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Estimation of Dynamic Labor Demand Schedules Under Rational Expectations

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Robert Litterman most ably performed the rather involved calculations reported in this paper.

Both Keynes and various classical writers asserted that real wages would move countercyclically as employers moved along downward sloping schedules relating the employment-capital ratio to the real wage. Dunlop [1938] and Tarshis [1939] described evidence which they interpreted as failing to confirm a countercyclical pattern of real wage movements. That and subsequent evidence of a similar nature helped to stimulate attempts to describe aggregate employment and real wages by "disequilibrium models," the work of Barro and Grossman [1971] and Solow and Stiglitz [1968] being two prominent examples. However, most of that empirical evidence stemmed from fitting static regressions in attempts to test static theories of the demand for employment. The recent paper by Salih Neftci [1977], which computes long two-sided distributed lags between employment and real wages, indicates that there are complicated and statistically significant dynamic interactions between real wages and employment, at least in the post-World War II U.S. data.

This paper estimates a dynamic aggregative demand schedule for employment for post-war U.S. data. While the demand model makes employment depend inversely on the appropriate real wage, as does the static theory, a potentially rich dynamic structure is introduced into that dependence because firms are assumed to face costs of rapidly adjusting their labor force and so find it optimal to take into account future expected values of the real wage in determining their current employment. The model imposes overidentifying restrictions on the bivariate real wage-employment stochastic process and therefore can be tested given a single sample of data.

The model is formed by blending the costly adjustment model of Lucas [1967], Treadway [1969], and Gould [1969] with Lucas's static model of overtime work and capacity [1970]. The model is formulated so that it delivers linear decision rules relating the demand for straight-time employment and overtime employment each to the real wage process. The model imposes the rational expectations hypothesis, since firms are supposed to use the true moments of the real wage process in forming forecasts. The rational expectations hypothesis is a main source of the overidentifying restrictions imposed by the model.

In addition hopefully to providing some new evidence in the Dunlop-Tarshis tradition, this paper illustrates a technology for maximum likelihood estimation of decision rules under the hypothesis that expectations are rational. That technology potentially has a variety of applications.[1/]

1. The Demand for Employment

The model is formed by taking Lucas's model of overtime work and capacity [1970] and amending it to permit potentially different adjustment costs to be associated with rapidly changing straight-time and overtime labor.[2/] It is widely asserted that it is much cheaper to adjust the overtime labor force quickly than it is to adjust the straight-time labor force; consequently, it is alleged that overtime labor responds rapidly to the market signals that the firm receives, while the straight-time labor force responds more sluggishly. The model is designed to represent this phenomenon and to provide a framework for estimating its dimensions and testing it.

I shall work with a representative firm, although as I shall remark below, the model can handle certain kinds of diversity across firms. Following Lucas, suppose that the representative firm faces the instantaneous production function

$$y(t+\tau) = f(n(t+\tau),k(t+\tau)),\ f_n,\ f_k,\ f_{nk} > 0;\ f_{nn},\ f_{kk} < 0$$

$$t=0,\ 1,\ 2,\ 3,\ \ldots$$

$$\tau\varepsilon[0,1).$$

Here $y(t+\tau)$ is the rate of output per unit time at instant $t+\tau$, $n(t+\tau)$ is the number of employees at instant $t+\tau$, and $k(t+\tau)$ is the stock of capital at $t+\tau$. The length of the "day" is 1, so that t indexes days and τ indexes moments within the day. The firm is assumed to have a constant capital stock over the day so that

$$k(t+\tau) = k(t) \equiv k_t \text{ for } \tau\varepsilon[0,1).$$

The firm is assumed to be able to hire workers for a straight-time shift of fixed length $h_1 < 1$ at the real wage w_t during day t. During the overtime shift of length $h_2 = 1 - h_1$, the firm can hire all the labor it wants during day t at the real wage aw_t, where $a \approx 1.5$ is an overtime premium. Thus, for the first h_1 moments of day t the firm must pay workers w_t, while for the remaining h_2 moments it must pay aw_t. Confronted with these market opportunities it is optimal for the firm to choose to set $n(t+\tau) = n_{1t}$ for $\tau \varepsilon [0,h_1]$ and $n(t+\tau) = n_{2t}$ for $\tau \varepsilon (h_1,1)$. That is, it is optimal for the firm to choose a single level of straight-time employment n_{1t} during t and a single level of overtime employment of n_{2t} during the day t.

The firm's output over the "day" is then

$$y_t = \int_0^1 y(t+\tau)d\tau$$

$$= h_1 f(n_{1t},k_t) + h_2 f(n_{2t},k_t).$$

I take several steps to specialize this setup further. First, to simplify things, I assume that capital is constant over time so that k_t can be dropped as an argument from $f(\cdot,\cdot)$. (In the econometric work below, steps are taken to detrend the data prior to estimation partly in order to minimize the damage caused by this approximation.) Second, I assume a quadratic production function so that we write instantaneous output on the first and second shifts as

$$f(n_{1t},k) = (f_0+a_{1t})n_{1t} - \frac{f_1}{2}n_{1t}^2$$

$$f(n_{2t},k) = (f_0+a_{2t})n_{2t} - \frac{f_1}{2}n_{2t}^2$$

where f_0, $f_1 > 0$, and where a_{1t} and a_{2t} are exogenous stochastic processes affecting productivity of straight-time and overtime employment. We assume that $Ea_{1t} = Ea_{2t} = 0$. The stochastic processes a_{1t} and a_{2t} will be required to satisfy certain regularity conditions to be specified below.

The firm is assumed to bear daily costs of adjusting its straight-time labor force of $\frac{d}{2}(n_{1t}-n_{1t-1})^2$ and to bear daily costs of adjusting its overtime labor force of $\frac{e}{2}(n_{2t}-n_{2t-1})^2$. It is widely believed that it is substantially more expensive to adjust the straight-time labor force so that $d >> e$. The firm faces an exogenous stochastic process for the real wage of $\{w_t\}$. The firm's straight-time and overtime wage bills are, respectively, $w_t h_1 n_{1t}$ and $aw_t h_2 n_{2t}$.

The firm chooses contingency plans for n_{1t} and n_{2t} to maximize its expected real present value[3/]

$$(1) \qquad v_t = E_t \sum_{j=0}^{\infty} b^j [(f_0+a_{1t+j}-w_{t+j})h_1 n_{1t+j} - \frac{f_1}{2} h_1 n_{1t+j}^2$$
$$- \frac{d}{2}(n_{1t+j}-n_{1t+j-1})^2 + (f_0+a_{2t+j}-aw_{t+j})h_2 n_{2t+j}$$
$$- \frac{f_1}{2} h_2 n_{2t+j}^2 - \frac{e}{2}(n_{2t+j}-n_{2t+j-1})^2]$$

$$f_0,\ f_1,\ d,\ e > 0,\ a > 1,\ 0 < b < 1$$

where n_{1t-1} and n_{2t-1} as well as the stochastic processes for w, a_1, and a_2 are given to the firm. Here b is a real discount factor that lies between zero and one. The operator E_t is defined by $E_t x \equiv Ex|\Omega_t$ where x is a random variable, E is mathematical expectation, and Ω_t is an information set available to the firm at time t. I assume that Ω_t includes at least $\{n_{1t-1}, n_{2t-1}, a_{1t}, a_{1t-1}, \ldots, a_{2t}, a_{2t-1}, \ldots, w_t, w_{t-1}, \ldots\}$.

The firm is assumed to maximize (1) by choosing stochastic processes for n_1 and n_2 from the set of stochastic processes that are (nonanticipative) functions of the information set Ω_t. (Below, I will further restrict the class of stochastic processes over which the optimization is carried out.) I assume that the stochastic processes w_t, a_{1t}, and a_{2t} are of exponential order less than $(\frac{1}{b})$, which means that for some $K > 0$

$$|E_t w_{t+j}| < K(\tfrac{1}{b})^j$$

$$|E_t a_{1t+j}| < K(\tfrac{1}{b})^j$$

$$|E_t a_{2t+j}| < K(\tfrac{1}{b})^j$$

for all t and all $j \geq 0$.

First-order necessary conditions for the maximization of (1) consist of a set of "Euler equations" and a pair of transversality conditions.[4/] The Euler equations for $\{n_{1t}\}$ and $\{n_{2t}\}$ are

$$bE_{t+j}n_{1t+j+1} + \phi_1 n_{1t+j} + n_{1t+j-1} = \frac{h_1}{d}(w_{t+j} - a_{1t+j} - f_0)$$

$$j=0,\ 1,\ 2,\ \ldots$$

(2)

$$bE_{t+j}n_{2t+j+1} + \phi_2 n_{2t+j} + n_{2t+j-1} = \frac{h_2}{e}(aw_{t+j} - a_{2t+j} - f_0)$$

$$j=0,\ 1,\ 2,\ \ldots$$

where

$$\phi_1 = -(\frac{f_1 h_1}{d} + (1+b))$$

(3)

$$\phi_2 = -(\frac{f_2 h_2}{e} + (1+b)).$$

The transversality conditions are

$$(4) \qquad \lim_{T\to\infty} b^T E_t n_{1t+T} = \lim_{T\to\infty} b^T E_t n_{2t+T} = 0.$$

To solve the Euler equations for the optimum contingency plans, first obtain the factorizations

$$(5) \qquad (1 + \frac{\phi_1}{b}z + \frac{1}{b}z^2) = (1-\delta_1 z)(1-\delta_2 z)$$

$$(6) \qquad (1 + \frac{\phi_2}{b}z + \frac{1}{b}z^2) = (1-\mu_1 z)(1-\mu_2 z).$$

Given the assumptions about the signs and magnitudes of the parameters composing b, ϕ_1, and ϕ_2, it follows that factorizations exist with $0 < \delta_1 < 1 < \frac{1}{b} < \delta_2$ and $0 < \mu_1 < 1 < \frac{1}{b} < \mu_2$. It then follows that solutions of the Euler equations that satisfy the transversality conditions and the initial conditions are given by 5/

$$(7) \quad \begin{aligned} &\text{(a)} \quad n_{1t} = \delta_1 n_{1t-1} - \frac{\delta_1 h_1}{d} \sum_{i=0}^{\infty} (\frac{1}{\delta_2})^i E_t(w_{t+i} - a_{1t+i} - f_0) \\ &\text{(b)} \quad n_{2t} = \mu_1 n_{2t-1} - \frac{\mu_1 h_2}{e} \sum_{i=0}^{\infty} (\frac{1}{\mu_2})^i E_t(aw_{t+i} - a_{2t+i} - f_0). \end{aligned}$$

It can be verified directly that these solutions satisfy the Euler equations and the transversality conditions. The polynomial equation (5) implicitly defines δ_1 and δ_2 as functions of $\frac{f_1 h_1}{d}$. By studying this polynomial,6/ it is possible to show that δ_1 is a decreasing function of $\frac{f_1 h_1}{d}$ and that $\frac{1}{\delta_2} = b\delta_1$. It follows that δ_1 and $\frac{1}{\delta_2}$ both increase with increases in the adjustment cost parameter d. Reference to equation (7a) then shows that increases in the adjustment cost parameter d, by increasing δ_1 and $\frac{1}{\delta_2}$, decrease the speed with which the firm responds to the real wage and productivity signals that it receives. Similarly, μ_1 and $\frac{1}{\mu_2}$ are decreasing functions of $\frac{f_1 h_2}{e}$, and $\frac{1}{\mu_2} = b\mu_1$.

Equations (7) are decision rules for setting n_{1t} and n_{2t} as linear functions of n_{1t-1}, n_{2t-1}, and the conditional expectations $E_t w_{t+i}$, $E_t a_{1t+i}$, and $E_t a_{2t+i}$, i=0, 1, 2, ... However, in general, these conditional expectations are nonlinear functions of the information in Ω_t. Given particular stochastic processes for w_t, a_{1t}, and a_{2t}, equations (7) can be solved for decision rules expressing n_{1t} and n_{2t} as, in general, nonlinear functions of Ω_t.

For the purposes of empirical work, it is convenient to restrict ourselves to the class of decision rules that are linear functions of Ω_t. The optimal linear decision rules can be obtained by replacing the conditional mathematical expectations in (7) with the corresponding linear least squares projections on the information set Ω_t. Accordingly, henceforth, I will interpret E as the linear least squares projection operator.

To derive from (7) explicit decision rules for n_{1t} and n_{2t} as functions of Ω_t, it is necessary further to restrict the stochastic processes w_t, a_{1t}, and a_{2t}. I assume that a_{1t} and a_{2t} are each first-order Markov processes for which

$$
(8)\qquad \begin{aligned} E_t a_{1t+i} &= \rho_1^i a_{1t} \qquad i \geq 0 \\ E_t a_{2t+i} &= \rho_2^i a_{2t} \qquad i \geq 0 \end{aligned}
$$

where $|\rho_1| < \frac{1}{b}$, $|\rho_2| < \frac{1}{b}$. I assume that w_t is an n^{th}-order Markov process

$$
(10)\qquad w_t = v_0 + v_1 w_{t-1} + v_2 w_{t-2} + \ldots + v_n w_{t-n} + \xi_{3t}
$$

where ξ_{3t} is a least squares disturbance that satisfies $E_{t-1}\xi_{3t} \equiv E\xi_{3t}|\Omega_{t-1} = 0.$[7/] It is convenient to represent the n^{th}-order process

(10) as the (n+1)-vector first-order Markov process

$$x_t = Ax_{t-1} + \varepsilon_t$$

where

$$x_t = \begin{bmatrix} w_t \\ w_{t-1} \\ w_{t-2} \\ \cdot \\ \cdot \\ \cdot \\ w_{t-n} \\ 1 \end{bmatrix} \qquad \varepsilon_t = \begin{bmatrix} \xi_{3t} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n & v_0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 0 & \ldots\ 1 & 0 & 0 \\ 0 & 0 & \ldots\ 0 & 0 & 1 \end{bmatrix} .$$

We can write

$$x_{t+1} = Ax_t + \varepsilon_{t+1}$$

$$x_{t+2} = A^2 x_t + \varepsilon_{t+2} + A\varepsilon_{t+1}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$x_{t+j} = A^j x_t + \varepsilon_{t+j} + A\varepsilon_{t+j-1} + \ldots A^{j-1}\varepsilon_{t+1}.$$

Since $E_t \varepsilon_{t+k} = 0$ for $k \geq 1$, we have

$$E_t x_{t+j} = A^j x_t.$$

Assume that the eigenvalues of A are distinct so that A can be written as

$$A = P \Lambda P^{-1}$$

where the columns of P are the eigenvectors of A and Λ is the diagonal matrix whose elements are the eigenvalues of A.[8] Then we have

$$E_t x_{t+j} = P \Lambda^j P x_t.$$

Finally, let c be the 1x(n+1) row vector (1,0,0,...,0) so that $w_t = cx_t$. We thus have that

$$E_t w_{t+j} = cP\Lambda^j P^{-1} x_t. \tag{11}$$

Substituting from (8) and (11) into (7a) gives[9]

$$n_{1t} = \delta_1 n_{1t-1} - \frac{\delta_1 h_1}{d} cP \sum_{i=0}^{\infty} \left(\frac{1}{\delta_2}\Lambda\right)^i P^{-1} x_t$$
$$+ \frac{\delta_1 h_1}{d}\left(\frac{f_0}{1 - \frac{1}{\delta_2}}\right) + \frac{\delta_1 h_1}{d}\left(\frac{1}{1 - \frac{\rho_1}{\delta_2}}\right) a_{1t}.$$

Let λ_i be the iith element of Ω. Since $\delta_2 = \frac{1}{\delta_1 b}$, we have that $\left|\frac{\lambda_1}{\delta_2}\right| = |\lambda_i \delta_1 b| < 1$ by virtue of the assumption that w_t is of exponential order less than $\frac{1}{b}$, i.e., that $|\lambda_i \cdot b| < 1$. Then the infinite sum above converges and we can write

$$(12) \qquad n_{1t} = \delta_1 n_{1t-1} - \frac{\delta_1 h_1}{d} cP\left[\frac{1}{1-\frac{\lambda_i}{\delta_2}}\right]_{ii} P^{-1} x_t$$

$$+ \frac{\delta_1 h_1}{d}\left(\frac{f_0}{1-\frac{1}{\delta_2}}\right) + \frac{\delta_1 h_1}{d}\left(\frac{1}{1-\frac{\rho_1}{\delta_2}}\right) a_{1t}$$

where $\left[\frac{1}{1-\frac{\lambda_i}{\delta_2}}\right]_{ii}$ is a diagonal matrix with $(1 - \frac{\lambda_i}{\delta_2})$ as the i^{th} diagonal element.

Let us write (12) as[10/]

$$(13) \qquad n_{1t} = \delta_1 n_{1t-1} + \alpha_1 w_t + \alpha_2 w_{t-1} + \ldots + \alpha_n w_{t-n+1} + \alpha_0$$

$$+ \frac{\delta_1 h_1}{d}\left(\frac{f_0}{1-\delta_1 b}\right) + a_{1t}$$

where

$$(14) \qquad (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_0) = - \frac{\delta_1 h_1}{d} cP\left[\frac{1}{1-\lambda_i \delta_1 b}\right]_{ii} P^{-1}$$

$$a'_{1t} = \frac{\delta_1 h_1}{d}\left(\frac{1}{1-\rho_1 \delta_1 b}\right) a_{1t}.$$

Proceeding in the same way, we can write the decision rule for n_{2t} as

$$(15) \qquad n_{2t} = \mu_1 n_{2t-1} + \beta_1 w_t + \beta_2 w_{t-1} + \ldots + \beta_n w_{t-n+1} + \beta_0$$

$$+ \frac{\mu_1 h_2}{e}\left(\frac{f_0}{1-\mu_1 b}\right) + a'_{2t}$$

where

$$(16) \qquad (\beta_1, \beta_2, \ldots, \beta_n, \beta_0) = - \frac{\mu_1 h_2}{e} cP\left[\frac{1}{1-\lambda_i \mu_1 b}\right] P^{-1}$$

$$a'_{2t} = \frac{\mu_1 h_2}{e}\left(\frac{1}{1-\rho_2 \mu_1 b}\right) a_{2t}.$$

Equations (14) and (16) succinctly summarize how the distributed lag coefficients, the α's and β's, reflect the combination of forecasting (through the parameters of P and Λ) and optimization (through the parameters d, δ, and μ) elements. Clearly, the decision rules (13) and (15) are <u>not</u> invariant with respect to the stochastic process for real wages (8), a general characteristic of optimum decision rules whose far reaching implications for econometric policy evaluation have been stressed by Robert E. Lucas, Jr., [1976].

Since I will fit the model to data that are deviations from means and trends, I shall henceforth drop the constants from (13), (15), and (10). Substitute (10) for w_t and subtract $\rho_1 a'_{1t-1}$ from both sides of (13) to get

$$
\begin{aligned}
(17) \qquad n_{1t} = {} & (\delta_1+\rho_1)n_{1t-1} - \rho_1\delta_1 n_{1t-2} + (\alpha_2+\alpha_1 v_2-\alpha_2\rho_1)w_{t-1} \\
& + (\alpha_3+\alpha_1 v_2-\alpha_2\rho_1)w_{t-2} + \ldots + (\alpha_n+\alpha_1 v_{n-1}-\alpha_{n-1}\rho_1)w_{t-n+1} \\
& + (\alpha_1 v_n-\alpha_n\rho_1)w_{t-n} + [\alpha_1\xi_{3t}+(a'_{1t}-\rho_1 a'_{1t-1})].
\end{aligned}
$$

From our earlier assumptions, $E_{t-1}[\alpha_1\xi_{3t}+(a'_{1t}-\rho_1 a_{1t-1})] = 0$, so that (17) is the (vector) autoregression for n_{1t}. In particular, we have

$$
\begin{aligned}
(18) \qquad E_{t-1}n_{1t} = {} & (\delta_1+\rho_1)n_{1t-1} - \rho_1\delta_1 n_{1t-1} + (\alpha_2+\alpha_1 v_2-\alpha_2\rho_1)w_{t-1} \\
& + (\alpha_3+\alpha_1 v_2-\alpha_2\rho_1)w_{t-2} + \ldots + (\alpha_n+\alpha_1 v_{n-1}-\alpha_{n-1}\rho_1)w_{t-n+1} \\
& + (\alpha_1 v_n-\alpha_n\rho_1)w_{t-n}.
\end{aligned}
$$

Similarly, we have for n_{2t}

$$(19)\qquad n_{2t} = (\mu_1+\rho_2)n_{2t-1} - \rho_2\mu_1 n_{2t-2} + (\beta_2+\beta_1 v_2-\beta_2\rho_2)w_{t-1} + (\beta_3+\beta_1 v_2-\beta_2\rho_2)w_{t-2} +\ldots+ (\beta_n+\beta_1 v_{n-1}-\beta_{n-1}\rho_2)w_{t-n+1} + (\beta_1 v_n-\beta_n\rho_2)w_{t-n} + [\beta_1\xi_{3t}+(a'_{2t}-\rho_2 a'_{2t-1})].$$

We can now write the complete three-variate vector autoregression for n_{1t}, n_{2t}, w_t as

$$\text{(a)}\quad n_{1t} = (\delta_1+\rho_1)n_{1t-1} - \rho_1\delta_1 n_{1t-2} + (\alpha_2+\alpha_1 v_2-\alpha_2\rho_1)w_{t-1} + (\alpha_3+\alpha_1 v_2-\alpha_2\rho_1)w_{t-1} +\ldots+ (\alpha_n+\alpha_1 v_{n-1}-\alpha_{n-1}\rho_1)w_{t-n+1} + (\alpha_1 v_n-\alpha_n\rho_1)w_{t-n} + u_{1t}$$

$$(20)\ \text{(b)}\quad n_{2t} = (\mu_1+\rho_2)n_{2t-1} - \rho_2\mu_1 n_{2t-2} + (\beta_2+\beta_1 v_2-\beta_2\rho_2)w_{t-1} + (\beta_3+\beta_1 v_2-\beta_2\rho_2)w_{t-2} +\ldots+ (\beta_n+\beta_1 v_{n-1}-\beta_{n-1}\rho_2)w_{t-n+1} + (\beta_1 v_n-\beta_n\rho_2)w_{t-n} + u_{2t}$$

$$\text{(c)}\quad w_t = v_1 w_{t-1} + v_2 w_{t-2} +\ldots+ v_n w_{t-n} + u_{3t}$$

where

$$u_t \equiv \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix} \equiv \begin{bmatrix} \alpha_1\xi_{3t}+(a'_{1t}-\rho_1 a'_{1t-1}) \\ \beta_1\xi_{3t}+(a'_{2t}-\rho_2 a'_{2t-1}) \\ \xi_{3t} \end{bmatrix} = \begin{bmatrix} n_{1t}-E_{t-1}n_{1t} \\ n_{2t}-E_{t-1}n_{2t} \\ w_t-E_{t-1}w_t \end{bmatrix}.$$

Here u_t is the vector of innovations, i.e., errors in predicting (n_{1t},n_{2t},w_t) from past information. There are (3n+4) regressors in (20), i.e.,

w_{t-1}, ..., w_{t-n}, each of which appear three times, and n_{1t-1}, n_{1t-2}, n_{2t-1}, and n_{2t-2}, each of which appears once. The free parameters of the model are f_1, d, e, ρ_1, ρ_2, v_1, ..., v_n, so that there are (n+5) parameters to be estimated. As it turns out, the model is overidentified for $n \geq 1$.

Collecting the equations that summarize the restrictions that the model imposes on the vector autoregression (20), we have

$$
(21)\qquad \begin{cases}
\phi_1 = -(\frac{f_1 h_1}{d} + (1+b)) \\
\phi_2 = -(\frac{f_1 h_2}{e} + (1+b)) \\
(1 + \frac{\phi_1}{b}z + \frac{1}{b}z^2) = (1-\delta_1 z)(1-\delta_2 z) \\
(1 + \frac{\phi_2}{b}z + \frac{1}{b}z^2) = (1-\mu_1 z)(1-\mu_2 z) \\
(\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_0) = -\frac{\delta_1 h_1}{d} cP[\frac{1}{1-\lambda_i \delta_1 b}]_{ii} P^{-1} \\
(\beta_1, \beta_2, \ldots, \beta_n, \beta_0) = -\frac{\mu_1 h_2}{e} cP[\frac{1}{1-\lambda_i \mu_1 b}]_{ii} P^{-1} \\
A = P \Lambda P^{-1}.
\end{cases}
$$

Estimates of the free parameters $\phi = (f_1, d, e, \rho_1, \rho_2, v_1, \ldots, v_n)$ are obtained by using the method of maximum likelihood to estimate the vector autoregression (20), subject to (21).[11/] Let $\hat{u}_t = (\hat{u}_{1t}, \hat{u}_{2t}, \hat{u}_{3t})$ be the sample residual vector associated with the parameter values θ. Under the assumption that u_t is a trivariate normal vector with $Eu_t u_t' = V$, the likelihood function of a sample of observations on the residuals extending over t=1, ..., T is

$$
(22)\qquad L(\theta) = (2\Pi)^{-\frac{1}{2}3T} |V|^{-\frac{1}{2}} \exp(-\frac{1}{2}\sum_{t=1}^{T} \hat{u}_1' V^{-1} \hat{u}_t).
$$

As shown by Wilson [1973] and Bard [1974], maximum likelihood estimates of θ with V unknown can be obtained by minimizing $|\hat{V}|$ with respect to θ, where $\hat{V}$ is the sample covariance matrix of u_t,

$$\hat{V} = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t \hat{u}_t'.$$

The matrix $\hat{V}$ is the maximum likelihood estimator of V (see Wilson [1973] or Bard [1974]).

Now consider the unconstrained version of the vector autoregression (20) in which each of the (3n+4) regressors has its own free parameter. Let L_u be the value of the likelihood function at its unrestricted maximum, i.e., the maximum obtained by permitting each of the (3n+4) regressors to have its own free parameter. Let L_r be the value of the likelihood under the restrictions (21). Then $-2 \log_e(L_r/L_u)$ is asymptotically distributed as $\chi^2(q)$ where $q = (3n+4) - (n+5)$ is the number of restrictions imposed by the theory. High values of the likelihood ratio lead to rejection of the restrictions that the theory imposes on the vector autoregression. Using the calculations of Wilson [1973, p. 80] or Bard [1974], it can be shown that the likelihood ratio is equal to

$$T\{\log_e|\hat{V}_r| - \log_e|\hat{V}_u|\}$$

where $\hat{V}_r$ and $\hat{V}_u$ are the restricted and unrestricted estimates of V, respectively.

2. Alternative Estimation Strategies

It should be stressed that the vector autoregression (20) which builds in the cross-equation restrictions implied by the model has been obtained under the assumption (8) that the productivity shocks a_{1t} and a_{2t} are first-order Markov processes. The forms of the vector autoregressions (20) would be altered had we assumed other forms for the a_{1t} and a_{2t} processes, as the reader can verify by calculations paralleling those above.

An alternative estimation strategy is available that avoids the necessity to make specific assumptions about the forms of the stochastic processes for the disturbances a_{1t} and a_{2t}, only requiring that these processes be covariance stationary. The alternative estimator requires instead that the w_t process be strictly econometrically exogenous with respect to n_{1t} and n_{2t}, in particular requiring that $Ew_t a_{1s} = Ew_t a_{2s} = 0$ for all t and s. Under that assumption, the model (7a) and (7b) can readily be shown to place restrictions on the projections of n_{1t} and n_{2t}, respectively, on the entire $\{w_s\}$ process. The structure of those restrictions parallels those worked out by Sargent [1977a] for a consumption function example. An asymptotically efficient estimator such as "Hannan's efficient estimator," which allows for complicated serial correlation patterns in the disturbances, could then be applied to estimating the projections with and without the restrictions imposed by the model.

This alternative estimation strategy gets along with much weaker assumptions about the serial correlation properties of the disturbance processes $\{a_{1t}\}$ and $\{a_{2t}\}$ at the cost of making somewhat more stringent assumptions about the exogeneity of w_t, i.e., about the correlation between w_t and the a_{js}'s. The original estimator proposed that operates on (20) does assume that $\{w_t\}$ is a process that is not caused in Granger's [1969] sense by n_{1t} or n_{2t}, i.e., that $E[w_t | w_{t-1}, w_{t-2}, \ldots, n_{1t-1}, n_{1t-2}, \ldots, n_{2t-1}, n_{2t-2}, \ldots] = E[w_t | w_{t-1}, w_{t-2}, \ldots]$. Now Sims' [1972] theorems assure us that if w_t is not Granger-caused by n_{1t} or n_{2t}, then there exists a statistical representation in which w_t is strictly econometrically exogenous with respect to n_{1t} or n_{2t}. However, this statistical representation need not correspond with the appropriate economic behavioral relationship. It is possible for n_{1t} or n_{2t} to fail to cause w_t, and yet for "instantaneous causality" to flow from n_{1t} or n_{2t} to w_t so that w_t is not strictly exogenous in the appropriate model. See Sargent [1977b] for an example of this phenomenon within the context of Cagan's model of hyperinflation. The "autoregressive estimator" based on (20) permits arbitrary correlation between the innovations to n_{1t} or n_{2t} and w_t and makes no assumption about which pattern of instantaneous causality explains that correlation. On the other hand, the alternative "projection estimator" attributes all of that correlation to the workings of the demand schedules for n_{1t} and n_{2t}, (7a) and (7b). For the present application, I prefer the estimator that makes the weaker assumption about the correlation between innovations to employment and the real wage.

The reader by now will have understood that optimizing, rational expectations models do not entirely eliminate the need for side assumptions not grounded in economic theory. Some arbitrary assumptions about the nature of the serial correlation structure of the disturbances and/or about strict econometric exogeneity are necessary in order to proceed with estimation.

Perhaps I should conclude this section by pointing to another source of arbitrariness, namely the latitude at our disposal in specifying the firm's optimization problem. For example, adding terms like $-\frac{d_2}{2}(n_{1t-1}-n_{1t-2})^2$ to the firm's daily profits would lead to Euler equations that are fourth-order stochastic difference equations and would lead to decision rules that depend on two lagged values of employment. Such specifications would seem plausible and would lead to materially different restrictions than those above on vector autoregressions (or projections of n on w, as the case may be). There are clearly limits set by the requirements of econometric identification on our ability to estimate such complicated adjustment cost parameterizations. Identification problems in such models have as yet received little attention at a general level.

The general theme of this section has been to issue a warning that rational expectations, optimizing models will not be able to save us entirely from the <u>ad hoc</u> assumptions and interpretations made in applied work. However, this is not to deny that the rational expectations hypothesis seems promising as a device for organizing restrictions on parameterizations of econometric models.

3. Parameter Estimates

The model was estimated using quarterly data on total civilian employment and a straight-time real wage index, with the period of observation for the dependent variables extending from 1947I through 1972IV. The variable n_{1t} was in the first instance measured by the seasonally adjusted BLS series "Employees on Nonagricultural Payrolls, Private and Government." To get a measure of n_{2t}, the following procedure was used. I defined the variable $\bar{h}_t$ to be average weekly hours, a series measured by the seasonally adjusted BLS series "Average Weekly Hours in Manufacturing." I then estimated total manhours by $\bar{h}_t n_{1t}$. Finally, I measured n_{2t} by

$$n_{2t} = \frac{\bar{h}_t n_{1t} - h_1 n_{1t}}{h_2}$$

where h_1 and h_2 were set a priori at 37 and 17, respectively.[12/] The real wage w_t was measured by deflating the seasonally unadjusted BLS series "Average Hourly Earnings: Straight-time Manufacturing Production Workers" by the seasonally unadjusted Consumer Price Index (1967=100).

I also created seasonally unadjusted measures of n_{1t} and n_{2t} by taking as a measure of n_{1t} the seasonally unadjusted BLS series "Employees on Private Nonagricultural Payrolls" and then using the preceding procedure to create estimates of n_{2t} by using the seasonally unadjusted average weekly hours series. The data are quarterly averages of monthly data. Notice that h_1 and h_2 are constants that are independent of time.

I begin by describing the estimates obtained using the seasonally adjusted employment series together with the seasonally unadjusted real wage series. (Later I will describe the results obtained with the seasonally unadjusted series for all variables.) Before estimating the model, the data on n_{1t} and n_{2t} were each detrended by regressing them on a constrant, linear trend, and trend squared, and then using the residuals from those regressions as the data for estimating the model. The data on w_t were formed as the residuals from a regression on a constant, linear trend, trend squared, and three seasonal dummies. Two reasons can be given for detrending in this way prior to fitting the model. First, the model ignores the effects of capital on employment, except to the extent that these can be captured by the productivity processes a_{1t} and a_{2t}. Second, the theory predicts that any deterministic components of the employment and real wage processes will not be related by the same distributed lag model as are their indeterministic parts. Detrending prior to estimation is a device designed to isolate the indeterministic components. The real wage is measured in 1967 dollars, while employment is measured in millions of men.

Table 1 reports estimates of the vector autoregressions for (n_{1t},n_{2t},w_t) both unconstrained and constrained by the model. Each set of estimates was obtained by the method of maximum likelihood. We have set n equal to four, a fourth-order autoregression being used to model the real wage process. This means that the likelihood ratio statistic is asymptotically distributed as chi-square with $q = (3n+4) - (n+5) = 7$ degrees of freedom. The likelihood ratio is 7.3172, which has a marginal confidence level of .6034. The model thus passes the likelihood ratio test of its overidentifying restrictions at the usual significance levels.

The parameter estimates for the model are reported in Table 2. The free parameters were f_1, d, e, ρ_1, ρ_2, v_1, v_2, v_3, and v_4 with b being fixed at .95, h_1 at 37, and h_2 at 17. The decision rules associated with the maximum likelihood estimates are:

$$n_{1t} = .5782n_{1t-1} - 1.3781w_t + .0580w_{t-1} + .1098w_{t-2}$$

$$+ .2929w_{t-3} + a'_{1t}$$

$$n_{2t} = .1979n_{2t-1} - 4.2723w_t - .0065w_{t-1} - .0217w_{t-2}$$

$$+ .1725w_{t-3} + a'_{2t}.$$

Notice how both the shape of the distributed lag and the magnitude of the response to the real wage differs between straight-time and overtime employment. Overtime employment is more responsive to the real wage. Further, the straight-time adjustment cost parameter d is estimated to be much larger than the overtime adjustment parameter e. That is why n_{1t} depends more strongly on n_{1t-1} than does n_{2t} on n_{2t-1}.

Table 2 also reports the estimated covariance matrix of the innovations $V = Eu_t u'_t$. Recall that

$$\begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & \beta_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \end{bmatrix} \equiv B\xi_t$$

where $\xi_{1t} = a'_{1t} - \rho_1 a'_{1t-1}$, $\xi_{2t} = a'_{2t} - \rho_2 a'_{2t-1}$, and where

$$B = \begin{bmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & \beta_1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \xi_t = \begin{bmatrix} 1t \\ 2t \\ 3t \end{bmatrix}.$$

Then, since $\xi_t = B^{-1}u_t$, the covariance matrix of ξ_t can be estimated from $E\xi_t\xi_t' = B^{-1}VB^{-1'}$, an estimate of which is also reported in Table 2. The correlation between the innovations to a_{1t}' and to a_{2t}', i.e., ξ_{1t} and ξ_{2t}, is estimated to be .759. The correlation between the innovations to a_{1t} and w_t, i.e., ξ_{1t} and ξ_{3t} is only .0215, while that between ξ_{2t} and ξ_{3t} is .0127. I had expected ξ_{1t} and ξ_{2t} to be even more highly correlated than they are.

As it happens, the estimates reported in Tables 1 and 2 correspond to the higher of two local maxima of the likelihood function which I found. The parameter estimates associated with the lower of these two local maxima are reported in Table 3. In view of the form the vector autoregression (20), it is not at all surprising that the likelihood function should exhibit multiple maxima. In particular, notice that the coefficients in (20) on n_{1t-1}, n_{1t-2}, n_{2t-1}, n_{2t-2} are, respectively, $(\delta_1+\rho_1)$, $-\delta_1\rho_1$, $(\mu_1+\rho_2)$, and $-\mu_1\rho_2$. <u>If it were not</u> for the constraints across μ_1 and the β's and across δ_1 and the α's and the appearance of ρ_1 and ρ_2 elsewhere on the right side of (20), the parameters δ_1, ρ_1, μ_1, and ρ_2 would not be identified, since it would be impossible to distinguish the effects of δ_1 from ρ_1 and the effects of μ_1 from ρ_2. The presence of lagged w's on the right side of (20) and the aforementioned constraints resolve this identification problem but leave a vestige of it in the form of probable multiple peaks in the likelihood function with small samples. Comparing the parameter estimates in Tables 2 and 3 shows that Table 2 is a high (ρ_1,ρ_2) - low (δ_1,μ_1) solution, while Table 3 reports the high (δ_1,μ_1) - low (ρ_1,ρ_2) solution. Notice that for the Table 3 estimates, $\rho_1 + \delta_1 = 1.534$ and $\rho_1\delta_1 = .538$, while for the Table 2 estimates, $\rho_1 + \delta_1 = 1.516$ while $\rho_1\delta_1 = .542$.

The presence of multiple maxima of the likelihood function means that caution is called for in interpreting the test statistics reported, since the asymptotic distribution on which the test is computed does not predict multiple maxima for the likelihood function and so does not provide a very good approximation for the sample size that we are studying. The presence of multiple maxima of the likelihood function also argues for starting the nonlinear estimation from several different initial parameter estimates. I followed this practice in each case reported below and, in each case, found another lower local maximum of the likelihood function in addition to the one reported below. In each case there was a high (δ_1,μ_1) - low (ρ_1,ρ_2) solution and a high (ρ_1,ρ_2) low (δ_1,μ_1) solution.

Table 4 reports the estimates associated with the higher likelihood of two maxima found for the seasonally unadjusted data[13/] with n=4. In this case, the high (δ_1,μ_1) - low (ρ_1,ρ_2) solution had the higher likelihood. The estimates indicate d >> e and are qualitatively similar to those described above. The marginal confidence level is .6206, which indicates that the sample does not contain strong evidence against the null hypothesis.

Table 5 reports estimates with the seasonally unadjusted data with n=8. In this case, the likelihood function calls for high values of δ_1 and μ_1. The likelihood ratio statistic is now distributed asymptotically as chi-square with fifteen degrees of freedom under the null hypothesis that the model is correct. Once again the likelihood ratio statistic does not call for rejecting the model.

Table 6 shows the estimates obtained for the seasonally adjusted data for n=8. It is interesting that with n=8 the high δ_1, μ_1 estimates are the ones that maximize the likelihood function. The likelihood ratio test again fails to reject the model.

I also estimated a single-shift version of the model in which all of the terms in n_{2t} are dropped from the objective function (1). The result is a model restricting only n_{1t} and w_t and consisting of equations (20a) and (20c). The right side of (20a) and (20c) contain $2n + 2$ regressors, while the model possesses the $(n+3)$ free parameters $d, f_1, \rho_1, v_1, \ldots, v_n$. So the model places $q = (2n+2) - (n+3)$ overidentifying restrictions on the bivariate vector autoregression (20a), (20c).

Tables 7 and 8 report the parameter estimates for the seasonally adjusted and unadjusted data. The seasonally adjusted data indicate that the overidentifying restrictions are marginally to be rejected at the .95 confidence level, while they are marginally rejected at the .90 confidence level for the seasonally unadjusted data. The parameter point estimates continue to indicate that adjustment costs exert an important influence on the demand for employment.

Conclusions

The simple contemporaneous correlations that formed the evidence in the original Dunlop-Tarshis-Keynes exchange, and also in much of the follow-up empirical work done to date, are not sufficient to rule on the question of whether the time series are compatible with a model in which firms are always on their demand schedules for employment. This is true according to virtually any dynamic and stochastic theory of the demand for employment. In this paper, I have tried to indicate one way in which the time series evidence can be brought to bear on the question in the context of a simple dynamic, stochastic model. The empirical results are moderately comforting to the view that the employment-real wage observations lie along a demand schedule for employment. It is important to emphasize that this view has content (i.e., imposes overidentifying restrictions) because I have *a priori* imposed restrictions on the orders of the adjustment cost processes and on the Markov processes governing disturbances. At a general level without such restrictions, it is doubtful whether the equilibrium view has content.

Table 1

Vector Autoregressions, Seasonally Adjusted Data (n=4)

(20a)	Unconstrained	Constrained by (20)
n_{1t-1}	1.5128	1.5159
n_{1t-2}	-.5372	-.5422
w_{t-1}	-2.0287	.0225
w_{t-2}	-1.9667	.0512
w_{t-3}	2.9944	.0969
w_{t-4}	-1.0538	-.0343
(20b)		
n_{2t-1}	.9667	.9730
n_{2t-2}	-.1596	-.1534
w_{t-1}	-5.6155	-.8117
w_{t-2}	.5847	-.0297
w_{t-3}	1.9400	-.0988
w_{t-4}	-4.8444	.6114
(20c)		
w_{t-1}	.9307	.9635
w_{t-2}	-.0290	.0031
w_{t-3}	.1146	.0674
w_{t-4}	-.1907	-.1744
$\|V\|$	.5113199E-05	.55014E-05

$$T\{\log|V_r| - \log|V_u|\} = 7.3172$$

Marginal confidence level = .6034

Let X be a random variable distributed chi-square seven degrees of freedom, and let x be the value of the computed test statistic. Then the marginal confidence level is defined as Prob $\{X<x\}$ under the null hypothesis.

Table 2

Parameter Estimates
Seasonally Adjusted Data (n=4)

$f_1 = .2794$	$\rho_1 = .9377$
$d = 31.4283$	$\rho_2 = .7751$
$e = 1.4429$	$\mu_1 = .1979$
$\delta_1 = .5782$	
$\alpha_1 = -1.3781$	$\beta_1 = -4.2723$
$\alpha_2 = .0580$	$\beta_2 = -.0065$
$\alpha_3 = .1098$	$\beta_3 = -.02174$
$\alpha_4 = .2929$	$\beta_4 = .1725$

$$V = \begin{pmatrix} .9291E-01 & .2011E+00 & .1294E-02 \\ & .7746E+00 & .2089E-02 \\ & & .1939E-03 \end{pmatrix}$$

$$B^{-1}VB^{-1'} = \begin{pmatrix} .0968E+00 & .21061E+00 & .1561E-02 \\ & .7960E+00 & .5382E-03 \\ & & .1939E-03 \end{pmatrix}$$

Table 3

Seasonally Adjusted Data--Second Solution
Likelihood Equations

$f_1 = .03402$	$v_1 = .9461$
$d = 2367.94$	$v_2 = .0162$
$e = 10.4558$	$v_3 = -.1148$
$\rho_1 = .5426$	$v_4 = .0139$
$\rho_2 = .1552$	
$\delta_1 = .9910$	$\mu_1 = .8077$
$\alpha_1 = -3.4140$	$\beta_1 = -6.6658$
$\alpha_2 = -.3800$	$\beta_2 = .1880$
$\alpha_3 = -.3483$	$\beta_3 = .3530$
$\alpha_4 = -.7619$	$\beta_4 = -.3053$

$$V = \begin{pmatrix} .9491E-01 & .2053E+00 & .1242E-02 \\ & .7701E+00 & .1982E-02 \\ & & .1999E-03 \end{pmatrix}$$

$$B^{-1}VB^{-1'} = \begin{pmatrix} .10572E+00 & .2249E+00 & .19241E-02 \\ & .8054E+00 & .3315E-02 \\ & & .1999E-03 \end{pmatrix}$$

$|V_r| = .56353E-05, \; |V_u| = .511320E-05$

$T\{\log|V_r| - \log|V_u|\} = 9.7221$

Marginal confidence level = .7951

Table 4

Seasonally Unadjusted Data (n=4)

$f_1 = .4358$ $\qquad$ $v_1 = .8498$

$d = 3266.29$ $\qquad$ $v_2 = .1215$

$e = 75.6748$ $\qquad$ $v_3 = -.0219$

$\rho_1 = .3849$ $\qquad$ $v_4 = -.1198$

$\rho_2 = .0909$

$\delta_1 = .9512$ $\qquad$ $\mu_1 = .7475$

$\alpha_1 = -.0110$ $\qquad$ $\beta_1 = -.5620$

$\alpha_2 = .0091$ $\qquad$ $\beta_2 = .0409$

$\alpha_3 = .0114$ $\qquad$ $\beta_3 = .1259$

$\alpha_4 = .0124$ $\qquad$ $\beta_4 = .1650$

$$V = \begin{pmatrix} .1414E+00 & .2688E+00 & .1234E-02 \\ & .8189E+00 & .1631E-02 \\ & & .2340E-03 \end{pmatrix}$$

$$B^{-1}VB^{-1'} = \begin{pmatrix} .1414E+00 & .2695E+00 & .1237E-02 \\ & .8208E+00 & .1762E-02 \\ & & .2340E-03 \end{pmatrix}$$

$|V_r| = .96438,\ |V_u| = .89475E-05$

$T\{\log|V_r| - \log|V_u|\} = 7.4934$

Marginal confidence level = .6206

Table 5

Seasonally Unadjusted Data (n=8)

$f_1 = .3721$	$v_1 = .8388$	$v_5 = .2718$
$d = 3266.29$	$v_2 = .1512$	$v_6 = -.1369$
$e = 75.6750$	$v_3 = .0204$	$v_7 = -.0306$
$\rho_1 = .39930$	$v_4 = -.2786$	$v_8 = -.0340$
$\rho_2 = .0522$		
$\delta_1 = .9560$	$\mu_1 = .7651$	
$\alpha_1 = -.0282$	$\beta_1 = -.7194$	
$\alpha_2 = .0045$	$\beta_2 = -.0316$	
$\alpha_3 = .0092$	$\beta_3 = .0653$	
$\alpha_4 = .0108$	$\beta_4 = .1045$	
$\alpha_5 = .0040$	$\beta_5 = -.0567$	
$\alpha_6 = .0121$	$\beta_6 = .1176$	
$\alpha_7 = .0094$	$\beta_7 = .0633$	
$\alpha_8 = .0095$	$\beta_8 = .0650$	

$$V = \begin{pmatrix} .1370E+00 & .2756E+00 & .9515E-03 \\ & .8473E+00 & .1321E-02 \\ & & .2157E-03 \end{pmatrix}$$

$$B^{-1}VB^{-1'} = \begin{pmatrix} .1370E+00 & .2764E+00 & .9576E-03 \\ & .8494E+00 & .1476E-02 \\ & & .2157E-03 \end{pmatrix}$$

$|V_r| = .83442E-05$, $|V_u| = .79754E-05$

$T\{\log|V_r| - \log|V_u|\} = 4.3394$

Marginal confidence level = .0036

Table 6

Seasonally Adjusted Data (n=8)

f_1 = .2274	v_1 = .9117	v_5 = .0484
d = 2367.87	v_2 = .0727	v_6 = .0665
e = 67.3950	v_3 = .1036	v_7 = -.1518
ρ_1 = .5632	v_4 = -.2535	v_8 = .0337
ρ_2 = .1532		
δ_1 = .9608	μ_1 = .8044	
α_1 = -.5162	β_1 = -1.0803	
α_2 = -.0784	β_2 = -.0305	
α_3 = -.0484	β_3 = .0385	
α_4 = .0004	β_4 = .1624	
α_5 = -.1304	β_5 = -.0614	
α_6 = -.1178	β_6 = -.0280	
α_7 = -.0948	β_7 = .0351	
α_8 = -.1822	β_8 = -.1180	

$$V = \begin{pmatrix} .9443E\text{-}01 & .2156E\text{+}00 & .1152E\text{-}02 \\ & .8021E\text{+}00 & .2463E\text{-}02 \\ & & .1789E\text{-}03 \end{pmatrix}$$

$$B^{-1}VB^{-1'} = \begin{pmatrix} .9567E\text{-}01 & .2183E\text{+}00 & .1244E\text{-}02 \\ & .8076E\text{+}00 & .2656E\text{-}02 \\ & & .1789E\text{-}03 \end{pmatrix}$$

$|V_r|$ = .48162E-05, $|V_u|$ = .440883E-05

$T\{\log|V_r| - \log|V_u|\}$ = 8.484

Marginal confidence level = .0971

Table 7

Seasonally Adjusted, One-Shift Model

$f_1 = .0059$	$v_1 = .9342$
$d = 3.4108$	$v_2 = -.0063$
$\rho_1 = .7834$	$v_3 = -.0539$
$\lambda_1 = .7934$	$v_4 = -.0029$

$\alpha_1 = -26.3880$

$\alpha_2 = 1.0662$

$\alpha_3 = 1.2488$

$\alpha_4 = .2356$

$$V = \begin{pmatrix} .8359E-01 & .1204E-02 \\ & .1986E-03 \end{pmatrix}$$

$|V_r| = .15154E-04, \ |V_u| = .13983E-04$

$T\{\log|V_r| - \log|V_u|\} = 8.0402$

Marginal confidence level = .9548

Table 8

Seasonally Unadjusted, One-Shift Model

$f_1 = .0144$	$v_1 = .8476$
$d = 5.1839$	$v_2 = .0634$
$\rho_1 = .7002$	$v_3 = -.0388$
$\lambda_1 = .7417$	$v_4 = -.0147$

$\alpha_1 = -13.3288$

$\alpha_2 = -.1070$

$\alpha_3 = .6930$

$\alpha_4 = .4663$

$$V = \begin{pmatrix} .1267E+00 & .1224E-02 \\ & .2369E-03 \end{pmatrix}$$

$|V_r| = .28511E-04$, $|V_u| = .26757E-04$

$T\{\log|V_r| - \log|V_u|\} = 6.3492$

Marginal confidence level = .9042

Footnotes

1/ Applications of related methods are contained in Sargent [1977a, 1977b].

2/ Restrictions on the production function required to permit Lucas's static model to account for the cyclical behavior of labor productivity and real average hourly earnings were discussed by Sargent and Wallace [1974]. Adding differential costs for adjusting straight-time and overtime labor would widen the class of production functions that could lead to procyclical movements of average hourly earnings and labor productivity.

3/ Optimization problems of this form are discussed by Holt, Modigliani, Muth, and Simon [1960], Graves and Telser [1971], and Kwakernaak and Sivan [1972]. The treatment here closely follows that of Sargent [1977c]. It would be straightforward to carry along n firms, each facing the <u>same</u> wage process and operating under the <u>same</u> functional form for its objective function (1), yet each having <u>different</u> values for the parameters f_0, f_1, d, and e. It would then be straightforward to aggregate the Euler equations and their solutions (7). Thus, assuming a representative firm is only a convenience, as the model admits a tidy theory of aggregation.

4/ See Sargent [1977c], Chapters IX and XIV.

5/ See Sargent [1977c].

6/ See Sargent [1977c]. The solution (7) clearly exhibits the certainty-equivalence or separation property. That is, the same solution for n_{1t} and n_{2t} would emerge if we maximized the criterion formed by replacing $(a_{1t+j}, a_{2t+j}, w_{t+j})$ by $(E_t a_{1t+j}, E_t a_{2t+j}, E_t w_{t+j})$ and dropping the operator E_t from outside the sum in (1).

7/ The condition that $E\xi_{3t}|\Omega_{t-1} = 0$ is equivalent with the condition that w_t is not caused, in Granger's [1969] sense, by n_2 or n_1.

8/ The assumption that w_t is of exponential order less than $(\frac{1}{b})$ implies that the max $|\lambda_i| < (\frac{1}{b})$ where λ_i is the i^{th} element of Λ.

9/ Here we are using that $\left(\sum_{i=0}^{\infty} (\frac{1}{\mu_2})^i \rho_1^i\right) a_{1t} = \frac{1}{1 - \frac{1}{\mu_2}\rho_1} a_t$ since $|\rho_1| < \frac{1}{b}$ and $|\mu_2| > \frac{1}{b}$, so that the infinite sum converges.

10/ Engineers directly obtain solutions of the form (13) by solving matrix Ricatti equations, e.g., see Kwakernaak and Sivan [1972].

Footnotes, continued

In their jargon, our system is <u>not</u> "controllable" but is "stabilizable" and "detectable" so that convergence of iterations on the Ricatti equation is assured. The stabilizability of our system depends on $\{a_{1t}\}$, $\{a_{2t}\}$, and $\{w_t\}$ being of exponential order less than $(\frac{1}{b})$.

11/ The parameters f_0 and v_0 are dropped because the data are in the form of deviations from means and trend terms. The parameters b, h_1, and h_2 will be fixed <u>a priori</u>.

12/ That these values for h_1 and h_2 do not add to unity, as in the theoretical presentation of the model, amounts only to a harmless renormalization.

13/ With the seasonally unadjusted employment data, I first regressed each of n_{1t}, n_{2t}, and w_t against a constant, trend, trend squared, and three seasonal dummies and used the residuals from those regressions as the data.

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