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**Optimal Capital Income Taxation With
Incomplete Markets, Borrowing Constraints,
and Constant Discounting**

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ABSTRACT

For a wide class of dynamic models, Chamley (1986) has shown that the optimal capital income tax rate is zero in the long run. Lucas (1990) has argued that for the U.S. economy there is a significant welfare gain from switching to this policy. We show that for the Bewley (1986) class of models with heterogeneous agents and incomplete markets (due to uninsured idiosyncratic shocks), and borrowing constraints the optimal tax rate on capital income is positive even in the long run. Quantitative analysis of a parametric version of such a model suggests that one cannot dismiss the possibility that the observed tax rates on capital and labor income for the U.S. economy are fairly close to being (long run) optimal. We also provide an existence proof for the dynamic Ramsey optimal tax problem in this environment.

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I. Introduction

For a wide class of dynamic models, Chamley [1986] has shown that the optimal capital income tax rate is zero in the long run. The capital income tax rate for the U. S. economy appears to be quite far from zero¹. One possible response to this is to accept the prescription of theory and recommend a change in tax policy. Lucas [1990] took this route and, using a representative agent model, has argued that for the U.S. economy there is a significant welfare gain from switching to this policy. In Lucas' words (p.314), "...supply-side economists...have delivered the largest genuinely free lunch I have seen in 25 years in this business." According to Lucas' estimates eliminating the capital income tax can result in a welfare gain across steady states of over 5 percent of consumption, and about 1 percent when transitional costs are taken into account. He suggests that this welfare gain is about twice as large as the gain from eliminating a 10 percent inflation rate, 20 times as large as the gain from eliminating post-war business cycles, and 10 times the gain from eliminating all product market monopolies as estimated by Harberger.

One way to summarize this argument is to say that the capital stock in the U. S. economy is too low and that it ought to be higher, and, further, that there are large welfare gains from making it so.

In this paper we take the contrary view that perhaps there are good reasons for why the capital income tax is what it is, and, hence, that cutting it would lead to welfare losses rather than welfare gains. To put it another way, if the capital income tax were cut to zero, the capital stock would be too high and it ought to be lower.

¹Lucas [1990] calculates the capital income tax rate for the U. S. economy to be 0.36.

We present a class of environments together with a market structure such that the optimal capital income tax rate is *not* zero, but strictly positive, even in the long run. Specifically, we show that for the Bewley [1986] class of models with incomplete markets, heterogeneous agents and borrowing constraints the optimal tax rate on capital income is positive even in the long run². If we regard such models as providing a good description of reality, then we need to reassess the presumed welfare gains of reducing the capital income tax to zero. The presumed welfare gains may well turn into losses.

We should emphasize that the result in this paper is not just that the capital income tax rate is different from zero in the long run, but that it is always positive for the type of environment/market structure considered. In overlapping generations models with pure life-cycle consumers the long run capital income tax is not generally zero; however, it may be positive or negative.

In the Bewley [1986] class of models considered in this paper there is a continuum of agents subject to idiosyncratic shocks which are uninsured. Due to the absence of insurance markets agents become heterogeneous *ex-post*³. Because of the idiosyncratic nature of the shocks there is uncertainty at the individual level but there is no aggregate uncertainty.

²Jones, Manuelli and Rossi [1990] show that if government expenditures are endogenous and productive, then the capital income tax rate can be positive in the long run. Earl Thompson [1974] made an argument based on national defense for capital taxation as part of an optimal tax structure.

³Presumably private information and the resulting problems due to moral hazard and adverse selection have a lot to do with "missing markets" and incomplete risk sharing. While it would be desirable to take explicit account of these features of the environment as in some recent literature (see, for eg., Green 1987, Phelan and Townsend 1991, Atkeson and Lucas 1992) this is beyond the scope of the present paper. Instead, we simply impose a particular market structure as in Bewley [1986].

The intuition behind why the above features lead to a positive tax rate on capital income may be explained as follows. Because of incomplete insurance markets there is a precautionary motive for accumulating capital. In addition, the possibility of being borrowing constrained in some future periods also leads agents to accumulate additional capital. These two features lead to increases in their saving and hence capital accumulation, and thereby lower the return on capital below the rate of time preference⁴. That is, the above features lead to excess (i.e., greater than the optimal level of) capital⁵. As we will show, a positive tax rate on capital income will be needed to reduce capital accumulation and bring the pre-tax return on capital to equality with the rate of time preference.

It is well known from the study of overlapping generations models (Diamond 1965) that competitive equilibria may be characterized by capital overaccumulation and that government debt (equivalently, interest bearing money) can be used to soak up excess saving and reduce capital accumulation. This suggests the possibility that in the Bewley [1986] class of models also government debt may serve to eliminate excess capital accumulation and bring the return on capital to equality with the rate of time preference without a tax on capital income. However, this turns out to be infeasible, due to a crucial feature of this class of models. This feature is that the demand for assets on the part of households for precautionary saving purposes tends to infinity as the return on the assets approaches the rate of time preference. However, the supply of capital is bounded because there is a maximum sustainable capital stock in the economy; further, the supply of government

⁴See, Bewley [no date], Laitner [1979, 1992], and Aiyagari [1992].

⁵This should be distinguished from the standard notion of capital overaccumulation which refers to an inefficiently high level of capital.

debt is bounded above because tax revenues from labor and capital are bounded above. Therefore, the supply of assets in the economy (capital plus debt) is bounded above. Consequently, it is not possible to support as an equilibrium an interest rate that is arbitrarily close to the time preference rate. Making this argument rigorous and showing that it implies that the capital income tax rate must be strictly positive even in the long run is the main goal of the theoretical part of the paper.

The quantitative part of the paper is motivated by the following question: Assuming that a reasonably parameterized version of such a model is a reasonably good approximation to reality, can we easily dismiss the possibility that the observed labor and capital income tax rates in the U. S. economy are approximately optimal? We will suggest that the answer to this question is: No⁶.

We also provide a general theorem regarding the existence of a solution for the dynamic Ramsey optimal taxation problem for this environment.

The outline of the paper is as follows. In Section II we describe the dynamic Ramsey optimal tax problem in a Bewley type model with a continuum of agents, stochastic and idiosyncratic shocks to labor productivities, and borrowing constraints. In section III we try to provide some intuition for the results by conducting steady state analysis. In section IV we prove the result that the optimal capital income tax must be positive even in the long run. Section V contains the quantitative results. Appendix A contains many

⁶Recently, models of this type have been used to address quantitatively a variety of questions. See, for example, Imrohoroglu [1988,1989] on the costs of inflation and the costs of the business cycle, Diaz-Gimenez and Prescott [1989] on monetary policy and asset returns, Huggett [1989] on the risk free interest rate, Aiyagari and Gertler [1991] on asset returns with transactions costs, and Aiyagari [1992] on the contribution of precautionary saving to aggregate saving.

of the proofs and Appendix B contains the details of computation involved in the quantitative exercise.

II. A Bewley Type Model

In this section we consider the problem of optimal capital income taxation in a Bewley [1986] type model with a continuum of agents receiving stochastic, idiosyncratic shocks to labor productivities which are uninsured.

The Environment

We assume that there is a continuum of agents of size unity. Per capita variables (or averages across individuals) are distinguished from individual specific variables by using upper case letters for the former and lower case letters for the latter.

Endowments and Technology

Agents are endowed with one unit of perfectly divisible labor each period which can be used either in the market sector or in the home sector. Let n_t and $1-n_t$ be an agent's market work at time t , and home work at time t , respectively. Home production is given by a production function $\theta_t H(1-n_t)$ where $H: [0,1] \rightarrow \mathbb{R}_+$, is bounded, continuously differentiable, strictly increasing, and strictly concave. In addition $H(\cdot)$ satisfies: $H(0) = 0$, $H'(0) = \infty$, $H'(1) > 0$.

θ_t denotes an idiosyncratic shock to the home production of an agent in period t . We assume that θ_t is i.i.d. across agents so that there is no

aggregate uncertainty⁷. Further, θ_t follows a Markov process over time with probability transition function denoted by $P(\theta', \theta) = \text{prob}[\theta_{t+1} = \theta' | \theta_t = \theta]$. We assume that the Markov process has a unique non-degenerate stationary distribution (denoted by F) to which it converges strongly and that the stationary distribution has bounded support. Let θ_{\min} and θ_{\max} denote the lower and upper ends of the support. We assume $\theta_{\min} > 0$.

In the market sector production is governed by a neoclassical production function $f(K_t, N_t)$, where K_t is the per capita amount of capital in the economy, N_t is the per capita amount of market work, and $f(\cdot)$ is the per capita market output net of capital depreciation. We assume that $f(\cdot)$ is homogeneous of degree one, and twice continuously differentiable. Further, $f(\cdot)$ satisfies: (i) $f(0, N) = f(K, 0) = 0$, (ii) for $(K, N) \gg 0$, $f_{11} < 0$, $f_{22} < 0$, (iii) for $K > 0$, $f_2 > 0$ and $\lim_{N \rightarrow 0} f_2 = \infty$, (iv) for $N > 0$, $\lim_{K \rightarrow 0} f_1 = \infty$, $\lim_{K \rightarrow \infty} f_1 = -\delta < 0$.

Preferences

An agent consumes the amount c_t of goods in period t , and the government consumes the amount G_t of goods (per capita) in period t . An agent's preferences are described by the following expected value of the sum of discounted utilities of private consumption and public consumption: $E_0 \{ \sum_{t=0}^{\infty} \beta^t [u(c_t) + U(G_t)] \}$, where $\beta \in (0, 1)$, $u(\cdot)$ is the utility from private consumption and $U(\cdot)$ is the utility from public consumption⁸. The functions

⁷The technical difficulties arising from a continuum of i.i.d. random variables (see Judd 1985, Feldman and Gilles 1985) will be finessed in this paper in the same way as in Bewley [1986].

⁸It should be emphasized that Chamley's [1986] result that the capital income tax is zero in the long run holds for general recursive preferences, not just time additive preferences, as is assumed here. However, the environment here, unlike Chamley's, contains uncertainty at the individual

$u(\cdot)$ and $U(\cdot)$ are each assumed to be bounded, continuously differentiable, strictly increasing, and strictly concave.

Markets

There are competitive markets in labor, capital services, the output good, and one period consumption loans.

Competitive Equilibrium

Firms

Competition in product and factor markets and profit maximization on the part of firms implies that $w_t = f_2(K_t, N_t)$, and $r_t = f_1(K_t, N_t)$, where w_t denotes the pre-tax market real wage and r_t denotes the pre-tax real rental on capital services, respectively.

Government

The government consumes the amount G_t (per capita) in period t , issues new debt in the (per capita) amount $(B_{t+1} - B_t)$ where B_t is the per capita debt outstanding at the beginning of period t , and taxes market labor income and interest income at the rates τ_{nt} and τ_{kt} , respectively. A crucial assumption here is that while market work can be taxed, home work cannot be

level and results on the "income fluctuation problem" (Schechtman and Escudero 1977) which we will rely on are only available for time additive preferences. The separability of utility in private and public consumption is convenient but not essential. It enables us to pose the consumer's optimization problem by simply ignoring the utility from public consumption. Together with the specification of market labor supply this turns the problem into a standard income fluctuation problem and enables us to use results developed in that literature directly. See Bewley [undated, 1986], Laitner [1979, 1992], Chamberlain and Wilson [1984], Clarida [1987, 1990].

taxed⁹. Let \bar{w}_t and \bar{r}_t be the after-tax market real wage and the after tax real rental on capital services, respectively. Note that $\bar{w}_t = (1-\tau_{nt})w_t$, and $\bar{r}_t = (1-\tau_{kt})r_t$. Since there is no aggregate uncertainty capital and consumption loans are perfect substitutes. Therefore, the pre-tax interest rate on one period consumption loans (and government debt) must equal r_t .

The government budget constraint is as follows.

$$(2.1) \quad G_t + r_t B_t = B_{t+1} - B_t + \tau_{nt} w_t N_t + \tau_{kt} r_t (K_t + B_t).$$

Note that $\tau_{nt} w_t = w_t - \bar{w}_t = f_2(K_t, N_t) - \bar{w}_t$, and that $\tau_{kt} r_t = r_t - \bar{r}_t = f_1(K_t, N_t) - \bar{r}_t$. Making these substitutions into (2.1) and using the first degree homogeneity of $f(\cdot)$ we can rewrite (2.1) in the following form.

$$(2.2) \quad G_t + \bar{r}_t B_t = B_{t+1} - B_t - \bar{w}_t N_t - \bar{r}_t K_t + f(K_t, N_t).$$

Consumer

An agent starts with some assets a_0 and a realized productivity shock θ_0 in period 0, and solves the following problem.

Maximize $E_0 \{ \sum_{t=0}^{\infty} \beta^t u(c_t) \}$ subject to the sequence of budget constraints and borrowing constraints given by

$$(2.3a) \quad c_t + a_{t+1} = \theta_t H(1-n_t) + \bar{w}_t n_t + (1+\bar{r}_t) a_t,$$

$$(2.3b) \quad 0 \leq n_t \leq 1, \quad c_t \geq 0, \quad a_t \geq 0.$$

⁹Our specification of market labor supply is equivalent to assuming that there is no income effect on non-market work (home work or leisure) in the more conventional specification of preferences where non-market work is also an argument of the utility function $u(\cdot)$.

We will now reformulate the consumer's optimization problem in a dynamic programming framework.

To simplify the reformulation we start by noting that the solution to the labor allocation problem is obtained by maximizing $[\theta_t H(1-n_t) + \bar{w}_t n_t]$ over $n_t \in [0,1]$. This yields a supply function for market work denoted $n(\bar{w}_t/\theta_t)$. Using this we can define an agent's total (market plus home) earnings function (denoted by $y(\theta_t, \bar{w}_t)$) as: $y(\theta_t, \bar{w}_t) = \theta_t H(1-n(\bar{w}_t/\theta_t)) + \bar{w}_t n(\bar{w}_t/\theta_t)$. Note that $y(\theta_t, \bar{w}_t) \geq \theta_{\min} H(1) > 0$.

Now, let

$$(2.4a) \quad \bar{w}^t = \{\bar{w}_t, \bar{w}_{t+1}, \bar{w}_{t+2}, \dots\}, \quad t \geq 0,$$

$$(2.4b) \quad \bar{R}_t = 1 + \bar{r}_t, \quad t \geq 0,$$

$$(2.4c) \quad \bar{R}^t = \{\bar{R}_t, \bar{R}_{t+1}, \bar{R}_{t+2}, \dots\}, \quad t \geq 0.$$

An agent's decision problem can now be expressed in terms of the following Bellman's equation, where v is the value function.

$$(2.5a) \quad v(a_t, \theta_t, \bar{w}^t, \bar{R}^t) = \text{Max}\{u(c_t) + \beta E_t v(a_{t+1}, \theta_{t+1}, \bar{w}^{t+1}, \bar{R}^{t+1})\},$$

subject to:

$$(2.5b) \quad c_t + a_{t+1} = y(\theta_t, \bar{w}_t) + (1 + \bar{r}_t) a_t, \quad c_t \geq 0, \quad a_t \geq 0, \quad t \geq 0.$$

Note that in (2.5a) the sequences \bar{w}^t and \bar{R}^t are deterministic.

Equilibrium

Let $J_t(a, \theta)$ be the cross-section distribution (c.d.f.) of agents according to asset holdings and θ in period t , and let $J_0(a, \theta)$ be given as an initial condition. The evolution of $J_t(\cdot)$ over time will have to be

determined as part of the equilibrium.

The solution to the consumer's problem (2.5) will consist of the following decision rules.

$$(2.6a) \quad c_t = c(a_t, \theta_t, \bar{w}^t, \bar{R}^t),$$

$$(2.6b) \quad a_{t+1} = a(a_t, \theta_t, \bar{w}^t, \bar{R}^t).$$

Using (2.6b) and the probability transition function for θ , we can update the given initial distribution $J_0(a, \theta)$ to obtain $J_t(\cdot)$ for all t . Note that these distributions for $t \geq 1$, will depend on the sequence of after-tax prices. To make this dependence explicit, we will denote them by $J_t(a, \theta, \bar{w}^0, \bar{R}^0)$. Per capita consumption C_t is then given by the following.

$$(2.7) \quad C_t = \int c(a, \theta, \bar{w}^t, \bar{R}^t) dJ_t(a, \theta, \bar{w}^0, \bar{R}^0) \equiv \chi_t(\bar{w}^0, \bar{R}^0).$$

Per capita market work (denoted by N_t previously) is given by the following.

$$(2.8) \quad N_t = \int n(\bar{w}_t/\theta) dF(\theta) \equiv \nu(\bar{w}_t).$$

We can also write per capita output of home produced goods (denoted by H_t) as follows.

$$(2.9) \quad H_t = \int \theta H(1 - n(\bar{w}_t/\theta)) dF(\theta) \equiv \eta(\bar{w}_t).$$

Note that $H_t = \eta(\bar{w}_t) \leq \theta_{\max} H(1) < \infty$.

The resource constraint for this economy can now be written as follows.

$$(2.10) \quad f(K_t, \nu(\bar{w}_t)) + \eta(\bar{w}_t) + K_t - K_{t+1} - G_t - \chi_t(\bar{w}^0, \bar{R}^0) \geq 0, \quad t \geq 0.$$

In (2.10), $\nu(\bar{w}_t)$ is per-capita market work (from 2.8), $\eta(\bar{w}_t)$ is per capita home production (from 2.9), and $\chi_t(\bar{w}^0, \bar{R}^0)$ is per capita consumption (from 2.7).

Given time paths for \bar{w}_t and \bar{r}_t , and the stochastic process for θ_t , individuals choose processes for consumption and asset accumulation to solve the problem (2.5). This results in a time path for per capita consumption and per capita assets. Together with a time path for G_t , the government budget constraint (2.2) then determines the time path for government debt, since K_t must equal per capita assets at time t (denoted A_t) minus B_t . The time paths for G_t , \bar{w}_t and \bar{r}_t are consistent with equilibrium if the resulting time paths for per capita capital and consumption clear the goods market at each date, i.e., satisfy the resource constraint (2.10).

The above description is now formally summarized in the following definition of a competitive equilibrium.

Definition. For given initial conditions K_0 and $J_0(\cdot)$, and time paths $\{G_t, \bar{w}_t, \bar{r}_t\}$, a **competitive equilibrium** consists of a value function $v(\cdot)$, consumer's decision rules $c(\cdot)$ and $a(\cdot)$, and sequences $\{J_t(\cdot), K_t\}$ such that the following hold:

- (i) $v(\cdot)$ solves the Bellman equation (2.5a),
- (ii) $c(\cdot)$ and $a(\cdot)$ attain $v(\cdot)$,
- (iii) $\{J_t(\cdot)\}$ is generated from $J_0(\cdot)$ and $a(\cdot)$,
- (iv) $\{K_t\}$ satisfies (2.10).

The Optimal Tax Problem

The government's optimal tax problem is to choose time paths for G_t , \bar{w}_t and \bar{r}_t consistent with equilibrium such that the utilitarian social welfare $[\int v(a, \theta, \bar{w}^0, \bar{R}^0) dJ_0(a, \theta) + \sum \beta^t U(G_t)]$ is maximized.

More formally, the government's optimal tax problem may be written as follows.

(2.11) Max $[\int v(a, \theta, \bar{w}^0, \bar{R}^0) dJ_0(a, \theta) + \sum \beta^t U(G_t)]$ subject to (2.10) and $(\bar{w}_t, \bar{R}_t, G_t, K_{t+1}) \geq 0, t \geq 0$, by choice of $\{\bar{w}_t, \bar{R}_t, G_t, K_{t+1}\}$ for $t \geq 0$.

Note that in the above problem, the only constraint (aside from non-negativity constraints) is the resource constraint (2.10). The government budget constraint need not be included as an additional constraint since the individual decision rules automatically satisfy individual budget constraints, which together with the resource constraint implies the government budget constraint.

III. Steady State Analysis

In this section we try to provide some intuition for the way this class of models work and why they necessarily imply a positive tax on capital income, by analyzing steady states. The formal analysis of the Ramsey optimal tax problem is postponed to the next section.

For the steady state analysis of this section only the following additional assumptions are made: (i) θ_t is i.i.d. over time¹⁰, (ii) u is twice differentiable and there exist positive numbers μ^* and c^* such that

¹⁰ Steady state results for this class of models are available in the literature only for i.i.d. shocks.

$(-cu''/u') \leq \mu^*$ for all $c \geq c^*$ ¹¹.

It is convenient to index a steady state by r (the pre-tax return to capital), τ_n (the wage tax), and government consumption G . The condition $f_1(K,N) = r$, fixes the K/N ratio and, hence, the pre-tax wage $w = f_2(K,N)$. Therefore, \bar{w} (the after tax wage) is given by $(1-\tau_n)w$. Further, market work N is determined by \bar{w} in accordance with the steady state version of (2.8). Since K/N is determined by r , it follows that K is determined. Individual optimization and asset market clearing will then be used to determine B , and \bar{r} (the after-tax return to capital). The capital income tax rate τ_k is then given by $(1-\bar{r}/r)$.

The value of \bar{r} is found in the following way. The steady state version of the government budget constraint (2.2) can be manipulated to express $(K+B)$ as a function of \bar{r} (the after-tax return to capital) as follows.

$$(3.1) \quad K + B = (f(K,N) - \bar{w}N - G)/\bar{r}.$$

Note that $(f(K,N) - \bar{w}N - G)$ is completely determined by the given values of r , τ_n and G . We assume that this expression is positive. Therefore, the graph of $K + B$ versus \bar{r} looks as shown in Figure 1.

For a given value of \bar{r} (assumed less than $\rho \equiv 1/\beta - 1$) the consumer's problem is a stationary problem in the steady state and is described as follows.

¹¹This condition ensures that the asset accumulation process for an individual remains bounded so long as the return on assets is less than the rate of time preference. Further, there exists a unique long run distribution of assets which is stable, i.e., starting from any initial distribution of assets the sequence of distributions of assets converges to the unique long run distribution. See Schechtman and Escudero [1977], Clarida [1987, 1990].

Maximize $E_0\{\sum_{t=0}^{\infty} \beta^t u(c_t)\}$ subject to:

$$(3.2) \quad c_t + a_{t+1} = y(\theta_t, \bar{w}) + (1+\bar{r})a_t, \quad c_t \geq 0, \quad a_t \geq 0, \quad t \geq 0.$$

The solution to the consumer's decision problem yields a stationary decision rule for asset accumulation, $a_{t+1} = a(a_t, \theta_t; \bar{r})$. This decision rule together with the distribution of θ_t determines a Markov process for assets a_t . This Markov process has a unique stationary distribution (see Clarida 1990, proposition 2.2 and Corollary 2.3), denoted $J(a; \bar{r})$. Average asset holdings (denoted $A(\bar{r})$) are given by $A(\bar{r}) = \int a dJ(a; \bar{r})$. From proposition 2.4 in Clarida [1990] we know that $A(\cdot)$ is a continuous function of \bar{r} and that $A(\cdot)$ tends to infinity as \bar{r} tends to ρ . A possible graph of $A(\bar{r})$ versus \bar{r} is also shown in Figure 1. The value of \bar{r} is determined as the solution to the asset market equilibrium condition

$$(3.3) \quad K + B = A(\bar{r}),$$

that is, by the intersection of the two curves in Figure 1. Note that by virtue of the properties of the two curves in Figure 1, a solution is guaranteed to exist. Further, since K is determined by r and τ_n , once \bar{r} is known B can be found from (3.3).

We now give some intuition for why with incomplete markets the capital income tax rate is strictly positive even in the steady state. If there were no idiosyncratic shocks (equivalently, if markets were complete) then the consumers' asset demand function $A(\cdot)$ would coincide with the vertical axis for $\bar{r} < \rho$, and would be perfectly elastic at $\bar{r} = \rho$. Therefore, \bar{r} will equal

ρ in a steady state, regardless of the values of r , τ_n or G . In particular, if there were no capital income tax, then $r = \rho$ which is the standard result that the capital stock satisfies the modified golden rule.

However, when there are idiosyncratic shocks (and markets are incomplete) the individual has a precautionary motive for accumulating assets and will hold positive amounts of assets on average even when $\bar{r} < \rho$ in order to buffer earnings shocks and smooth consumption¹². The borrowing constraint also plays a role since the possibility of being borrowing constrained in future periods serves to enhance the individual's desire for current assets¹³.

More crucially, asset demand $A(\cdot)$ tends to infinity as \bar{r} tends to ρ from below. The intuition is that when \bar{r} equals ρ , the individual would like to maintain a smooth marginal utility of consumption profile. However, since there is some probability of receiving a sufficiently long string of bad θ 's, the only way to maintain a smooth marginal utility of consumption profile is to have infinite assets¹⁴.

¹²If \bar{r} is sufficiently low (close to negative unity, for example) then the individual will not hold any assets ever and will simply consume his earnings in each period.

¹³Even though we have ruled out borrowing this is not essential to the analysis. If $\bar{r} > 0$, then the present value budget constraint and non-negativity of consumption imply that $a_t \geq -y(\theta_{\min}, w)/\bar{r}$. That is, there is always a borrowing limit in this class of models. The intuition is that if ever $a_t < -y(\theta_{\min}, w)/\bar{r}$, then a sufficiently long series of bad θ 's will force the consumer to increase his debt level to such an extent that from then on even if he received the best θ 's forever he would never be able to pay off his debt. See proposition 1 (p. 34) in Aiyagari [1992].

¹⁴Individual assets go to infinity (a.s.) if $\bar{r} > \rho$. In this case the individual wants to be a lender and postpone consumption to the future. Therefore, per capita assets are infinite when $\bar{r} > \rho$. This holds also when $\bar{r} = \rho$. The intertemporal first order condition for individual optimization is: $u'(c_t) \geq \beta(1+\bar{r})E_t\{u'(c_{t+1})\}$, with equality if $a_{t+1} > 0$. If $\beta(1+\bar{r}) \geq 1$, then $u'(c_t) \rightarrow 0$ a.s. and, hence, $c_t \rightarrow \infty$ a.s. Therefore, $a_t \rightarrow \infty$ a.s. See

It follows that with incomplete markets the steady state equilibrium value of \bar{r} is always less than ρ , again regardless of the values of r , τ_n or G . As we vary r , the curve marked (K+B) in figure 1 shifts (since K in the numerator on the right side of (3.1) depends on r) and leads to different steady state values of \bar{r} all of which will be less than ρ . Therefore, it must be the case that the return on capital r consistent with zero capital income tax is strictly less than ρ . Consequently, under incomplete markets, there will always be capital overaccumulation if there is no tax on capital, i.e., the capital stock will be higher than the modified golden rule level. The additional capital accumulation and the implied higher saving rate may be attributed to precautionary saving.

As we will prove in the next section, the solution to the Ramsey optimal tax problem has the feature that (in the steady state) the modified golden rule holds, i.e., the pre-tax return on capital equals ρ (proposition 1). From the above discussion, this can only be achieved by having a positive tax on capital income, thereby eliminating capital overaccumulation.

Proving that the limiting pre-tax interest rate equals the time preference rate and that the limiting after-tax interest rate is strictly less than the time preference rate in the Ramsey optimal tax problem is the main goal of the analysis in the next section.

IV. The Optimal Capital Income Tax in the Long Run

In this section we return to the analysis of the Ramsey optimal tax problem formulated in section II and show that the optimal capital income

tax rate must be positive even in the long run. First, we provide an existence result for the optimal tax problem.

Theorem 1. A solution to the optimal tax problem exists.

Proof. See Appendix A (part 1). ■

Theorem 1 does not guarantee that in the solution, market production is necessarily positive, i.e., $N_t > 0$ ¹⁵, or that, in the long run, it is bounded away from zero, i.e., $\liminf_{t \rightarrow \infty} N_t > 0$. The proof works by showing that there always exists a feasible policy which involves zero market production and autarky for individuals. Given the existence of such a feasible policy, an optimum policy is shown to exist by using continuity and compactness arguments.

In what follows it is assumed that a solution to the optimal tax problem converges to a steady state in which factor prices, per capita capital, per capita private and government consumption, per capita market work, and per capita home work converge to limiting values which are all strictly positive and finite¹⁶. This is formalized as assumption 1 below.

¹⁵Note that this guarantees that $K_t > 0$. Otherwise, $w_t = f_2(K_t, N_t) = 0$, implying that $\bar{w}_t = 0$, which is inconsistent with $N_t > 0$.

¹⁶It seems quite difficult to guarantee that a solution to the optimal tax problem converges to a steady state. Even for the simpler version of the model without a government sector results are only available for steady states with i.i.d. over time shocks. See Bewley [undated], Laitner [1979, 1992], Clarida [1990]. There is no existence result or convergence to a steady state result for an arbitrarily given initial condition nor even an example. The technical difficulty is that the distribution of assets across individuals is an aggregate state variable which is, in general, changing over time. In any case this assumption is also made by Chamley [1986] and Lucas [1990]. However, Chamley does provide an example which exhibits convergence to the unique steady state.

Assumption 1: The solution to the optimal tax problem (2.14) is such that $(\bar{R}_t, G_t, N_t, 1-N_t) \rightarrow (1+\bar{r}^*, G, N, 1-N) \gg 0$ and finite. ■

The Long Run Capital Income Tax

From assumption 1 it follows immediately that $\bar{w}_t \rightarrow \bar{w}^* > 0$ and finite.

Proposition 1. $r_t \rightarrow \rho \equiv (1-\beta)/\beta$.

Proof: Let $\beta^t \lambda_t$ be the non-negative multiplier associated with the constraint (2.10) in the planning problem (2.11). The first order necessary conditions with respect to K_{t+1} and G_t for this problem are (for $K_{t+1} > 0$ and $G_t > 0$),

$$(4.1a) \quad -\lambda_t + \beta \lambda_{t+1} [f_1(K_{t+1}, \nu(\bar{w}_{t+1})) + 1] = 0.$$

$$(4.1b) \quad U'(G_t) - \lambda_t = 0.$$

From (4.1b) it follows that λ_t converges to some $\lambda > 0$ and finite. Therefore, $r_{t+1} = f_1(K_{t+1}, \nu(\bar{w}_{t+1})) \rightarrow \rho \equiv (1-\beta)/\beta$. ■

The nature of the variational experiment underlying the proof of proposition 1 is the following. Imagine that the government increases investment at date t by one unit (i.e., $\Delta K_{t+1} = 1$) and decreases government consumption by one unit ($\Delta G_t = -1$). The reduced public consumption is met by a reduction in new debt, also by one unit ($\Delta B_{t+1} = -1$). As a consequence, the resource constraint and the government budget constraint continue to be satisfied at date t and per capita assets do not change ($\Delta A_{t+1} = 0$).

Further, individuals are unaffected by these changes so that per capita consumption, per capita market work, per capita home production and per capita assets do not change. At time $t+1$ suppose the government increases government consumption by the amount of the increment in output due to increased investment, i.e., $\Delta G_{t+1} = f_1(K_{t+1}, \nu(\bar{w}_{t+1})) + 1$, and increases new debt issue so as to maintain B_{t+2} at the same level as before the experiment. It is easy to verify that the resource constraint and the budget constraint continue to be satisfied at date $t+1$ as well. The first term in (4.1a) measures the utility loss from reduced government consumption at date t and the second term in (4.1a) measures the utility gain from increased government consumption at date $t+1$ discounted by β^{17} .

Proposition 1 says that in the long run the pre-tax return to capital must equal the rate of time preference. Therefore, to show that the capital income tax is strictly positive even in the long run we need to show that $\bar{r}^* \equiv \lim_{t \rightarrow \infty} \bar{r}_t < \rho$. This is shown in Appendix A (part 2) via a series of claims. The proof is by contradiction, i.e., we rule out $\bar{r}^* \geq \rho$ by showing that per capita assets go to infinity. Since, per capita capital is bounded (there is a maximum sustainable capital stock) and per capita government debt is bounded (since tax revenues are bounded) the result follows.

Proposition 2. $\bar{r}^* < \rho$.

Proof. In Appendix A (part 2). ■

It remains to show that in a complete markets version of this model the

¹⁷The argument underlying the proof of proposition 1 is intuitively easier when government consumption is endogenous but the result does not hinge on this modeling feature. The result would still follow from the steady state version of (4.1a) without appealing to (4.1b).

capital income tax is zero in the long run, i.e., $\bar{r}^* = r$. This is done in the following proposition.

Proposition 3. Under complete markets the capital income tax rate is zero.

Proof. This follows because under complete markets the model in section II is a special case of that in Chamley [1986]. The complete markets case is equivalent to eliminating the idiosyncratic uncertainty, i.e., setting $\theta_t = E(\theta)$, across agents as well as t . For simplicity assume that initially all agents have the same assets, i.e., we have identical agents. In this case, the intertemporal Euler condition for an agent is given by $u'(c_t) = \beta(1+\bar{r}_{t+1})u'(c_{t+1})$. Therefore, in the steady state, $\bar{r}^* = \rho$. Proposition 1 continues to hold in the complete markets case implying that $r = \rho$. Hence, it follows that the capital income tax is zero in the long run. ■

V. Quantitative Results

The goal of this section is to suggest that the observed labor and capital income tax rates in the U. S. economy cannot be easily dismissed as being far from optimal relative to a reasonably parameterized version of the model described earlier. To put this another way, the question is whether it is possible to interpret the observed tax rates as being (approximately) the limiting values of the tax rates associated with the solution path of a dynamic Ramsey optimal tax problem for the type of model described earlier. To address this question we proceed as follows.

We try to construct a locus of points where each point represents a pair (τ_n, τ_k) , where τ_n and τ_k are the long run labor and capital income tax rates associated with a solution of the dynamic Ramsey optimal tax problem

for particular initial conditions. The locus is generated by varying the initial conditions. A possible example of such a locus is shown in figure 2¹⁸. For comparison, in the types of environments considered by Chamley [1986], the locus would coincide with the portion of the horizontal axis in figure 2 representing a value of τ_k equal to zero, and values of τ_n between zero and some upper bound.

A locus such as the one in figure 2 can be used to address the question posed in the following way. If the observed labor and capital income tax rates are quite far from the locus then one can conclude that the actual tax policy is quite far from being long run optimal. That is, no conceivable initial conditions could have led to the observed tax rates as being long run optimal. Therefore, a minimum condition for not dismissing the observed tax rates as being long run optimal is that they lie on the locus. Of course, the observed tax rates being on the locus does not imply that actual tax policy is long run optimal. This latter question can only be resolved by computing the solution path for the optimal tax problem and the associated limiting values. This computational problem is very hard, and it requires

¹⁸The reason that there is a locus of points rather than a single point in figure 1 is that, in general, different initial conditions will lead to different steady state values of the tax rates. For example, assume that government consumption is exogenous, and that two economies differ in initial per capita assets (A_0), and initial capital (K_0), but have the same initial debt (B_0), assumed positive. The economy with the higher K_0 (say, economy 1) has available a higher capital levy at date zero. Consequently, we would expect economy 1 to reach a steady state with a lower level of debt and taxes than the other economy. A similar argument holds if the two economies have the same initial capital K_0 , but differ in terms of initial debt B_0 (assumed positive), provided that we do not permit $\bar{r}_0 = -1$, i.e., we do not permit all of initial assets of consumers to be taxed away. Note that we are assuming here that the solution to the optimal tax problem converges to a steady state.

taking a stand on initial conditions¹⁹. In contrast, constructing the locus only involves computing steady states which is a lot easier. However, the drawback is that we are only able to provide a more limited answer.

We will show that there exist reasonable parameter values for which the observed labor and capital income tax rates are quite close to the locus depicted in figure 2.

If the solution path of the dynamic Ramsey optimal tax problem converges to a steady state then the limiting values must be consistent with the steady state analysis of Section III. Therefore, the locus of steady state tax rate pairs (τ_n, τ_k) can be constructed by following the procedure outlined in the steady state analysis of section III with the pre-tax return r being set equal to ρ by virtue of proposition 1²⁰. The steady state value of government consumption (G) is taken as given. For each given value of τ_n the corresponding value of \bar{r} , and, hence, τ_k can be determined from the asset market equilibrium condition. Thus, we can find steady state pairs (τ_n, τ_k) .

We now describe model specification and parameterization, and the results. Details of computation are described in Appendix B.

Model Specification and Parameterization

The model period is taken to be 1 year and the utility discount factor

¹⁹The computational problem is hard because the consumer's problem is non-stationary and one of the state variables for the economy is the cross-section distribution of asset holdings, which is an infinite dimensional variable.

²⁰It is possible that some steady states described in Section III (with r equal to ρ) may not be approachable from any initial conditions. There is no way to get a handle on this without doing the explicit dynamic analysis which is beyond the scope of this paper.

β is set at 0.96. The utility function over private consumption is of the CRRA form, $u(c) = [c^{1-\mu}-1]/(1-\mu)$. Two different values of the risk aversion coefficient $\mu \in \{3,5\}$ are considered.

The market sector net production function $f(K,N)$ is specified as $K^\alpha N^{1-\alpha} - \delta K$ with the capital share parameter (α) taken to be 0.36 and the depreciation rate of capital (δ) taken to be 0.08.

The specification of home production is as follows,

$$(5.1) \quad h_t = (e_t D)^{-1/\lambda} (1-n_t^{1+1/\lambda}) / (1+1/\lambda), \quad \lambda > 0, \quad D > 0, \quad 0 \leq n_t \leq 1,$$

which leads to the following individual market labor supply function²¹.

$$(5.2) \quad n_t = e_t D(\bar{w})^\lambda.$$

Thus, λ represents the labor supply elasticity. We experimented with three different values of $\lambda \in \{2, 1.5, 1\}$. The mean of e_t is normalized to unity and the value of D is chosen such that when the tax rate on labor (τ_n) is 0.35, per capita market work $N = D(\bar{w})^{1/\lambda} = 1/3$. According to Lucas [1990] the labor and capital income tax rates are both 0.36.

The stochastic process for e_t is specified as a Markov chain with seven states to match the following first order autoregressive representation for $\log(e_t)$.

$$(5.3a) \quad \log(e_t) = (\text{Const.}) + \rho_e \log(e_{t-1}) + \sigma_e (1-\rho_e^2)^{1/2} \varepsilon_t,$$

²¹Note that by interpreting $e_t^{-1/\lambda}$ as θ_t and writing n_t as $[1-(1-n_t)]$ the specification in (5.1) can be seen to be a special case of the specification in Section II.

$$(5.3b) \quad \sigma_e \in \{0.2, 0.4\}, \rho_e \in \{0, 0.6\}, \varepsilon_t - \text{Normal}(0,1).$$

The constant in (5.3a) is chosen so that $E(e_t)$ equals unity. The coefficient of variation equals σ_e and the serial correlation coefficient equals ρ_e . We then follow the procedure described in Deaton [1991, p.1232] and Tauchen [1986] to approximate the above autoregression by a seven state Markov chain. Table 1 below reports the σ_e and ρ_e values implied by the Markov chain and shows that the approximation is quite good.

Table 1

Markov Chain Approximation to the Labor Endowment Shock

	Markov Chain σ_e / Markov Chain ρ_e	
$\sigma_e \backslash \rho_e$	0	.6
.2	.21/0	.21/.59
.4	.43/0	.44/.58

The values of σ_e and ρ_e were chosen from various studies of individual market hours and individual market earnings, since (5.2) implies that individual market hours and individual market earnings are proportional to e_t . Kydland [1984] reports that the standard deviation of annual hours worked from PSID data is about 15 per cent. Abowd and Card [1987, 1989] use data from the PSID and NLS and calculate that the standard deviations of per cent changes in real earnings and annual hours are about 40 per cent and 35 per cent, respectively. The implied value for the coefficient of variation (c.v.) in earnings depends on the serial correlation in earnings. If earnings are i.i.d. this yields a figure of 28 per cent for the c.v. of

earnings. Positive correlation would lead to a larger figure²². The covariances reported in Abowd and Card [1987, Table 3, p.727 and 1989, Tables IV, V, VI, pp.418-422] suggest a first order serial correlation coefficient of about 0.3. This would give a figure of 34 per cent for the c.v. of earnings. Heaton and D. Lucas [1992] also use PSID data to estimate several versions of equation (5.3a). Their estimates (see their Tables A.2 - A.5) indicate a range of 0.23 to 0.53 for ρ_e and a range of 0.27 to 0.4 for σ_e . These studies suggest that a c.v. of 20-40 percent in earnings at an annual rate may be reasonable.

Note that we have made no allowance for the possibility that the reported earnings variabilities contain significant measurement error. As the discussion in the papers by Abowd and Card suggests, this is a serious possibility, and the relevant degree of idiosyncratic earnings variability may be somewhat lower. However, this is balanced by the possibilities that the data do not include uninsured losses and taste shocks. In addition since the agents in the model are infinitely rather than finitely lived a larger value of σ_e may be needed to capture the relevant degree of variability in permanent income²³.

²²Let y be the log of earnings, σ_y be the standard deviation (s.d.) of y , and σ_g be the s. d. of $(y_t - y_{t-1})$. Suppose that y follows the first order process: $y_t = \text{trend}_t + \rho y_{t-1} + \varepsilon_t$, where ε is i.i.d. It is straightforward to calculate that $\sigma_y / \sigma_g = [2(1-\rho)]^{-1/2}$.

²³Suppose that earnings (y_t) follow the process: $y_{t+1} = (1-\rho)y^a + \rho y_t + \sigma(1-\rho^2)^{1/2} \varepsilon_{t+1}$, where ε_t is i.i.d. with zero mean. Let $\gamma = 1/(1+r)$ be the market discount factor and T be the horizon. Then permanent income (y_t^p) is given by: $y_t^p = y^a + (y_t - y^a)[(1-\gamma)/(1-\gamma\rho)][1-(\gamma\rho)^T]/(1-\gamma^T)$. Therefore, the variability of permanent income as measured by the standard deviation (s.d.) is higher when the horizon is finite as compared to when the horizon is

Government consumption is chosen to be 20 percent of gross market output, i.e., $G/[f(K,N)+\delta K] = 0.2$, when the labor tax rate is 0.35^{24} .

The computational details are described in Appendix B. We now describe the results.

Results

Tables 2, 3 and 4 below contain the results of the above computational exercise for three different sets of parameter values. For comparison the values of the labor and capital income tax rates for the U. S. as calculated by Lucas [1990] are reported in Table 2. A reasonable range of values for the labor income tax rate is chosen since the value calculated by Lucas [1990] is probably only an approximation.

In Table 2, labor supply is fairly elastic, the idiosyncratic shock has relatively low variability and is i.i.d. over time. The capital income tax rates are quite close to zero suggesting that the results of Chamley [1986] and Lucas [1990] would continue to hold (approximately) for this economy. It

infinite. For illustrative purposes if we take $r = 0.04$, $\rho = 0$, and $T = 50$, then the s.d. of permanent income is higher by a factor of 1.16 as compared to the infinite horizon case. Note that higher values of ρ reduce this adjustment factor. This suggests that to capture the effects of the observed variability in earnings in a model with infinitely lived agents the standard deviation of earnings in the model needs to be scaled up by a factor of about 1.2. The Markov chain approximation that we use tends to deliver this automatically for the high value of σ_e (0.4); see Table 1 in the text.

²⁴Most of the above functional forms and parameter values are commonly employed in quantitative analyses of aggregative models of growth and business cycles. See, for example, Prescott [1986]. Our specification of home production is not so common but the values of the labor supply elasticity and the fraction of total time spent in market work are quite commonly employed in the above mentioned studies.

appears that one can easily reject the notion that the observed capital income tax rate is anywhere close to being optimal.

Table 2 ($\mu = 3$, $\lambda = 2$, $\sigma_e = 0.2$, $\rho_e = 0$)

	U. S. DATA (LUCAS 1990)			
τ_n	0.30	0.35	0.4	0.36
τ_k	0.004	0.005	0.007	0.36

However, Tables 3 and 4 below show that modest changes in parameter values are sufficient to generate quite large values for the capital income tax rate; values that cluster around the actual value from the data. In these tables labor supply is less elastic and market earnings are more variable and persistent. With the parameter values of Tables 3 and 4, one cannot now easily dismiss the possibility that the observed capital income tax rate is close to being optimal. One could even conclude that the observed tax rate is too low rather than too high.

Table 3 ($\mu = 5$, $\lambda = 1.5$, $\sigma_e = 0.4$, $\rho_e = 0.6$)

τ_n	0.30	0.35	0.4
τ_k	0.25	0.28	0.32

Table 4 ($\mu = 3$, $\lambda = 1$, $\sigma_e = 0.4$, $\rho_e = 0.6$)

τ_n	0.30	0.35	0.4
τ_k	0.44	0.44	0.45

Risk aversion, serial correlation in earnings, and the labor supply elasticity have quite a significant impact on the capital income tax rate. A

higher risk aversion coefficient leads individuals to desire to accumulate a larger quantity of assets requiring a higher tax on capital income in order to maintain the modified golden rule level of capital. The intuition for the effect of serial correlation is that a high persistence in earnings implies a much larger variability in the consumer's permanent income which is the relevant measure for precautionary saving. As a consequence, high persistence leads to much larger desired saving and capital accumulation requiring a larger tax on capital income to maintain the modified golden rule capital. A lower elasticity of labor supply makes total (market plus non-market) earnings more variable also leading to larger desired asset holdings and thereby requiring a higher capital income tax. The influence of the labor supply elasticity is quite strong. For example, if the labor supply elasticity is reduced from 1.5 to 1 with the other parameter values as in Table 3, the capital income tax rates rise from around 28 percent to around 58 percent.

The above results are consistent with the results found in Aiyagari [1992] in which the impact of precautionary saving on aggregate saving (with no taxes) was studied. In that paper, high values of risk aversion, variability and serial correlation in earnings led to significant increases in the aggregate capital stock and the aggregate saving rate, and lowered the return to capital significantly below the rate of time preference rate. This suggests that a large tax on capital income would indeed be needed to bring capital accumulation back to the modified golden rule capital. This is confirmed by Tables 3 and 4.

APPENDIX A

This appendix is divided into two parts, part 1 (proof of Theorem 1) and part 2 (Proof of Proposition 2).

Part 1 (Proof of Theorem 1)

We start by providing bounds for K_t , G_t , \bar{w}_t , \bar{r}_t .

Claim 1. (i) $\{K_t\}$ is bounded above, (ii) $\{G_t\}$ is bounded above.

Proof. (i) From the resource constraint (2.10) we have that $K_{t+1} \leq f(K_t, 1) + K_t + H_t \leq f(K_t, 1) + K_t + \theta_{\max} H(1)$. Let K' satisfy the following equation: $f(K', 1) + \theta_{\max} H(1) = 0$. Such a K' exists since $f(\cdot)$ is output net of depreciation and by assumption $\lim_{K \rightarrow \infty} f_1 < 0$. Define $K_{\max} = \max [K_0, K']$, where K_0 is the initial per capita capital. Then it is obvious that $K_t \leq K_{\max}$ for all t .

(ii) $G_t \leq f(K_{\max}, 1) + K_{\max} + \theta_{\max} H(1) \equiv G_{\max}$. ■

Now let $w_{\min} \equiv \theta_{\min} H'(1) > 0$ and let

$$(A.1) \quad \phi(\bar{w}_t) \equiv f_2(K_{\max}, \nu(\bar{w}_t)).$$

$\phi(\bar{w}_t)$ is a continuous and strictly decreasing function, tends to ∞ as \bar{w}_t tends to w_{\min} , and tends to $f_2(K_{\max}, 1)$ as \bar{w}_t tends to ∞ . Therefore, there exists a unique positive value, denoted w_{\max} , such that $w_{\max} = \phi(w_{\max})$. Let $r > -\delta$ be the return to capital and let $z(r)$ be the corresponding capital-labor ratio, i.e., $f_1(z(r), 1) \equiv r$. Let $w(r) \equiv f_2(z(r), 1)$ be the corresponding market real wage. Note that $w(r)$ is strictly decreasing, tends to ∞ as r tends to $(-\delta)$ and tends to zero as r tends to ∞ . Let r_{\max} be defined by $w(r_{\max}) = w_{\min}$.

Now we define the set $S \equiv [0, K_{\max}]^{\infty} \times [0, G_{\max}]^{\infty} \times [w_{\min}, w_{\max}]^{\infty} \times [0, 1+r_{\max}]^{\infty} \subseteq \mathbb{R}^{\infty}$ and endow S with the subspace product topology. Let $K^1 \equiv (K_1, K_2, K_3, \dots) \in [0, K_{\max}]^{\infty}$, $G^0 \equiv (G_0, G_1, G_2, \dots) \in [0, G_{\max}]^{\infty}$, $\bar{w}^0 \in [w_{\min}, w_{\max}]^{\infty}$, and $\bar{R}^0 \in [0, 1+r_{\max}]^{\infty}$.

Let $S^* = \{(K^1, G^0, \bar{w}^0, \bar{R}^0) \in S \mid (K^1, G^0, \bar{w}^0, \bar{R}^0) \text{ satisfy the constraint (2.10)}\}$.

Claim 2. The set S^* is compact.

Proof. S is compact by Tychonoff's theorem and S^* is a closed subset of S . ■

Claim 3. The maximand in (2.11) is continuous (in the product topology) over S .

Proof. Follows from the fact that $u(\cdot)$ and $U(\cdot)$ are bounded and continuous. Therefore, $v(\cdot)$ and $\Sigma \beta^t U(\cdot)$ are continuous in the product topology over S . ■

Theorem 1. A solution to the optimal tax problem exists.

Proof. In view of claims 2 and 3, we only need to show that S^* is nonempty. Choose $K_t = 0$, $t \geq 1$, $G_0 = (1-\delta)K_0$, $G_t = 0$, $t \geq 1$, $\bar{w}_t = w_{\min}$, $t \geq 0$, $\bar{r}_t = -1$, $t \geq 0$. Note that $N_t = v(\bar{w}_t) = 0$, $t \geq 0$. Under this policy, individuals never hold any positive assets, i.e., the asset distribution is completely concentrated at zero for $t \geq 1$. Individuals work only at home and eat whatever they produce. Therefore, per capita consumption equals per capita home production. Further, government debt $B_t = 0$, $t \geq 1$. ■

Part 2 (Proof of proposition 2)

We start by stating some simple properties of the solution to the agent's optimization problem (2.5) which will be needed later. Let l_∞ be the space of bounded sequences with the sup norm (denoted $[\cdot]_\infty$), and let l_∞^+ be the non-negative orthant of l_∞ . Let $\Theta = [\theta_{\min}, \theta_{\max}]$. Let $C = \{v: \mathbb{R}_+ \times \Theta \times l_\infty^+ \times l_\infty^+ \rightarrow \mathbb{R} \mid v \text{ continuous and bounded}\}$ and let the norm on C be the sup norm. The following proposition consists of easy extensions of standard results and, hence, the proof is omitted (see, for example, Stokey and Lucas with Prescott, 1989, chapter 9).

Claim 4. (i) There exists a unique $v \in C$ which solves the functional equation (2.5); further $v = \sup E_t \{ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \}$ subject to (2.5b); (ii) v is strictly increasing and strictly concave in a_t ; (iii) There exist unique decision rules (2.6) which attain v ; (iv) The decision rules (2.6) are continuous and non-decreasing in a_t ; (v) v is continuously differentiable in a_t and $v_1(a_t, \theta_t, \bar{w}^t, \bar{R}^t) = (1+\bar{r}_t)u'(c_t)$; (vi) The solution to the maximization problem on the right side of (2.5) is characterized by: $u'(c_t) \geq \beta E_t v_1(a_{t+1}, \theta_{t+1}, \bar{w}^{t+1}, \bar{R}^{t+1})$ with equality if $a_{t+1} > 0$, where E_t

denotes expectation conditional on information at time t . ■

Now we show that $\bar{r}^* \leq \rho$. This result uses a special case of Theorem 1 (p.12) and Theorem 2 (p.15) of Chamberlain and Wilson [1984].

Claim 5. $\bar{r}^* \leq \rho$.

Proof: Suppose if possible that $\bar{r}^* > \rho$. Let $\zeta_t = \beta^t \prod_{j=0}^t (1+\bar{r}_j)$, and note that $\zeta_t \rightarrow \infty$.

From Claim 4 (v) and 4 (vi) the following intertemporal Euler equation holds for a typical agent.

$$(A.2) \quad u'(c_t) \geq \beta(1+\bar{r}_{t+1})E_t\{u'(c_{t+1})\}, \text{ with equality if } a_{t+1} > 0.$$

By multiplying both sides of (A.2) by ζ_t we can rewrite it as follows.

$$(A.3) \quad \zeta_t u'(c_t) \geq \zeta_{t+1} E_t\{u'(c_{t+1})\}, \text{ with equality if } a_{t+1} > 0.$$

It follows that $\zeta_t u'(c_t)$ is a non-negative super martingale. Further, $y(\theta_t, \bar{w}_t) \geq \theta_{\min} H(1) > 0$, implies that $\zeta_0 u'(c_0) < \infty$. Therefore, $\zeta_t u'(c_t)$ converges with probability one (w. p. 1) to a finite random variable (Doob, p. 324, Theorem 4.1s). Since $\zeta_t \rightarrow \infty$, it follows that $u'(c_t) \rightarrow 0$ w. p. 1, and, hence, that $c_t \rightarrow \infty$ w. p. 1. Since it must hold for all individuals this implies that per capita consumption $C_t \rightarrow \infty$. However, assumption 1 and proposition 1 imply that $K_t \rightarrow K > 0$ and finite. The resource constraint (2.10) then implies that $C_t \rightarrow C$ finite, which is a contradiction. Therefore, $\bar{r}^* \leq \rho$. ■

Now we rule out the possibility that $\bar{r}^* = \rho$. This is done by showing that when \bar{r}^* equals ρ per capita assets go to infinity. However, since per capita capital is bounded (there is a maximal sustainable capital stock) and per capita government debt is bounded above (because tax revenues are bounded above) this leads to a contradiction. Thus, we establish that $\bar{r}^* \neq \rho$. Hence, $\bar{r}^* < \rho$.

Consider the following stationary problem (denoted $P(S, \rho)$).

$P(S, \rho)$: maximize $E_0\{\sum_{t=0}^{\infty} \beta^t u(c_t)\}$ subject to:

$$(A.4) \quad c_t + a_{t+1} = y(\theta_t, \bar{w}^*) + (1+\rho)a_t, \quad c_t \geq 0, \quad a_t \geq 0, \quad t \geq 0.$$

In contrast the original problem with constraints (2.5b) is non-stationary and is denoted P(NS). Note that when $\bar{r}^* = \rho$, problem P(S, ρ) is obtained by substituting the limiting values of \bar{w}_t and \bar{r}_t in problem P(NS). We will use the result that for problem P(S, ρ), $E(a_t | a_0=0, \theta_0, P(S, \rho)) \rightarrow \infty$. Using this we will show that when $\bar{r}^* = \rho$, $E(a_t | a_0, \theta_0, P(NS)) \rightarrow \infty$. This implies that per capita assets for P(NS) go to infinity because $A_t = \int \text{ad}J_t(a, \theta, \bar{w}^0, \bar{R}^0) = \int E[a_t | a_0, \theta_0, P(NS)] dJ_0(a_0, \theta_0)$.

Claim 6. $E(a_t | a_0=0, \theta_0, P(S, \rho)) \rightarrow \infty$.

Proof. See Corollary 2 (p.26) of Chamberlain and Wilson [1984]. They show that $\text{Prob}(\lim_{t \rightarrow \infty} c_t = \infty) = 1$. It follows that $\text{Prob}(\lim_{t \rightarrow \infty} a_t = \infty) = 1$, since $c_t \leq y(\theta_t, \bar{w}^*) + (1+\rho)a_t$. Therefore, $E(a_t | a_0=0, \theta_0, P(S, \rho)) \rightarrow \infty$. ■

Claim 7. If $\bar{r}^* = \rho$, then $E(a_t | a_0, \theta_0, P(NS)) \rightarrow \infty$.

Proof: Since the asset accumulation decision rule (2.6b) is non-decreasing in a_t (Claim 4 iv) it follows that $E(a_t | a_0, \theta_0, P(NS)) \geq E(a_t | a_0=0, \theta_0, P(NS))$. Therefore, it is sufficient to show that $E(a_t | a_0=0, \theta_0, P(NS)) \rightarrow \infty$. So suppose to the contrary that $E(a_t | a_0=0, \theta_0, P(NS)) \rightarrow x < \infty$. Then, it must be true that for any date τ , $E(a_{t+\tau} | a_\tau=0, \theta_\tau, P(NS)) \rightarrow x < \infty$. This is because $E(a_{t+\tau} | a_0=0, \theta_0, P(NS)) = E(E(a_{t+\tau} | a_\tau, \theta_\tau, P(NS)) | a_0=0, \theta_0, P(NS))$, and $E(a_{t+\tau} | a_\tau, \theta_\tau, P(NS)) \geq E(a_{t+\tau} | a_\tau=0, \theta_\tau, P(NS))$, again by Claim 4(iv). Therefore there exists a subsequence of dates $\{t_j + \tau\}$ and a number M such that

$$(A.5) \quad E(a_{t_j + \tau} | a_\tau=0, \theta_\tau, P(NS)) < M < \infty, \quad \text{for all } t_j > 0.$$

Since $E(a_t | a_0=0, \theta_0, P(S, \rho)) \rightarrow \infty$ (Claim 6), there exists $T < \infty$ such that for any τ ,

$$(A.6) \quad E(a_{t+\tau} | a_\tau=0, \theta_\tau, P(S, \rho)) > M+1, \quad t \geq T.$$

In view of (A.5) we can choose T in such a way that

$$(A.7) \quad E(a_{T+\tau} | a_\tau=0, \theta_\tau, P(NS)) < M < \infty.$$

Let $\bar{w}(S) = (\bar{w}^*, \bar{w}^*, \bar{w}^*, \dots)$, and $\bar{R}(S) = (1+\rho, 1+\rho, 1+\rho, \dots)$. Let A_T be an upper bound on asset holdings that can be attained in T periods starting from zero assets at any date τ in $P(NS)$. Such an upper bound exists (independently of the starting date τ) since $\{\bar{w}_t\}$, $\{\bar{R}_t\}$, and $y(\theta_t, \bar{w}_t)$ are bounded. It follows that $E\{a_{T+\tau} | a_\tau=0, \theta_\tau, P(NS)\} \leq A_T$ for all τ . Now note that $E\{a_{T+\tau} | a_\tau=0, \theta_\tau, P(NS)\}$ depends only on \bar{w}^τ and \bar{R}^τ . Therefore, by making τ suitably large we can make $|\bar{w}^\tau - \bar{w}(S)|_\infty$ and $|\bar{R}^\tau - \bar{R}(S)|_\infty$ as small as we like. Hence, by the continuity of the asset accumulation decision rule (2.6b) - Claim 4(iv) - we can choose a τ sufficiently large such that the following holds.

$$(A.8) \quad |E\{a_{T+\tau} | a_\tau=0, \theta_\tau, P(NS)\} - E\{a_{T+\tau} | a_\tau=0, \theta_\tau, P(S, \rho)\}| < 1.$$

However, (A.6), (A.7) and (A.8) are mutually contradictory. Therefore, $E\{a_t | a_0=0, \theta_0, P(NS)\} \rightarrow \infty$. Hence, $E\{a_t | a_0, \theta_0, P(NS)\} \rightarrow \infty$. ■

Proposition 2. $\bar{r}^* < \rho$.

Proof. By Claim 5, $\bar{r}^* \leq \rho$. So, suppose, if possible, that $\bar{r}^* = \rho$.

From the government budget constraint (2.2) we have

$$(A.9) \quad B_t \leq [B_{t+1} + f(K_t, 1)] / (1 + \bar{r}_t).$$

Let $\bar{\gamma}_t = \prod_{j=0}^t (1 + \bar{r}_j)^{-1}$. Since $\bar{r}_t \rightarrow \rho > 0$, consumer optimization implies that $\lim_{j \rightarrow \infty} \bar{\gamma}_{t+j} a_{t+j+1} = 0$ (a.s.), and, hence, that $\lim_{j \rightarrow \infty} \bar{\gamma}_{t+j} A_{t+j+1} = 0$. Since, $B_t = A_t - K_t$, and $\{K_t\}$ is bounded (claim 1), it follows that $\lim_{j \rightarrow \infty} \bar{\gamma}_{t+j} B_{t+j+1} = 0$. Using this in (A.9) and noting that $\{f(K_t, 1)\}$ is bounded above we can conclude that $\{B_t\}$ is bounded above. Therefore, $\{K_t + B_t\}$, and, hence, per capita assets are bounded above. This contradicts Claim 7 and shows that $\bar{r}^* \neq \rho$. This fact together with Claim 5 establishes that $\bar{r}^* < \rho$. ■

APPENDIX B

Details of Computation*Markov Chain Approximation to (5.3)*

We divide the real line into seven intervals as follows. $I_1 = (-\infty, -5\sigma_e/2)$, $I_2 = (-5\sigma_e/2, -3\sigma_e/2)$, $I_3 = (-3\sigma_e/2, -\sigma_e/2)$, $I_4 = (-\sigma_e/2, \sigma_e/2)$, $I_5 = (\sigma_e/2, 3\sigma_e/2)$, $I_6 = (3\sigma_e/2, 5\sigma_e/2)$, and $I_7 = (5\sigma_e/2, \infty)$. The state space of $\ln(e_t)$ is taken to be the finite set $\{-3\sigma_e, -2\sigma_e, -\sigma_e, 0, \sigma_e, 2\sigma_e, 3\sigma_e\}$ so that $e_i = \exp[(i-4)\sigma_e]$, $i = 1, 2, \dots, 7$. We then compute the transition probabilities $\pi_{ij} = \text{prob}\{\ln e_{t+1} \in I_j : \ln e_t = \log e_i\}$ by numerical integration using the Normal $(0, 1)$ density for ε_t assumed in (5.3b). We then compute the stationary probability vector p associated with the probability transition matrix π and the expected value $Ee = \sum_i p_i e_i$. The per capita value of e is normalized to unity by scaling the support of its distribution by Ee . That is, we define $e'_i = e_i/Ee$. The Markov chain for $\{e_t\}$ is defined by the state space $\{e'_i\}$ together with the probability transition matrix π . Note that e' will have the same coefficient of variation and serial correlation coefficient as e . Table 1 in the text shows that the approximation is quite good for moderate values of σ_e , though for high values of σ_e the Markov chain has a somewhat higher coefficient of variation. We also tried the following alternative for calculating the transition probabilities: $\pi_{ij} = \text{prob}\{\ln e_{t+1} \in I_j : \ln e_t \in I_i\}$. This procedure yielded a very good approximation to σ_e even for high values. However, its approximation to ρ_e (especially for the high values) was not so good. The values of ρ_e based on the Markov chain were somewhat lower.

Solving (3.3) for \bar{r}

Recall that in the quantitative analysis the idiosyncratic shock e_t may be serially correlated over time. The Bellman equation for the consumer's problem can be written as follows.

$$(B.1) \quad v(z_t, e_t) = \max\{u(z_t - a_{t+1}) + \beta E_t[v(\bar{w}\theta_{t+1} + (1+\bar{r})a_{t+1}, e_{t+1}) | e_t]\},$$

subject to:

$$(B.2) \quad 0 \leq a_{t+1} \leq z_t \leq y(e_t, \bar{w}) + (1+\bar{r})a_t,$$

where $y(e_t, \bar{w})$ is total earnings and the maximum on the right side of (B.1)

is taken over a_{t+1} . This leads to an asset demand function of the form

$$(B.3) \quad a_{t+1} = a(z_t, e_t),$$

which together with the definition of z_t and the Markov chain for e_t determines the stochastic evolution of a_t .

We approximate the asset demand as a function of z_t (for each of seven possible current values of e) by a continuous, piece-wise linear function over an interval with 27 sub-intervals not of equal length. Finer sub-intervals were chosen at the lower end of the interval and coarser sub-intervals at the upper end of the interval²⁵.

The algorithm for finding the value of \bar{r} that solves (3.3) uses simulated series and the bisection method. To initialize the process, let r_1 equal ρ and let r_2 equal zero. Let r_3 equal $(r_1+r_2)/2$. We then compute the asset demand function as described above corresponding to r_3 . We then simulate the Markov chain for the labor endowment shock using a random number generator and obtain a series of 50,000 draws²⁶. These are used with the asset demand function to obtain a simulated series of assets. The sample mean of this is taken to be equal to per capita assets A_3 . If A_3 exceeds $K+B$ evaluated at r_3 (see equation 3.1), then r_1 is replaced by r_3 , a new r_3 is calculated and the process is repeated. If A_3 is less than $K+B$ evaluated at r_3 , then r_2 is replaced by r_3 , a new r_3 is calculated and the process is repeated. Note that by construction r_1 and r_2 are always on opposite sides of the steady state interest rate \bar{r}^* , and that with each iteration $|r_1-r_2|$ is getting halved (see figure 1). Typically, this yields an excellent approximation to the steady state within ten iterations.

²⁵The reason is that for low levels of total resources assets will be zero since the borrowing constraint will bind. At some critical level of total resources assets will become positive. This introduces a high degree of nonlinearity in the asset demand function. Consequently, it is important to have a finer grid at the lower end of the interval to obtain a good approximation. It turned out that throughout the upper half of the interval the asset demand function was very nearly linear so that a small number of grid points was adequate to obtain a good approximation in this region.

²⁶The results were about the same even when we used 20,000 draws.

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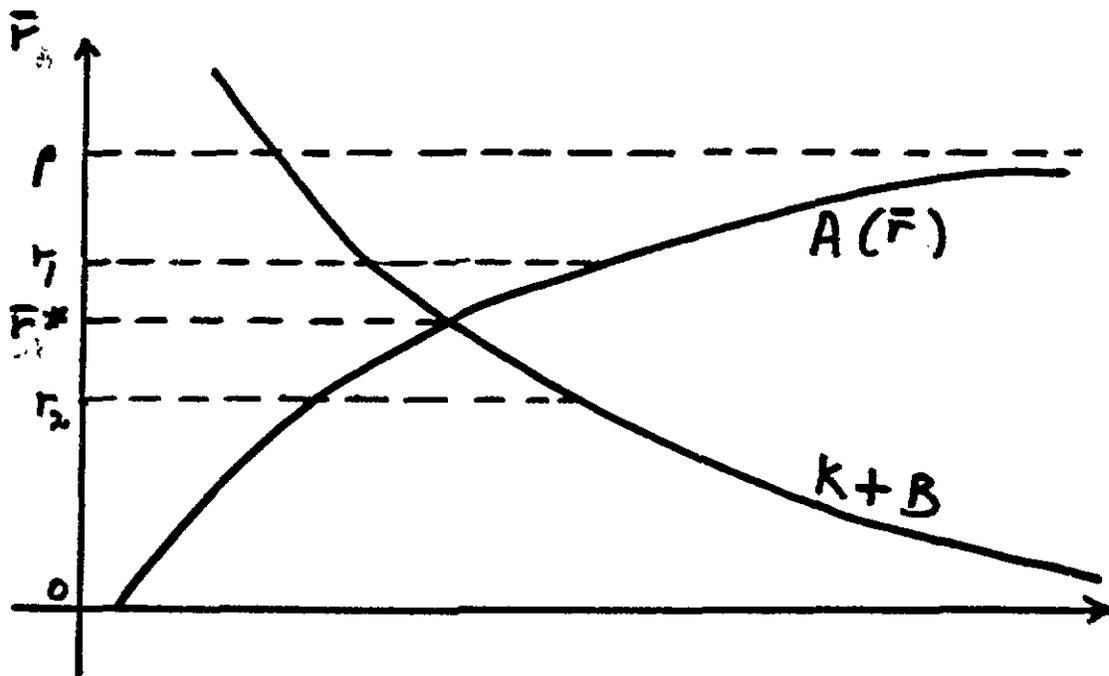


FIGURE 1

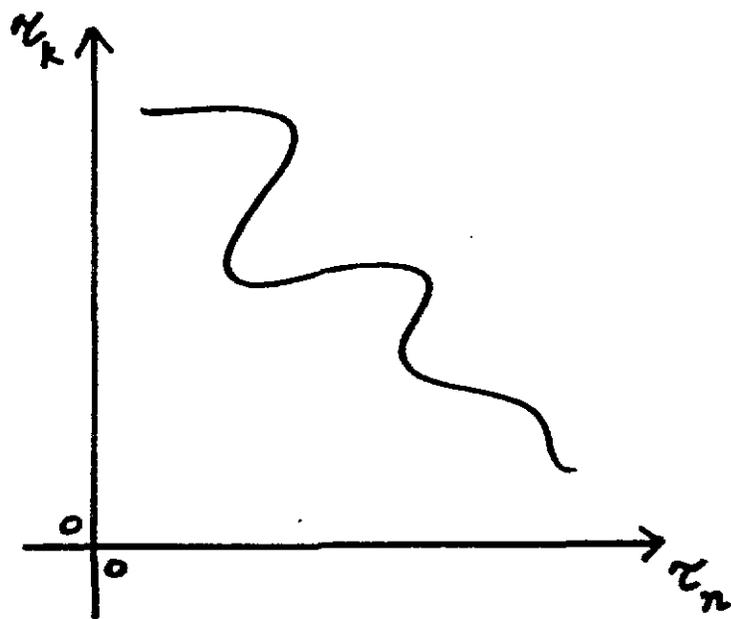


FIGURE 2