

Historical

"Rational" Forecasts From "Nonrational" Models

Paul A. Anderson

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Abstract

This paper puts forward a method for simulating an existing macroeconomic model while maintaining the additional assumption that individuals form their expectations rationally. This simulation technique is a first response to Lucas' criticism that standard econometric policy evaluation allows policy rules to change but doesn't allow expectations rules to change as economic theory predicts they will. The technique is applied to a version of the St. Louis Federal Reserve Model with interesting results. The rational expectations version of the St. Louis Model exhibits the same neutrality with respect to certain policy rules as small, analytic rational expectations models considered by Lucas, Sargent, and Wallace.

"Rational" Forecasts from "Nonrational" Models

by Paul A. Anderson

The inability of large-scale macroeconomic models to predict the quantitative effects of alternative policy rules has been given a new explanation by Lucas [6]. He argues that the problem stems from the static expectations mechanisms embedded in most structural models. Theory implies that when a change in policy is undertaken, agents in the economy will revise the expectations rules which guide many of their current decisions. However, as Lucas pointed out, standard models are simulated under the assumption that producers and households will continue to act on the basis of forecasts generated by the outmoded rules which were considered optimal under the previous regime.

Whether these hypothesized changes in forecasting rules are important enough to invalidate current simulation methodology is an empirical question which should be of great concern to model builders and model users. Empirical investigation of this point involves some rather difficult inference problems. However, it seems reasonable to maintain that macroeconomic models which incorporate adjusting forecasting rules will represent the responses of the real economy more accurately than the static expectations models now employed.

A simple way to model adjusting forecast rules is to incorporate the rational expectations hypothesis, initially framed by Muth [9] and applied in the empirical work of Shiller [17] and Sargent [13], among others. Basically, the hypothesis is that individuals' expectations of future quantities are, on average, the true expected values of those quantities based on the data available. Careful empirical testing of

the rational expectations hypothesis and the closely allied natural rate hypothesis has not called for outright rejection, though much testing remains to be done. (See Sargent [] for an excellent survey.)

If the rational expectations hypothesis stands up to further empirical testing, it should be included in the behavioral specifications of models. The estimation of a large, rational expectations macro-econometric model is a costly, time-consuming project. While research on the rational expectations hypothesis continues, it seems useful to develop a method for simulating existing models under the added assumption that expectations adjust in an optimal manner.

We shall present one possible simulation method which incorporates a rationality postulate. This method can be implemented in any existing macroeconomic model. Policy simulations using our method may provide better forecasts than standard simulations. At the very least, comparisons of the two types of simulations will provide an indication of the extent to which the policy responses implied by the standard simulations depend on the "slow-learning" of economic agents.

This paper includes an application of our method to the St. Louis Federal Reserve Model. Policy simulations with this model indicate that the real effects of policy in standard simulations with the St. Louis Model derive almost solely from the slowness of agents to perceive policy changes.

Rational Forecasts from Nonrational Models

We propose a numerical method for producing forecasts subject to an additional constraint on the structure, viz. that expectations be rational. Let us consider a typical equation of a model, an equation which contains an explicit price expectation term.^{4/} Consider

$$(1) \quad ay_t + bz_t + cp_t^e = e_t$$

where y_t and z_t are vectors of endogenous and predetermined variables, respectively, p_t^e is the agent's forecast of price in period t , based on information available at time t ; a , b , c are conformable with y_t , z_t and p_t^e , and e_t is a white-noise error term. In existing models an equation like (1) is usually combined with an autoregressive expectations mechanism of the form of equation (2)

$$(2) \quad p_t^e = \sum_{s=1}^n d_s p_{t-s}$$

to yield an equation of the form of equation (3).^{5/}

$$(3) \quad ay_t + bz_t + c \sum_{s=1}^n d_s p_{t-s} = e_t.$$

Equation (2) is an example of a static expectations rule which may have described the price process well over the sample period. If a policy change alters the price process in the real economy, the agents represented by equation (3) would adopt a new forecasting rule for p . However, a simulation of that policy would leave equation (3) unchanged and, hence, would have an internal inconsistency. The agents of equation (3) would be assumed to act on forecasts generated by (2), whereas the entire model would imply a different price forecast.

The essence of our method for simulating the model subject to rational expectations is to eliminate that inconsistency between forecasts. We change the computer coding so that p_t^e , instead of being predetermined each quarter, is determined simultaneously with the endogenous variables and is equal to the forecast of price made by the model as a whole.

Simulations of alternative policies using such an altered structure would have the feature that the expectations present in behavioral relations are, in fact, the forecasts implied by the model as a whole. Such a feature may be viewed as almost a verbatim translation of Muth's [9] original characterization of rational expectations.

Comparison of such policy simulations with similar ones using the original structure should be of interest to a model builder. From such comparisons he could gauge, to some extent, the robustness of his structure to specification error in his modeling of expectations. It may be that the two structures yield similar results for certain types of experiments and widely differing results for others. If the experiments in this second group were of central interest to the model builder, he might reconsider his original assumptions about expectations--assumptions which may have been made rather casually during the initial modeling. Macro models which depend upon Phillips curve relationships for a good share of their real dynamics will almost surely require reexamination of their expectations mechanisms.

The use of simulations from a rational-expectations-augmented structure as a guide to policy should be approached with caution. If the rational expectations version of a model seems to dominate the original model as an accurate predictor of the effects of policy changes,^{6/} the original model was probably misspecified. If a full-information estimator was used, none of the coefficient estimators is consistent. If limited information simultaneous estimation was used, some of the equations may be estimated consistently. In either case, augmenting expectations as we do produces mongrel models whose properties are hard to characterize with generality. The policy maker who considers using

one of these mongrels for policy is probably well advised to respecify and reestimate.

Rational Policy Simulations with the St. Louis Model

As an example of how this method of simulation yields very different results from those obtained by standard procedures, we present two sets of policy simulations using the model developed by Anderson and Jordan and first reported in the Review of the Federal Reserve Bank of St. Louis [1]. The St. Louis Model is a useful vehicle for this demonstration because it is of small size and is familiar to a wide audience.

The simulation experiments we will describe were carried out using the version of the model described in the original paper. However, for purposes of illustration, we will consider the following simplified version which contains five endogenous variables, three exogenous variables, and three random disturbances.

$$(SL1) \quad \Delta p_t^e = A(L) \frac{p_{t-1}}{u_{t-1}}$$

$$(SL2) \quad \Delta y_t = B(L)m_t + C(L)e_t + v_{1t}$$

$$(SL3) \quad \Delta p_t = D(L)(\Delta y_t - x f_t + x_{t-1}) + .86 \Delta p_t^e + v_{2t}$$

$$(SL4) \quad x_t = y_t / p_t$$

$$(SL5) \quad u_t = G(L) \left(\frac{x f_t - x_t}{x f_t} \right) + v_{3t}$$

where $A(L), B(L) \dots$ are one-sided polynomials in the lag operator L . The five endogenous variables are nominal GNP (y), constant dollar GNP (x), the implicit GNP deflator (p), the unemployment rate (u), and the expected change in the price level (Δp^e). The three exogenous variables

are the money supply (m), government expenditures (e), and full-employment output (xf). The v_i 's are the random disturbances.

Equation (SL1) is an expectations equation where expected inflation is a weighted sum of past inflation rates. The weights, however, are variable and vary inversely with the unemployment rates in the past periods. Equation (SL2) is termed the total spending equation. Equation (SL3) is referred to by the authors as the price equation. Equation (SL4) is the identity for real output, nominal output, and the price level. Equation (SL5) relates the unemployment rate to capacity utilization, a rough empirical approximation sometimes called "Okun's Law." Equation (SL1) and (SL3) correspond to equations (2) and (1) above.

In order to simulate this model under the assumption of rational expectations, we simply drop equation (SL1) from the model and replace Δp_t^e in equation (SL3) by the expression Δp_t to yield:

$$\Delta p_t = C(L)(\Delta y_t - x f_t + x_{t-1}) + .86 \Delta p_t$$

or, more compactly

$$p_t = \frac{1}{1-.86} C(L)(\Delta y_t - x f_t + x_{t-1}).$$

The effects of 4, 6, and 8 percent constant money growth rates in the original and RE versions of the St. Louis Model were simulated from the initial conditions of 1960I by solving dynamically through 1965IV. Both sets of simulations used actual values of the exogenous variables (excluding m, of course). All of the coefficients were the same for both models. Tables 1 and 2 contain the inflation and unemployment rate paths from those simulations. The quarterly inflation rates have been converted to annual rates.

Table 1

Rate of Inflation
Constant Money Growth Simulations of St. Louis Model

	Original Model			Rational Expectations Model		
	<u>4%</u>	<u>6%</u>	<u>8%</u>	<u>4%</u>	<u>6%</u>	<u>8%</u>
1960 I						
II	1.9	2.0	2.0	1.0	1.5	2.0
III	1.7	1.9	2.1	1.3	2.6	3.8
IV	1.7	2.0	2.3	2.5	4.5	6.5
1961 I	1.7	2.2	2.7	4.1	6.7	9.4
II	1.8	2.5	3.2	5.4	8.5	11.6
III	2.0	2.9	3.7	6.2	9.5	12.7
IV	2.2	3.2	4.2	6.2	9.5	12.7
1962 I	2.3	3.6	4.8	5.8	8.8	11.8
II	2.5	3.9	5.4	5.2	8.0	10.8
III	2.7	4.3	6.1	4.4	7.0	9.5
IV	2.8	4.7	6.9	3.5	5.9	8.3
1963 I	3.0	5.0	7.9	2.7	5.0	7.3
II	3.0	5.3	9.2	2.0	4.4	6.7
III	3.1	5.7	10.8	1.7	4.1	6.4
IV	3.2	6.1	12.0	1.8	4.2	6.4
1964 I	3.3	6.5	13.8	2.3	4.5	6.9
II	3.4	7.0	15.3	3.1	5.3	7.4
III	3.5	7.3	16.7	3.5	5.6	7.6
IV	3.6	7.7	17.8	3.4	5.6	7.6
1965 I	3.6	8.1	18.8	3.1	5.3	7.3
II	3.7	8.4	19.4	2.9	5.1	7.2
III	3.8	8.6	19.6	3.1	5.3	7.4
IV	4.0	8.8	19.5	3.6	5.6	7.6

Table 2

Unemployment Rate
Constant Money Growth Simulations of St. Louis Model

	Original Model			Rational Expectations Model		
	<u>4%</u>	<u>6%</u>	<u>8%</u>	<u>4%</u>	<u>6%</u>	<u>8%</u>
1960 I	5.8	5.8	5.8	5.8	5.8	5.8
II	5.9	5.8	5.8	5.8	5.8	5.7
III	5.9	5.7	5.5	5.7	5.6	5.4
IV	5.6	5.3	4.9	5.5	5.2	5.0
1961 I	5.3	4.7	4.1	5.2	4.9	4.6
II	5.0	4.2	3.5	5.1	4.8	4.5
III	4.7	3.8	2.9	5.1	4.9	4.6
IV	4.5	3.5	2.4	5.3	5.0	4.8
1962 I	4.4	3.2	2.0	5.4	5.2	5.0
II	4.2	2.9	1.5	5.5	5.4	5.2
III	4.0	2.6	1.2	5.5	5.4	5.2
IV	4.0	2.5	1.0	5.6	5.5	5.3
1963 I	4.0	2.4	0.9	5.6	5.5	5.4
II	4.0	2.4	1.0	5.6	5.5	5.3
III	4.1	2.5	1.2	5.6	5.5	5.3
IV	4.1	2.5	1.4	5.5	5.4	5.2
1964 I	4.1	2.6	1.8	5.4	5.3	5.1
II	4.0	2.6	2.3	5.3	5.1	4.9
III	3.9	2.6	2.9	5.2	5.0	5.0
IV	4.0	2.9	3.7	5.3	5.1	5.0
1965 I	4.2	3.2	4.6	5.4	5.2	5.0
II	4.3	3.5	5.6	5.4	5.3	5.1
III	4.3	3.7	6.4	5.4	5.2	5.0
IV	4.1	3.8	7.2	5.2	5.1	4.9

The simulations using the original St. Louis Model demonstrate an exploitable trade-off between inflation and unemployment. Higher money growth rates not only increase the rate of inflation but also decrease the unemployment rate substantially. However, when the rational expectations adjustment is made to the structure of the St. Louis Model, that trade-off virtually disappears.

There is almost no change in the unemployment rate path when the money supply growth rate is increased from 4 percent to 8 percent. The unemployment rates for the 4 and 8 percent rational simulations never differ by more than six-tenths of 1 percent and the mean difference is only three-tenths. In contrast, the 4 and 8 percent simulation unemployment rates from the original model differ, at times, by over 3 percent and the mean difference is 1.2 percent, four times larger than that for the rational expectations simulations.

The large short-term decreases in the unemployment rate produced by increasing the money growth rate in the original model result from systematically mistaken expectations of the inflation rate. This can be seen by examining the difference in the rate of inflation and the expected rate of inflation implicit in different model simulations. Table 3 includes the values of the "expected forecast error" calculated as

$$(4) \quad FE = \left(\frac{\Delta p_t^e}{p_{t-1}} \right) - \left(\frac{\Delta p_t}{p_{t-1}} \right)$$

where Δp_t^e , Δp_t , and p_{t-1} are values from the 2 and 8 percent money growth simulations of the original model. The 2 percent growth rate was chosen for this illustration because 2 percent was approximately the average rate of money supply growth over the sample period.

Table 3

Errors in Forecasts of Inflation Implicit in
Simulation of St. Louis Model

	2% Money Growth	8% Money Growth
	Forecast errors* in percent at annual rates	
1960I	0.25%	0.20%
	0.34	0.10
	0.39	-0.16
	0.39	-0.62
1961I	0.31	-1.18
	0.22	-1.76
	0.14	-2.31
	0.07	-2.78
1962I	0.00	-3.16
	-0.08	-3.47
	-0.12	-3.65
	-0.11	-3.69
1963I	-0.07	-3.59
	0.00	-3.33
	0.07	-2.94
	0.13	-2.42
1964I	0.13	-1.82
	0.09	-1.15
	0.07	0.46
	0.10	0.46
1965I	0.14	1.31
	0.16	2.11
	0.10	2.82
	-0.01	3.46

* Negative value indicates inflation was underestimated by agents.

The 8 percent simulation shows a mean forecast error roughly 10 times that of the 2 percent simulation and a root mean square forecast error roughly 15 times larger. The pattern of the errors in the 8 percent simulation is especially instructive. The agents in this model are expected to underestimate the inflation rate by more than 3 percent for six consecutive quarters and by more than 2 percent for ten consecutive quarters. The absolute size of these errors and the slowness with which expectations "catch up" to actual inflation will seem "unrealistic" to many readers. But the belief, based on a simulation of this model, that a sustained high rate of money growth will drive down the unemployment rate is predicated on just such a pattern of forecasting errors.

The estimated version of the St. Louis Model does not share the strict neutrality property of the small, illustrative models considered elsewhere by Wallace and Sargent, Lucas, and others. However, the great reduction in the real impact of money growth rates resulting from imposing rational expectations on simulations using that model leads the author to speculate that a version of the St. Louis Model which is estimated using the restrictions implied by the rational expectations hypothesis may fit historical data well by current standards.

The experiment reported here illustrates that the method of simulation we propose is, indeed, feasible and may provide interesting perspectives on existing models. The author has also implemented this method of simulation in the context of the FRB-MIT Model with less dramatic, though interesting, results. Some of the questions that must be faced by an investigator wishing to apply this technique to a large model like the FRB-MIT Model are considered in the next section on

implementation. It may be skipped by a reader not interested in technical details.

Implementation

The preceding discussion has probably given the reader the impression that, whatever its relative merit as an analytical device, the rational expectations adjustment may be implemented in a straightforward, mechanical fashion. Unfortunately, with most large macroeconomic models this will not be the case.

Since the rational expectations hypothesis is a hypothesis about individual behavior, building a model which incorporates it typically involves more microeconomics than is usually used in macroeconomics.^{7/} It is, therefore, not too surprising that an attempt to graft rational expectations onto an existing structure may lead an investigator to consider issues which the original model builder either did not consider or failed to document. It will often be impossible to avoid using one's own judgment on points which will significantly influence the characteristics of the resulting hybrid model. In this section, we will describe some of the difficulties a model user must face in adapting a particular model.

The first task is to detect where (if anywhere) expectational elements were intended to enter into the model. If the model builder did not include any explicit expectational terms in the equations, the exercise we contemplate here is impossible. If, on the other hand, the builder specifically admits in the documentation or the equations include distributed lags of, for example, past prices in contexts where an interpretation as expectations is natural, one may proceed.

The second problem one must often face is a problem which the initial model builder may not have considered. Often models with static expectations rules are specified in such a way that a certain coefficient is not identified. (See Sargent [14].) For example, equation (3) (reproduced here)

$$(3) \quad ay_t + bz_t + c \sum_{s=1}^n d_s p_{t-s} = e_t$$

is actually estimated as

$$(3') \quad ay_t + bz_t + \sum g_s p_{t-s} = e_t.$$

From estimates of (3') one cannot infer the value of c . For using the model as initially estimated this lack of identification is not important. However, for making the substitution we propose, it is crucial that we separate the values of c and the d 's in equation (3). It is easy to see that the properties of the rational expectations structure will turn on the choice of c , since a larger or smaller value of c affects the degree to which observable behavior responds to a change in expectations.

The most common method for identifying c is to impose the a priori restriction

$$\sum_{s=1}^n d_s = 1.$$

Such a restriction would be useful if, for example, prices had followed the process such a restriction implies over the sample period of the model. This is rarely the case and some alternative identifying restriction is to be preferred for most post-war models. Failing to find cues

from the model builder or wishing to disregard them, an investigator might either choose a priori or use the following objective procedure. When faced with an estimated equation like (3'), estimate the h_i 's in the equation

$$p_t = \sum_{i=1}^n h_i p_{t-i} + u_t$$

and identify c as

$$c = \frac{\sum_{s=1}^n g_s}{\sum_{i=1}^n h_i} .$$

A third problem, that of specifying the time horizon of the expectations in certain equations, highlights the need for more micro-foundations in estimated macro models. The discussion to this point has assumed that each agent wishes to forecast only the current period's price level. However, that assumption is clearly erroneous for certain types of behavior. In an investment equation or a demand curve for consumer durables a distributed lag such as that in equation (3') is probably intended to reflect expectations over a considerably longer horizon.

The specification of the expectations horizon (and the form of the structural equation) should be derived from the statement of the individual decision problem and market aggregation. When this information is often not available, the investigator may posit a model of his own.

Often a complete dynamic model will yield a decision rule which depends upon expectations of certain quantities far (perhaps

infinitely far) into the future. The computational problem of implementing such a rule may be overcome by using a proxy for all future prices. I would recommend using a k-period-ahead forecast of price, where k is chosen by consideration of the specific individual choice problem. I would also recommend identifying the coefficient of this proxy by re-estimation including a k-period-ahead forecast of price based on some simple single-equation forecasting mechanism.

The computation of the rational expectations forecast in the case of a k-period horizon can be accomplished by an iterative scheme. We choose an initial guess for price expectations, and simulate the entire model for k-periods. The forecast of price for period k is then used as the next "guess" for a simulation beginning at the original starting point. We continue until the forecast for period k is sufficiently close to the expectation assumed by that forecast. At this point, we have produced the rational expectations forecast for the first period of our simulation.

There is no guarantee that this iterative procedure will converge for arbitrary initial conditions. However, if a "fixed-point" price path is achieved, that simulation will have not only the contemporaneous consistency of expectation and model forecast claimed earlier, but also a dynamic consistency. At each stage the expectation of future prices held by individuals are precisely the model forecasts of those prices. This method of simulation implicitly assumes that individuals know the processes which generate all of the exogenous variables. Such an assumption is consistent with the rational paradigm we are maintaining.

Conclusion

Lucas' theoretical objections to current econometric policy evaluation and the failure of empirical tests to reject the natural rate-rational expectations hypothesis cast doubt on the ability of standard policy simulations to represent the effects of different policies. This paper provides a method for simulating standard models under the assumption of rational expectations in cases where reestimation of the model under that assumption is considered too costly. The results of "rational expectations simulations" should indicate to what extent the effects of alternative policies in standard policy simulations depend on exploitation of the assumed naiveté of the agents in the economy.

Footnotes

1/ This section is based almost completely on Lucas [6].

2/ Here Shiller means fixed-weight distributed lags of past prices.

3/ See Cyert and DeGroot [4] and Taylor [19] for examples of Bayesian models.

4/ Price is used only as example of variable whose future values may be anticipated. This method could be applied wherever expectations are explicitly modeled.

5/ Examples of such equations may be found in [20] or the equation listing of almost any large model.

6/ Judging whether the new model is superior to the old model as a predictor of the effect of policy is generally difficult. This is chiefly because there are usually only a small number of policy shifts outside the estimation period of the model which can be used to contrast the forecasts of the two models.

7/ Fair [5] presents a detailed micro structure for his model.

8/ Another attractive feature of the St. Louis Model is its easy availability through NBER TROLL computer time-sharing network.

9/ The author has completed a similar demonstration with the MIT-PENN-SSRC Model though the rational expectations version of the MPS did not exhibit the strict neutrality demonstrated by the St. Louis Model considered here.

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