be such that

$$v(c_1,\underline{c}_2) \geq \frac{\underline{p}(k)}{\underline{r}} \text{ for all } k\varepsilon K \text{ if consumption is capital intensive}$$

$$v(c_1,\underline{c}_2) \geq \frac{\overline{p}(\underline{k})}{\underline{r}} \text{ if capital is capital intensive}$$

where

$$\underline{r} = \begin{cases} r(\overline{p}(\overline{k})) & \text{if consumption is capital intensive} \\ r(\underline{p}(\overline{k})) & \text{if capital is capital intensive} \end{cases}$$

$$\underline{c}_2 = \underline{r} \cdot \underline{k}$$

$$\underline{w} = \begin{cases} w(\underline{p}(k)) & \text{if consumption is capital intensive} \\ w(\overline{p}(\underline{k})) & \text{if capital is capital intensive} \end{cases}$$

$$c_1 = \begin{cases} \underline{w} - \underline{p}(k)\underline{k} & \text{if consumption is capital intensive} \\ \underline{w} - \overline{p}(\underline{k})\underline{k} & \text{if capital is capital intensive.} \end{cases}$$

To show that this condition is sufficient for assumption 3 to hold, let me discuss each case separately.

If consumption is capital intensive, then $D_{p(t+1)} \leq 0$. Therefore, to find the smallest k(t+1) demanded, it suffices to set p(t+1) to its highest value, $\overline{p}(\overline{k})$. Since $r' < 0$ in this case, r(t+1) will be the smallest possible. Now assume that $k(t+1) = \underline{k}$. Then $p(t) = \underline{p}(k)$. But the condition says that $v \geq \frac{\underline{p}(k)}{\underline{r}}$. Since v is a decreasing function of $k(t+1)$, the quantity demanded cannot be smaller than $\underline{k}$.

If capital is capital intensive, $D_{p(t)} < 0$, $D_{p(t+1)} \geq 0$. Thus, it is sufficient to set $p(t+1)$ to its smallest value, $\underline{p}(\overline{k})$ and $p(t)$ to its largest, $\overline{p}(\underline{k})$. In this case $r' > 0$, and again r obtains its smallest value. If $v \geq \frac{\overline{p}(\underline{k})}{\underline{r}}$, $k(t+1)$ demanded cannot be smaller than $\underline{k}$ for any $k \varepsilon K$.

Roughly speaking, this condition requires that the marginal utility of second-period consumption becomes big enough when second-period consumption is small. Thus, even though wages and first-period consumption become small, demand for second-period consumption is such that demand for second period's capital is still higher than $\underline{k}$.

I impose one further condition on the relationship between the various derivatives.

<u>Assumption 4</u>: For all $p(t)$, $p(t+1) \varepsilon P$, $k(t) \varepsilon K$,

(i) $$\left| \frac{D_{p(t+1)} S_{k(t)}}{D_{p(t)} - S_{p(t)}} \right| - 1 < 0$$

(ii) $$\left(\left| \frac{D_{p(t+1)} S_{k(t)}}{D_{p(t)} - S_{p(t)}} \right| - 1 \right)^2 - 4 \left| \frac{D_{p(t+1)} S_{p(t)}}{D_{p(t)} - S_{p(t)}} \right| \cdot \left| \frac{S_{k(t)}}{D_{p(t)} - S_{p(t)}} \right| \geq 0.$$

Note that when $D_{p(t+1)} \equiv 0$ in $p(t)$ and $p(t+1)$, assumption 4 holds. Therefore, one may view this assumption as roughly requiring that demand for next period's capital be fairly insensitive to next period's price or, in other words, that demand be unsensitive to the rate of return.

An example is presented in an appendix to show that the assumptions made are not mutually exclusive.

Equilibrium of the Model

The market equilibrium condition (5) together with (3) or (4) define a system of two first-order difference equations. The system has just one initial condition, $k(t)$. An equilibrium, in general, is a sequence of prices which satisfies this system of difference equations. However, we would like to restrict the class of solutions.

Definition. An equilibrium is a function which maps every period's capital labor ratio to the price at the same period.

Technically this condition means that a single initial condition is sufficient to generate the path of the economy. This seems to be a desirable feature. Any other solution would require a justification as to the second initial condition, which will have to be fairly artificial. One has to bear in mind also that this is a deterministic model, and, hence, the path of the economy should be perfectly predictable given the information about tastes, technology, and the initial capital labor ratio.

In order to prove the existence of such an equilibrium, we shall use (4) and (5) to define a mapping from a subset of a Banach space into itself. We shall then use Schauder's fixed point theorem to show that the mapping has a fixed point. Schauder's theorem says that every convex compact subspace of a Banach space is a fixed point space.

Theorem: There exists a continuous function α: $K \to P$ such that

$$D(\alpha(k(t)),\alpha(k(t+1))) - S(\alpha(k(t)),k(t)) = 0$$

$$k(t+1) = S(\alpha(k(t)),k(t)).$$

Proof: Define

$A = \{\alpha\colon\ K \to P;$ for each k, $\underline{p}(k) \leq \alpha(k) \leq \overline{p}(k)$, $\alpha \varepsilon C_1$,
$\alpha'(\cdot)$ is uniformly bounded$\}$

where

$K = [\underline{k}, \overline{k}]$,

P is related to K as discussed above,

$\underline{p}(k)$ solves $\underline{k} = S(p,k)$, and

$\overline{p}(k)$ is the price at which the country is on the verge of specializing in capital.

For each $\alpha(\cdot)$ and k, define $\beta(k)$ by

$$D(\beta(k), \alpha(S(\alpha(k),k))) - S(\beta(k),k) = 0. \tag{6}$$

Claim: For each $\alpha \varepsilon A$, (6) implicitly defines a function $\beta\colon\ K \to P$.

Proof: For every $k \varepsilon K$, $\alpha(k)$ gives a price. Due to the restrictions imposed on A, this price, when plugged into the supply function, gives a $k(t+1)$ such that $\underline{k} \leq k(t+1) \leq f_2(k)$. Assumption 1 and the choice of K with $\overline{k} > \tilde{k}$ assure us that $k(t+1) \varepsilon K$. Applying α again, we get another price, $p(t+1)$. As shown above, for each $k(t)$, $p(t+1)$ there exists a unique $p(t)$ which clears the market. This $p(t)$ is defined to be $\beta(k(t))$. By varying k, we get a function β as required.

Let $B = \{\beta\colon\ K \to P\}$.

Claim: (6) implicitly defines a mapping $T\colon\ A \to B$.

Proof: Every function α will yield for every $k(t)$ a unique $p(t)$. Hence, every α is mapped into a unique β.

Our goal now is to show that T has a fixed point.

The first step is to establish that T maps A into itself.

Claim: $T(A) \subseteq A$.

Proof: 1) By assumption 3, for all $p(t)$, $p(t+1) \varepsilon P$, $D(p(t),p(t+1)) \geq \underline{k}$. Hence $\underline{p}(k) \leq p(t) = \beta(k)$. We have also shown that at $p = \overline{p}(k)$ there is excess supply of next period's capital, therefore, $\beta(k) = p(t) \leq \overline{p}(k)$.

2) $\beta \varepsilon C_1$.

Consider $D(p(t),p(t+1)) - S(p(t),k(t)) = 0$.

The functions are continuously differentiable for every positive $P(t)$, $p(t+1)$, and $k(t)$. By assumption 2, $D_{p(t)} - S_{p(t)} \neq 0$. For every $p(t+1)$ and $k(t)$, there exists a unique market clearing $p(t)$. Therefore, there exists a unique function

$$G: \; P \times K \rightarrow P$$

such that $p(t) = G(p(t+1),k(t))$ and $G \varepsilon C_1$. Recall that $p(t+1) = \alpha[S(\alpha(k(t)),k(t)]$ so that

$$p(t) = G[\alpha[S(\alpha(k(t)),k(t)],k(t)] \equiv \beta(k(t)). \tag{7}$$

Clearly, $\beta \varepsilon C_1$.

3) β' is bounded by M.

β' may be obtained from (7):

$$\beta' = G_{p(t+1)}[\alpha'|_{k(t+1)}(S_{p(t)}\alpha'|_{k(t)}+S_{k(t)})] + G_{k(t)}$$

where

$$G_{p(t)} = -\frac{D_{p(t+1)}}{D_{p(t)} - S_{p(t)}}$$

$$G_{k(t)} = \frac{S_{k(t)}}{D_{p(t)} - S_{p(t)}}.$$

Since D and S are continuously differentiable and their domains are compact, the derivatives are bounded. Assumption 2 guarantees that $D_{p(t)} - S_{p(t)}$ is bounded away from 0. Therefore, β' is bounded.

We still have to show that there exists $M > 0$, such that $|\alpha'| \leq M \Rightarrow |\beta'| \leq M$.

$$|\beta'| \leq \left|\frac{D_{p(t+1)}S_{p(t)}}{D_{p(t)}-S_{p(t)}}\right| M^2 + \left|\frac{D_{p(t+1)}S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right| M + \left|\frac{S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right| \leq M.$$

Therefore, if for all $p(t)$, $p(t+1)\varepsilon P$, $k(t)\varepsilon K$

(i) $$\left|\frac{D_{p(t+1)}S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right| - 1 < 0$$

(ii) $$\left(\left|\frac{D_{p(t+1)}S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right| -1\right)^2 - 4\left|\frac{D_{p(t+1)}S_{p(t)}}{D_{p(t)}-S_{p(t)}}\right| \cdot \left|\frac{S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right| \geq 0$$

then M as required exists. Assumption 4 says that these conditions are fulfilled.

Define a norm on A by $||\alpha|| = \sup_{k\varepsilon K} |\alpha(k)|$.

<u>Claim</u>: The set A is convex, bounded, and equicontinuous.

<u>Proof</u>: 1) Convexity. Let $\alpha_1\varepsilon A$, $\alpha_2\varepsilon A$, $0 \leq \lambda \leq 1$. Define $\alpha_\lambda = \lambda\alpha_1 + (1-\lambda)\alpha_2$. Clearly, $\underline{p}(k) \leq \alpha_\lambda(k) \leq \overline{p}(k)$. Also, $\alpha_\lambda \varepsilon C_1$ and $|\alpha'_\lambda| \leq M$.

2) Boundedness. Since $\underline{p}(k)$ and $\overline{p}(k)$ are continuous functions defined on a compact set, the result is immediate.

3) Equicontinuity. Let $\alpha\varepsilon A$, k_1, $k_2\varepsilon K$ with $|k_1-k_2| \leq \frac{\varepsilon}{M}$.

Then $|\alpha(k_1)-\alpha(k_2)| = |\int_{k_1}^{k_2} \alpha'(k)dk| \leq M|k_2-k_1| = \varepsilon$.

The set A is a subset of C(K) (the space of all bounded continuous functions defined on K) which is a Banach space.

By Ascoli's theorem, the set A is conditionally compact, i.e., $\overline{A}$ (A closure) is compact in its relative topology.

Since A is not closed (limit points may not be differentiable), it is important to note that $\alpha \varepsilon \overline{A} \Rightarrow \underline{p}(k) \leq \alpha(k) \leq \overline{p}(k)$, α continuous. Hence, $T(\alpha)$ is well defined for $\alpha \varepsilon \overline{A}$.

<u>Claim</u>: T: $\overline{A} \to B$ is a continuous mapping.

<u>Proof</u>: Let $\{\alpha^n\} \subseteq \overline{A}$ be a uniformly convergent sequence with a limit α. Examine

$$\beta^n(k) = G(\alpha^n(S(\alpha^n(k(k)))$$

$$\beta(k) = G(\alpha(S(\alpha(k),k))).$$

We want to show that $\beta^n(\cdot) \to \beta(\cdot)$ uniformly.

$$|\beta^n(k)-\beta(k)| = |G(\alpha^n(S(\alpha^n(k),k)))-G(\alpha(S(\alpha(k),k)))| \leq$$

$$|G(\alpha^n(S(\alpha^n(k),k)))-G(\alpha^n(S(\alpha(k),k)))| +$$

$$|G(\alpha^n(S(\alpha(k),k)))-G(\alpha(S(\alpha(k),k)))|.$$

Start with the second expression on the right-hand side.

Let $k^* = S(\alpha(k),k)$.

Since α^n converge uniformly, for each $\delta>0$, there exists $K(\delta)$ such that for all $n>K(\delta)$, $|\alpha^n(k^*)-\alpha(k^*)|<\delta$. G is a continuous function defined on a compact set, and therefore it is uniformly continuous. Hence, for each $\varepsilon>0$ there exists $\delta(\varepsilon)$ such that if $|\alpha^n(k^*)-\alpha(k^*)|<\delta(\varepsilon)$ then $|G(\alpha^n(k^*))-G(\alpha(k^*))|<\varepsilon$. Thus we can find $K(\delta(\varepsilon))$ such that for all $n>K(\delta(\varepsilon))$ the second expression is less than ε.

Turn to the first expression on the right-hand side. By the same argument, using the fact that S is a continuous function defined on a compact set, we get that for any $\eta>0$, there exists $K(\eta)$ such that for all $n>K(\eta)$ $|S(\alpha^n(k),k)-S(\alpha(k),k)|<\eta$. Since $\{\alpha^n\}$ is uniformly convergent on a compact set, it is an equicontinuous family. Therefore, we can choose $\eta(\mu)>0$ such that if $|S(\alpha^n(k),k)-S(\alpha(k),k)|<\eta$ then $|\alpha^n(S(\alpha^n(k),k))-\alpha^n(S(\alpha(k)k))|<\mu$. The fact that G is uniformly continuous means that for each $\varepsilon>0$ we can choose $\mu(\varepsilon)$ such that if $|\alpha^n(S(\alpha^n(k),k))-\alpha(S(\alpha^n(k),k))|<\mu(\varepsilon)$ then $|G(\alpha^n(S(\alpha^n(k),k)))-G(\alpha^n(S(\alpha(k),k)))|<\varepsilon$. So, for each $\varepsilon>0$ we can find a $K(\eta(\mu(\varepsilon)))$ such that for all $n>K(\eta(\mu(\varepsilon)))$ the first expression is less than ε. This completes the proof that $\beta_n\to\beta$ uniformly.

Since T is continuous, $T(\overline{A}) \subseteq \overline{T(A)}$. But $\overline{T(A)} \subseteq \overline{A}$, so $T(\overline{A}) \subseteq \overline{A}$. As A is convex, so is $\overline{A}$.

$\overline{A}$ is a convex compact subspace of a Banach space, T is a continuous mapping of $\overline{A}$ into itself, and by Schauder's theorem there exists an $\alpha\varepsilon\overline{A}$ such that $T(\alpha) = \alpha$.

A remark on the assumptions is called for. Clearly they were made so that the proof could work. There is no pretense on my side that any economic theory underlines them. Furthermore, there may be a different set of assumptions that can yield the existence of an equilibrium since the mapping I chose to explore is just one out of many possibilities. My choice was based on an intuitive notion that an equilibrium should exist if we tell the young of the country that next period's price will be exogenously determined, say, by opening the economy to trade with a "big" country. This choice may well be responsible for the need to ask for assumption 4, since in this particular mapping the young are faced with arbitrary next-period prices. It is probably to be expected that they should not react in a sensitive manner to these prices if an equilibrium is to exist.

Appendix

Example

Let the utility function be

$$u = A[\alpha c_1(t)^{-\rho}+(1-\alpha)c_2(t)^{-\rho}]^{-1/\rho}; \quad 0 < \alpha < 1.$$

Then it is easy to get

$$v(c_1(t),c_2(t)) = \frac{u_2}{u_1} = \frac{(1-\alpha)}{\alpha}\left(\frac{c_1(t)}{c_2(t)}\right)^{\rho+1}.$$

We assume $c_1(t)$ and $c_2(t)$ to be gross substitutes, so $-1 < \rho < 0$.

On the production side we have two Cobb-Douglas functions:

$$c_1(t) = f_1(k_1(t)) = \mu_1 k_1(t)^{\beta_{11}}$$

$$k(t+1) = f_2(k_2(t)) = \mu_2 k_2(t)^{\beta_{21}}$$

where

$$\alpha < \beta_{11} < 1,\ 0 < \beta_{21} < 1,\ \mu_1 > 0,\ \mu_2 > 0.$$

Let $\beta_{12} = 1 - \beta_{11}$, $\beta_{22} = 1 - \beta_{21}$.

It can be shown that the supply for capital is given by:

$$k(t+1) = -K_1\frac{\beta_{12}}{\delta}p(t)^{-\frac{\beta_{22}}{\delta}}k(t) + K_2\frac{\beta_{11}}{\delta}p(t)^{\frac{\beta_{21}}{\delta}};$$

$$\underline{p}(k(t)) \leq p(t) \leq \overline{p}(k(t))$$

where

$$\delta = \beta_{11}\beta_{22} - \beta_{12}\beta_{21} = \beta_{11} - \beta_{21} = \beta_{22} - \beta_{12}.$$

K_1 and K_2 are functions of μ_1, μ_2, β_{11}, β_{21}. The rental and wage functions are given by

$$r(t) = K_1 \cdot p(t)^{-\frac{\beta_{12}}{\delta}}$$

$$w(t) = K_2 \cdot p(t)^{\frac{\beta_{11}}{\delta}}.$$

<u>Consumer Equilibrium</u>

$$v(c_1(t),c_2(t)) = \frac{p(t)}{r(t+1)}$$

$$\frac{(1-\alpha)}{\alpha}\left(\frac{w(t)-p(t)k(t+1)}{r(t+1)k(t+1)}\right)^{\rho+1} = \frac{p(t)}{r(t+1)}.$$

Assumption 1 puts a restriction on the productivity of the capital producing process.

$$k = \mu_2 k^{\beta_{21}} \Rightarrow k = \mu_2^{\frac{1}{\beta_{22}}}$$

is the unique positive solution.

Assumptions 2 and 3 deal with restrictions put across production and utility.

Assumption 2 requires the excess demand function for next period's capital to be negatively sloped with respect to p(t) at a neighborhood of equilibrium.

$$D_{p(t)} - S_{p(t)} = \frac{v_1(w'-k(t+1)) - \frac{1}{r(t+1)}}{v_1 p(t) - v_2 r(t+1)} - S_{p(t)}.$$

Notice that the supply can be written as

$$k(t+1) = r'k(t) + w'$$

so that

$$S_{p(t)} = r''k(t) + w''.$$

We can write

$$D_{p(t)} - S_{p(t)} = \frac{-v_1 r' k(t) - \frac{v}{p(t)} - (v_1 p(t) - v_2 r(t+1))(r'' k(t) + w'')}{v_1 p(t) - v_2 r(t+1)}$$

$$= \frac{-v_1 k(t)(r' + p(t) r'') - v_1 p(t) w'' - \frac{v}{p(t)} + v_2 r(t+1)(r'' k(t) + w'')}{v_1 p(t) - v_2 r(t+1)}.$$

In our case, $w'' > 0$ and therefore if $(r' + p(t) r'')$ is positive, then the whole expression is negative.

$$r' + p(t) r'' = K_1 \frac{\beta_{12}}{\delta} p(t)^{-\frac{\beta_{22}}{\delta}} \left(\frac{\beta_{22}}{\delta} - 1\right) = K_1 \left(\frac{\beta_{12}}{\delta}\right)^2 p(t)^{-\frac{\beta_{22}}{\delta}} > 0.$$

Therefore, assumption 2 holds.

Assumption 3 requires that there exists a $\underline{k}$ such that for some $\bar{k} > \tilde{k}$, $D(p(t), p(t+1)) \geq \underline{k}$ for all $p(t)$, $p(t+1)$ which belong to the price set defined by $K = [\underline{k}, \bar{k}]$.

As this assumption is more complicated than the previous one, let us study first an example where utility is also Cobb-Douglas. In this case one may obtain an explicit demand function:

$$k(t+1) = (1-\alpha) K_2 p(t)^{\frac{\beta_{21}}{\delta}}; \quad 0 < \alpha < 1.$$

Assume first that $\delta > 0$ (i.e., the consumption good is capital intensive). Then the demand is increasing in $p(t)$, so we have to study $p(t) = \underline{p}(\underline{k})$. For convenience we shall look at $\underline{\underline{p}}(\underline{k})$, which is the price at which the country specializes in producing the consumption good. Clearly, $\underline{\underline{p}}(\underline{k}) < \underline{p}(\underline{k})$.

One can show that

$$\underline{p}(\underline{k}) = \left(\frac{K_1}{K_2}\frac{\beta_{12}}{\beta_{11}}\underline{k}\right)^{\delta}.$$

Our condition then is

$$k(t+1) > (1-\alpha)K_2\left(\frac{K_1}{K_2}\frac{\beta_{12}}{\beta_{11}}\underline{k}\right)^{\beta_{21}} \geq \underline{k}$$

or

$$\underline{k} \leq ((1-\alpha)K_2)^{\frac{1}{\beta_{22}}}\left(\frac{K_1}{K_2}\frac{\beta_{12}}{\beta_{11}}\right)^{\frac{\beta_{21}}{\beta_{22}}}.$$

Clearly such a $\underline{k}$ exists.

If $\delta < 0$, then we have to look at the largest price which, in this case, is obtained at $\overline{p}(\underline{k})$.

$$\overline{p}(\underline{k}) = \left(\frac{K_1}{K_2}\frac{\beta_{22}}{\beta_{21}}\underline{k}\right)^{\delta}.$$

Now the condition is

$$k(t+1) = (1-\alpha)K_2\left(\frac{K_1}{K_2}\frac{\beta_{22}}{\beta_{21}}\underline{k}\right)^{\beta_{21}} \geq \underline{k}$$

or

$$\underline{k} \leq ((1-\alpha)K_2)^{\frac{1}{\beta_{22}}}\left(\frac{K_1}{K_2}\frac{\beta_{22}}{\beta_{21}}\right)^{\frac{\beta_{21}}{\beta_{22}}}.$$

Again, $\underline{k}$ as required by assumption 5 exists.

In the case of a C.E.S utility function, there is no explicit demand function, and the treatment becomes less elegant.

We shall use the sufficient condition suggested in the text.

When consumption is capital intensive, we want to show that we can find a $\underline{k}$ such that

$$v(c_1,\underline{c}_2) \geq \frac{\underline{p}(k)}{\underline{r}} \text{ for all } k\varepsilon[\underline{k},\overline{k}],\ p(t)\varepsilon[\underline{p}(\underline{k}),\overline{p}(\overline{k})]$$

where

$$\underline{r} = r(\overline{p}(\overline{k})),$$

$$\underline{c}_2 = \underline{r}\cdot\underline{k},$$

$$c_1 = w(p) - p\cdot\underline{k}, \text{ and}$$

$$\underline{p}(k) \text{ solves } \underline{k} = S(p,k).$$

For convenience we require even more; $\underline{p}(k)$ is increasing in k, so let the right-hand side be the highest value, namely $\frac{\underline{p}(\overline{k})}{\underline{r}}$.

Examine now $w(p) - p\cdot\underline{k}$.

$$c_1 = K_2 p^{\frac{\beta_{11}}{\delta}} - p\cdot\underline{k} > 0 \Rightarrow p > \left(\frac{\underline{k}}{K_2}\right)^{\frac{\delta}{\beta_{21}}}.$$

Let $p_0 = \left(\frac{\underline{k}}{K_2}\right)^{\frac{\delta}{\beta_{21}}}$.

Find now a $\underline{k}$ such that the price at which the country specializes in the consumption good is higher than p_0.

$$\underline{p}(\underline{k}) = \left(\frac{K_1}{K_2}\,\frac{\beta_{12}}{\beta_{11}}\underline{k}\right)^{\delta} > \left(\frac{\underline{k}}{K_2}\right)^{\frac{\delta}{\beta_{21}}}$$

$$\underline{k} < \left(\frac{K_1}{K_2}\,\frac{\beta_{12}}{\beta_{11}}\right)^{\frac{\beta_{21}}{\beta_{22}}}\cdot K_2^{\frac{1}{\beta_{22}}}.$$

For any such $\underline{k}$, solve $\underline{p}(\underline{k})$ from $\underline{k} = S(p,\underline{k})$. Since c_1 is an increasing function of p for all $p > p_0$, it suffices to require:

$$\frac{(1-\alpha)}{\alpha}\left(\frac{w(\underline{p}(\underline{k}))-\underline{p}(\underline{k})\cdot\underline{k}}{\underline{r}\cdot\underline{k}}\right)^{\delta+1} \geq \frac{\underline{p}(\overline{k})}{\underline{r}}.$$

So far we have fixed all parameters except α. For some α small enough the inequality will be obtained as needed.

When capital is capital intensive, we examine the demand at $p(t+1) = \underline{p}(\overline{k})$ and $p(t) = \overline{p}(\underline{k})$.

The condition is

$$v(\overline{c}_1, \underline{c}_2) \geq \frac{\overline{p}(\underline{k})}{\underline{r}}$$

where

$$\underline{r} = r(\underline{p}(\overline{k})),$$

$$\underline{c}_2 = \underline{r} \cdot \underline{k},$$

$$\overline{c}_1 = w(\overline{p}(\underline{k})) - \overline{p}(\underline{k}) \cdot \underline{k}, \text{ and}$$

$$\underline{p}(k) \text{ solves } \underline{k} = S(p,k).$$

We have again to guarantee that $c_1 > 0$.

$$K_2 p^{\frac{\beta_{11}}{\delta}} - p\underline{k} > 0 \Rightarrow p < \left(\frac{\underline{k}}{K_2}\right)^{\frac{\delta}{\beta_{21}}}; \; p_0 = \left(\frac{\underline{k}}{K_2}\right)^{\frac{\delta}{\beta_{21}}}.$$

Now we require that $\underline{k}$ be such that the price in which the country specializes in k be smaller than p_0.

$$\overline{p}(\underline{k}) = \left(\frac{K_1}{K_2} \frac{\beta_{22}}{\beta_{21}} \underline{k}\right)^{\delta} < \left(\frac{\underline{k}}{K_2}\right)^{\frac{\delta}{\beta_{21}}}$$

$$\underline{k} < \left(\frac{K_1}{K_2} \frac{\beta_{22}}{\beta_{21}}\right)^{\frac{\beta_{21}}{\beta_{22}}} K_2^{\frac{1}{\beta_{22}}}.$$

For any $\underline{k}$ which satisfies this requirement, calculate $\overline{p}(\underline{k})$ to find $\overline{c}_1$ and again fix α such that assumption 3 is obtained.

Note that in the procedure described above demand could be made bigger than any $\underline{k}$ small enough by making the marginal utility of c_2 dominate the marginal utility of c_1 at small values of consumption.

For assumption 4, I rewrite the derivatives as follows:

$$D_{p(t)} - S_{p(t)} =$$

$$\frac{-(\rho+1)\frac{k(t)}{c_1(t)}(r'+p(t)r'') - (\rho+1)\frac{p(t)}{c_1(t)}w'' - \frac{1}{p(t)} - (\rho+1)\frac{r(t+1)}{c_2(t)}(r''k(t)+w'')}{(\rho+1)(\frac{p(t)}{c_1(t)} + \frac{r(t+1)}{c_2(t)})}$$

$$D_{p(t+1)} = \frac{-\frac{r'}{r}\rho}{(\rho+1)(\frac{p(t)}{c_1(t)} + \frac{r(t+1)}{c_2(t)})}$$

$$S_{k(t)} = r'.$$

Condition i) of assumption 6 requires:

$$\left|\frac{D_{p(t+1)}S_{k(t)}}{D_{p(t)}-S_{k(t)}}\right| - 1 < 0 \quad \text{for all } p(t),\ p(t+1)\varepsilon P,\ k(t)\varepsilon K.$$

Clearly, when $\rho = 0$, the condition is fulfilled since then $D_{p(t+1)} = 0$. Due to continuity there exists an $\varepsilon_1 > 0$ such that $|\rho| < \varepsilon_1$ fulfills the condition.

Condition ii) requires:

$$\left(\left|\frac{D_{p(t+1)}S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right|-1\right)^2 - 4\left|\frac{D_{p(t+1)}S_{p(t)}}{D_{p(t)}-S_{p(t)}}\right|\left|\frac{S_{k(t)}}{D_{p(t)}-S_{p(t)}}\right| \geq 0.$$

Again, for $\rho = 0$ the condition is fulfilled, and there exists an $\varepsilon_2 > 0$ such that $|\rho| < \varepsilon_2$ still fulfills the condition. To fulfill both conditions, choose $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$ and let $|\rho| < \varepsilon$.