

Preliminary version

**Market Games and the Overlapping Generations
Model:
Existence and Stationary Equilibria***

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Abstract

This paper develops a dynamic model of general imperfect competition by embedding the Shapley-Shubik model of market games into an overlapping generations framework. Existence of an open market equilibrium where there is trading at each post is demonstrated when there are an arbitrary (finite) number of commodities in each period and an arbitrary (finite) number of consumers in each generation. The open market equilibria are fully characterized when there is a single consumption good in each period and it is shown that stationary open market equilibria exist if endowments are not Pareto optimal. Two examples are also given. The first calculates the stationary equilibrium in an economy, and the second shows that the on replicating the economy the stationary equilibria converge to the unique non-autarky stationary equilibrium in the corresponding Walrasian overlapping generations economy. Preliminary on-going work indicates the possibility of cycles and other fluctuations even in the log-linear economy.

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1 Introduction

The challenge for general equilibrium analysis is to extend the Arrow-Debreu-McKenzie model in such a way that features of actual markets can be explained. In this paper we combine two streams of literatures which in themselves have proved to be extremely fertile in this endeavor - the overlapping generations model and market games. The objective is to close a gap by developing a dynamic model of imperfect competition at the same level of generality as the Arrow-Debreu-McKenzie model. In addition the model developed is tractable and amenable to the program of studying the working of markets.

The overlapping generations model developed by Allais [1] and Samuelson [13] has been the leading infinite horizon general equilibrium model as it incorporates agent heterogeneity and finite lives of consumers (see also Balasko, Cass, and Shell [2], Balasko and Shell [3]). The overlapping generations model has been used extensively to not only increase our understanding of infinite horizon economies and economic fluctuations but also to study money, public finance, development issues, international economics, etc. (see Geanakoplos and Polemarchakis [8] and Shell and Smith [16] for surveys and more complete references). However, the model assumes Walrasian behavior on the part of agents which is not satisfactory in small economies and does not develop a process by which prices are determined.

To study strategic behavior while maintaining the methodology of general equilibrium theory, two main models have been used for a Cournotian foundation of the Walrasian model: the Cournot-Walras model of Gabszewicz and Vial [7], and the market game of Shapley and Shubik [15] (see also Shapley [14]). In both of the models as the number of agents becomes large, the equilibrium outcomes are Walrasian (see Gabszewicz and Vial [7], Dubey, Mas-Colell, and Shubik [5], Mas-Colell [9], and Postlewaite and Schmeidler [12]). We work with the market game model, as non-existence of equilibrium is a problem in the Cournot-Walras model (see Dierker and Grodal [4], Gabszewicz and Vial [7], and Mas-Colell [9]), and because the market game model has been helpful in studying market uncertainty (Peck and Shell [10]), monetary phenomenon, bankruptcy, etc. Unlike the Cournot-Walras model the market game model describes a rule by which prices are determined in markets. While in a large part of the literature a specific price determining rule has been used (which we adopt as well) it can be amended and generalized.

The models however that have been studied in the literature so far have been static except for the paper of Forges and Peck [6]. In this paper a similar economy with a single good in each period and a continuum of identical consumers in each generation is used to examine the relationship between correlated equilibria and sunspot equilibria.

We develop a general model where agents live for two periods ¹ and trade according to the rules of the market game. The market game is in the form modified to remove any inessential asymmetries (see Postlewaite and Schmeidler [12], Peck and Shell [10], and Peck, Shell, and Spear [11]). Each consumer offers commodities in the endowment for sale at a market or trading post (where only one commodity is traded), and bids a non-negative amount as well. There are no liquidity constraints and the general purchasing power can be transferred from one market to another through inside money. In the general formulation there is no restriction on number of commodities in each period, or the number of consumers in each generation. First, we study the existence of a perfect foresight Nash equilibrium where all the markets are open, i.e., a non-zero quantity is offered for sale and a non-zero amount (of inside money) is bid for the commodities. We show that there always exists such an equilibrium. Second, we study further properties of open market equilibria (we restrict here to a single consumption good in each period). We give a complete characterization of the equilibria. Using this characterization we show that if the endowments of the consumers are not Pareto efficient then there always exists open market stationary equilibria. These stationary equilibria exist when consumers are restricted to offer their entire endowments for sale, as well as when there are no restrictions. This points to

¹This is not a restriction for establishing the existence of a Nash equilibrium under our maintained assumptions. The argument of Balasko, Cass, and Shell [2] for competitive overlapping generations economies applies as well to the imperfectly competitive economies.

a potential multiplicity of stationary equilibria. Next, we give two examples. In the first example for a log-linear economy we calculate these stationary equilibria. In the second example we see if the result of convergence to the Walrasian equilibria still holds as the number of consumers (now in each generation) becomes large. We find that is indeed the case. The interesting thing is that all the open market stationary equilibria converge to the unique Walrasian non-autarky stationary equilibrium.

Our project extends beyond the results presented in this paper in three directions. First, as we can get explicit characterizations of equilibria we wish to do simulations which will give a clearer understanding of the equilibrium set. Secondly, we are studying the non-stationary dynamics of the model. In this, there are two directions. The first is to study dynamics under different forecasting rules, and the second is to get endogenous self-fulfilling fluctuations especially in terms of market liquidity. This has been done for exogenous randomizing devices (see Forges and Peck [6] and Peck and Shell [10]). The third direction is to generalize the allocation rule and weaken the strong restrictions placed on the characteristics of the consumers. We would like to obtain a convergence result for this general economy.

The plan of the paper is as follows. In section 2 the model is outlined. Section 3 contains the existence result, and section 4 covers characterization of perfect foresight open market equilibria, and the result on existence of stationary equilibria. The examples are in section 5.

2 The Model

There are $t = 1, 2, \dots$ periods, with an arbitrary finite number of commodities $2 \leq l_t + 1 < \infty$ in each period. The commodity 0 in each time period is inside fiat money. The commodities $l = 1, \dots, l_t$ are perishable and there is no production in the economy. In each period a finite number of consumers are born who live for two periods. Thus, each generation consists of $1 \leq \#G_t < \infty$ consumers. A consumer is indexed by (t, h) denoting the date of birth and name. In period 1 there is a generation of ‘old’ consumers who live for only one period, $h \in G_0, 1 \leq \#G_0 < \infty$.

The consumption sets of the consumers are the non-negative orthants.

$$x_h = x_h^1 = (x_h^{1,1}, x_h^{1,2}, \dots, x_h^{1,l_1}) \in \mathbb{R}_+^{l_1} \text{ for } h \in G_0$$

and

$$x_h = (x_h^t, x_h^{t+1}) = (x_h^{t,1}, \dots, x_h^{t,l_t}, x_h^{t+1,1}, \dots, x_h^{t+1,l_{t+1}}) \in \mathbb{R}_+^{l_t+l_{t+1}} \text{ for } h \in G_t, t \geq 1.$$

The endowment of each consumer $\omega_{t,h}$ lies in the interior of the consumption set. The utility function, $u_{t,h}$ of each consumer is defined over the consumption set, is strictly increasing, smooth, and strictly concave on the interior of the consumption set. Also, the closure in the consumption set of each indifference surface from the interior is contained in the interior. The boundary of the consumption set is the indifference surface of least utility.

We now define the *market game*. In each period t there are l_t trading posts. For each (Arrow-Debreu) consumption good there is a single trading

post where it is exchanged for money. Consumer (t, h) *offers* a non-negative quantity of commodity (s, l) , $q_{t,h}^{s,l}$, $s = t, t+1, l = 1, \dots, l_s$ at trading post (s, l) (consumers in generation 0 trade only in period 1). The consumer (t, h) also *bids* a non-negative quantity of money, $b_{t,h}^{s,l}$, $s = t, t+1, l = 1, \dots, l_s$ at trading post s, l . Thus, the offers and bids are given by the vectors $q_{t,h} = (q_{t,h}^t, q_{t,h}^{t+1}) = (q_{t,h}^{t,1}, \dots, q_{t,h}^{t,l_t}, q_{t,h}^{t+1,1}, \dots, q_{t,h}^{t+1,l_{t+1}})$ and $b_{t,h} = (b_{t,h}^t, b_{t,h}^{t+1}) = (b_{t,h}^{t,1}, \dots, b_{t,h}^{t,l_t}, b_{t,h}^{t+1,1}, \dots, b_{t,h}^{t+1,l_{t+1}})$ respectively for $h \in G_t$, $t \geq 1$. For $h \in G_0$ we write $q_{0,h} = (q_{0,h}^1) = (q_{0,h}^{1,1}, \dots, q_{0,h}^{1,l_1})$ and $b_{0,h} = (b_{0,h}^1) = (b_{0,h}^{1,1}, \dots, b_{0,h}^{1,l_1})$. As the offers are made in terms of the commodities they cannot exceed the endowment of the commodity, i.e., we have, $q_{t,h}^{s,l} \leq \omega_{t,h}^{s,l}$, $s = t, t+1, l = 1, \dots, l_s$. The *strategy set* of consumer (t, h) is given as follows.

$$S_h = \{(b_h, q_h) \in \mathbb{R}_+^{2l_1} : q_h \leq \omega_h\} \text{ for } h \in G_0 \quad (1)$$

and

$$S_h = \{(b_h, q_h) \in \mathbb{R}_+^{2(l_t+l_{t+1})} : q_h \leq \omega_h\} \text{ for } h \in G_t, t \geq 1. \quad (2)$$

We denote a strategy profile for all the consumers as $\sigma = (s_{t,h})_{h \in G_t, t \geq 0} = (b_{t,h}, q_{t,h})_{h \in G_t, t \geq 0}$, and the $\sigma_{-t,h}$ denotes the strategies of all consumers other than consumer (t, h) .

The trading process is as follows.

The total amount of the commodity offered at the trading post (t, l) , $Q^{t,l} = \sum_{h \in G_{t-1} \cup G_t} q_h^{t,l}$ is allocated to nonbankrupt consumers in proportion

to their bids for the commodities. Consumer (s, h) 's ($s = t - 1, t$) proportion of the bids for commodity (t, l) is $\frac{b_{s,h}^{t,l}}{B^{t,l}}$ where $B^{t,l} = \sum_{h \in G_{t-1} \cup G_t} b_h^{t,l}$. Thus, the gross allocation of the commodity to the consumer is $\frac{b_{s,h}^{t,l} Q^{t,l}}{B^{t,l}}$. Similarly, the total amount of money bid at trading post (t, l) , $B^{t,l} = \sum_{h \in G_{t-1} \cup G_t} b_h^{t,l}$ is allocated to consumers in proportion to their offers for the commodities. Consumer (s, h) 's ($s = t - 1, t$) proportion of the offers for commodity (t, l) is $\frac{q_{s,h}^{t,l}}{Q^{t,l}}$ where $Q^{t,l} = \sum_{h \in G_{t-1} \cup G_t} q_h^{t,l}$. Thus, the gross allocation of money in post (t, l) to the consumer is $\frac{q_{s,h}^{t,l} B^{t,l}}{Q^{t,l}}$. If we either have zero offers or zero bids at a trading post, then set $\frac{0}{0} = 0$.

Consumers do not face liquidity constraints. Each consumer faces a sequence of two budget constraints (except generation 0 consumers). However, the presence of inside money enables us to reduce these constraints into a single one. For the existence question it will help to work with the formulation with a single budget constraint. However, when we study dynamics the recursive formulation enabled by the sequence of constraints is more helpful. Thus, in the sequential formulation we have:

$$\begin{aligned} \sum_{l=1}^{l_t} \left(\frac{B^{t,l} q_{t,h}^{t,l}}{Q^{t,l}} \right) &\geq \sum_{l=1}^{l_t} (b_{t,h}^{t,l}) + m_{t,h} \\ \sum_{l=1}^{l_{t+1}} \left(\frac{B^{t+1,l} q_{t,h}^{t+1,l}}{Q^{t+1,l}} \right) &\geq \sum_{l=1}^{l_{t+1}} (b_{t,h}^{t+1,l}) - m_{t,h} \end{aligned}$$

for $h \in G_t$, $t \geq 1$, and $m_{t,h} \in \mathfrak{R}$ is the saving in the youth. This can be

collapsed into the single budget constraint:

$$\sum_{l=1}^{l_t} \left(\frac{B^{t,l} q_{t,h}^{t,l}}{Q^{t,l}} \right) + \sum_{l=1}^{l_{t+1}} \left(\frac{B^{t+1,l} q_{t,h}^{t+1,l}}{Q^{t+1,l}} \right) \geq \sum_{l=1}^{l_t} (b_{t,h}^{t,l}) + \sum_{l=1}^{l_{t+1}} (b_{t,h}^{t+1,l})$$

for $h \in G_t$, $t \geq 1$. For the consumers who are old at time period 1 ($h \in G_0$) we have the single budget constraint.

$$\sum_{l=1}^{l_1} \left(\frac{B^{1,l} q_{0,h}^{1,l}}{Q^{1,l}} \right) \geq \sum_{l=1}^{l_1} (b_{0,h}^{1,l}).$$

For a given a given strategy of a consumer, the consumption allocation is given as follows.

$$\begin{aligned} x_{t,h}^{s,l} &= \omega_{t,h}^{s,l} - q_{t,h}^{s,l} + \frac{b_{t,h}^{s,l} Q^{s,l}}{B^{s,l}} && \text{if budget constraint is satisfied} \\ x_{t,h}^{s,l} &= 0 && \text{if budget constraint is not satisfied} \end{aligned}$$

for $t = 0, 1, \dots$, $s = t, t+1$ if $t \geq 1$, and $s = t+1$ if $t = 0$.²

We have defined the set of players, strategies, and payoffs for all the player. This defines the market game, Γ . We also define an offer-constrained market game, $\Gamma(\bar{q})$, where the offers of each agent (t, h) is constrained to be equal to $\bar{q}_{t,h}$. The infinite dimensional vector \bar{q} is defined as vector $\bar{q} = (\bar{q}_{t,h})_{t \geq 0}$. As the solution concept we use Nash equilibrium. We also define a T -Nash equilibrium where the strategies are required to be an equilibrium strategies only for the first T periods. The definition of Nash equilibria and T -Nash equilibria can be applied to either the game Γ or the game $\Gamma(\bar{q})$.

²This is a credible mechanism as the allocation is feasible for all feasible strategies.

Definition 2.1.

A strategy profile $\sigma^* = (s_{t,h}^*)_{h \in G_t, t \geq 0} = (b_{t,h}^*, q_{t,h}^*)_{h \in G_t, t \geq 0}$ is a (Perfect Foresight) Nash Equilibrium for the market game Γ if:

$$u_{t,h}(s_{t,h}^*, \sigma_{-t,h}^*) \geq u_{t,h}(s_{t,h}, \sigma_{-t,h}^*) \quad \forall (t, h) \quad (3)$$

Definition 2.2.

A strategy profile $\sigma^* = (s_{t,h}^*)_{h \in G_t, t \geq 0} = (b_{t,h}^*, q_{t,h}^*)_{h \in G_t, t \geq 0}$ and where $s_{T,h}^* = (s_{T,h}^{T*}, s_{T,h}^{T+1*}) = ((b_{T,h}^{T*}, q_{T,h}^{T*}), (\alpha, \omega_{T,h}^{T+1}))$ where $\alpha \in (0, \infty)$ is a (Perfect Foresight) T - Nash equilibrium for the market game Γ , $T \geq 1$, if the following holds.³

$$u_{t,h}(s_{t,h}^*, \sigma_{-t,h}^*) \geq u_{t,h}(s_{t,h}, \sigma_{-t,h}^*) \quad \forall (t, h) \text{ with } t \leq T - 1 \quad (4)$$

and

$$u_{T,h}(s_{T,h}^*, \sigma_{-T,h}^*) \geq u_{T,h}(s_{T,h}, \sigma_{-T,h}^*) \quad \forall h \in G_T \quad (5)$$

and the budget constraint for $h \in G_T$ is

$$\sum_{l=1}^{l_T} \left(\frac{B^{T,l} q_{T,h}^{T,l}}{Q^{T,l}} \right) \geq \sum_{l=1}^{l_T} (b_{T,h}^{T,l}). \quad (6)$$

³If the utility function is time separable then no restriction needs to be placed on the next period strategies.

3 Existence of Nash equilibria

As in static market games, trivially a Nash equilibrium exists where all agents bid and offer zero. This equilibria is self enforcing. The more interesting question is whether an equilibrium exists where all the trading posts are open at all dates. We call such an equilibrium an interior equilibrium or an open markets equilibrium. We show that this is indeed the case. The strategy of the proof is to use the method of Balasko and Shell [3], and Balasko, Cass, and Shell [2] and work with truncation of economies. We show that if we truncate the economy at any date, T , we have a finite economy and using the method of Peck, Shell, and Spear [11] there exists an equilibria with all markets open. As we are also able to get bounds on the bids, by taking a sequence of the equilibria which form compact sets, (in the product topology) by increasing the date of truncation, we get a limit point which is an equilibrium in the entire economy. Before we present that result some auxillary results which characterize ‘interior’ Nash equilibria are given.

In the economy we consider one can define Pareto optimality and short run Pareto optimality. (See Balasko and Shell [3]).

Definition 3.1.

An allocation $(x_h)_{h \in G_t}$, $t \geq 0$ is short-run Pareto-optimal (SRPO) if there is

no other allocation $(y_h)_{h \in G_t}$, $t \geq 0$, and a $T \geq 0$ with the property,

$$\begin{aligned} \sum_{h \in (G_{t-1} \cup G_t)} y_h^t &= \sum_{h \in (G_{t-1} \cup G_t)} x_h^t, \\ y_{t,h} &= x_{t,h} \quad \forall t \geq T \\ \text{and } u_{t,h}(y_{t,h}) &\geq u_{t,h}(x_{t,h}) \quad \forall (t, h) \end{aligned}$$

with at least one strict inequality.

Essentially under short run Pareto optimality, we truncate the economy at some finite date, and consider optimality in the truncated economy (truncation here means holding allocations fixed after some date). We also know from the same paper that all Pareto Optimal allocations are SRPO, and Walrasian equilibria in the overlapping-generations model are SRPO. It will become apparent that the Nash equilibria are in general not SRPO.

Proposition 3.2.

- (i) *If the interior Nash Equilibrium allocation x for the market game Γ is autarky, then the endowments ω are SRPO.*
- (ii) *If the endowments ω are Pareto optimal for the market game Γ , then there is a unique interior Nash equilibrium allocation which is autarky.*

Proof

See Proposition 2.9 of Peck, Shell, and Spear [11].

Proposition 3.3.

Let $\sigma^* = (s_{t,h}^*)_{h \in G_t, t \geq 0}$ be a Nash Equilibrium profile for the offer constrained game $\Gamma(\bar{q})$. If the bids are strictly positive for all the consumers, i.e., $b_{t,h} \gg 0$, $\forall (t, h)$, then σ^* is also a Nash Equilibrium strategy profile for the market game Γ .

Proof See Proposition 2.11 of Peck, Shell, and Spear [11].

To demonstrate existence of an open market Nash equilibrium we will assume in addition that the endowments of each commodity for all consumers are uniformly bounded from below and above.

Assumption 3.4.

The endowments of each consumer satisfy the following condition.

$$0 < \underline{\omega}_h^{s,l} < \omega_h^{s,l} < \bar{\omega}_h^{s,l} < \infty \quad (7)$$

for all $h \in G_t$ for $t \geq 0$, $l = 1, \dots, l_s$, and $s = t, t+1$, if $t \geq 1$, and $s = t+1$ if $t = 0$.

To show existence of an open market T -Nash equilibria, we restrict offers to lie in a set of ‘sufficiently large’ offers. This set, $L(T)$, is a connected subset of offers with a non-empty interior yielding interior T -Nash equilibrium. For a definition of L when there is only one period see Peck, Shell, and Spear [11, pages 285-286]. This can be adapted to give us $L(T)$.

First we define $\xi(\omega(T)) > 0$ by the condition that for all $h \in G(t)$, $t \leq T$ and all commodities (t, l) , $t \leq T$, $x_{t,h}^{t,l} > \xi(\omega(T))$, $t = 1, \dots, T$, and $x_{t-1,h}^{t,l} > \xi(\omega(T))$, $t-1 = 1, \dots, T-1$ for all allocations $x_{t,h} \in \Pi_{t,h}$ where $\Pi_{t,h}$ is defined as:

$$\Pi_{t,h} = \{x_{t,h} \in \overset{\circ}{X}_{t,h} : u_{t,h}(x_{t,h}) \geq u_{t,h}(\omega_{t,h}) \text{ and } x_{t,h} \leq (\sum_{h \in (G_{t-1} \cup G_t)} \omega_h^t, \sum_{h \in (G_t \cup G_{t+1})} \omega_h^{t+1})\}$$

for all $t = 1, \dots, T$. For $h \in G_0$ the ‘pie- wedge’ is only relevant for the old age. This set is convex, compact, and bounded away from the axes. Hence there exists a scalar $\xi_{t,h}(\omega)$ such that $x_{t,h}^{s,l} > \xi_{t,h}$ for $t = 0, \dots, T$, $s = 1$ if $t = 0$, and $s = t, t+1$ if $t \geq 1$. Now define $\xi(T) = \inf_{\omega(T)} \xi_{t,h}(\omega)/2$. This scalar exists and is bounded away from zero. Finally, define $L(T)$ as:

$$L(T) = \{(q_{0,1}^1, \dots, q_{T, \#G_T}^T) \in \mathbb{R}^{l(T)\kappa(T)} : \omega_{t,h} + M > q_{t,h} > \omega_{t,h} - \xi(T), t = 0, \dots, T, \omega \in \Omega(T)\}$$

and where $l(T) = \sum_{t=1}^T l_t$ the total number of commodities through period T , $\kappa_T = \sum_{t=0}^T \#G_t$ the total number of consumers through period T , and $\xi(t) = (\xi, \dots, \xi)$ has the same dimension as the commodity space of consumer (t, h) , $t = 0, \dots, T$. Given this, we have the following result for T -Nash equilibria.

Lemma 3.5

There exist constants $\underline{\theta}$ and $\bar{\theta}$ such that for any trading post we have

$$0 < \underline{\theta} \leq B^{t,l} \leq \bar{\theta} < \infty \text{ for } t \leq T \quad (8)$$

for any T -Nash equilibrium of the market game Γ with $q \in L(T)$.

Proof

See Lemma 4.5 in Peck, Shell, and Spear [11].

The next result is again stated without proof. The results for existence of an open-market Nash equilibrium in a static model can be adapted to give the following.

Proposition 3.6

For any feasible $(q, \omega) \in L(T) \times \Omega$, there exists an interior T -Nash equilibrium. Let the set of interior T -Nash equilibria strategies associated with $L(T)$ be denoted as $E(L(T))$.

Proof

See Theorem 4.10 in Peck, Shell, and Spear [11].

First of all, we have $E(L(T+1)) \subset E(L(T))$, $T \geq 1$ as the restrictions are placed on strategies in period $T+1$ as the time horizon is extended. Secondly, the sets $E(L(T))$ are bounded. However, the sets $L(T)$ are open, making $E(L(T))$ open as well⁴. This is not a problem. Consider a closed,

⁴The sets $L(T)$ are taken to be open to be able to use degree theory for the existence proof of T -Nash equilibria

connected subset with non-empty interior, $L'(T) \subset L(T)$. We still get the bounds on the T -Nash equilibrium bids. Now consider the sets $E(L'(T))$ which is the set of T -Nash equilibria when the offers are restricted to lie in $L'(T)$. These are non-empty from the result above. The relationship $E(L'(T+1)) \subset E(L'(T))$, $T \geq 1$ will hold. This is now a nested sequence of non-empty compact sets (in the product topology), and hence we know that there is a point $\sigma^* \in \cap_{T=1}^{\infty} E(L'(T))$. This will be an open market equilibrium in the market game Γ . We have thus shown the following result.

Theorem 3.7

There exists an open market Nash equilibrium to the market game Γ .

4 Equilibria in the one good model

In this section we consider properties of open market equilibria when there is only one good (in addition to money) per period. First, we characterize the equilibria in terms of the first order conditions. As we are ultimately interested in studying stationarity properties we treat time as running from $-\infty$ to ∞ . Thus, there are no consumers who consume in only one period.

Proposition 4.1

The open market equilibria are solutions to the following set of equations:

$$-\frac{\partial u_{t,h}}{\partial x_{t,h}^t} \left(\frac{\hat{Q}_t \hat{B}_t}{(\hat{B}_t - m_{t,h})^2} \right) + \frac{\partial u_{t,h}}{\partial x_{t,h}^{t+1}} \left(\frac{\tilde{Q}_{t+1} \tilde{B}_{t+1}}{(\tilde{B}_{t+1} + m_{t,h})^2} \right) = 0 \quad (9)$$

for $t = \dots, t, \dots$, $h \in G_t$, and where $\hat{Q}_t = (\sum_{h \in (G_{t-1} \cup G_t)} q_h^t) - q_{t,h}^t$, $\hat{B}_t = (\sum_{h \in (G_{t-1} \cup G_t)} b_h^t) - b_{t,h}^t$, $\tilde{Q}_{t+1} = (\sum_{h \in (G_t \cup G_{t+1})} q_h^{t+1}) - q_{t,h}^{t+1}$, and $\tilde{B}_{t+1} = (\sum_{h \in (G_t \cup G_{t+1})} b_h^{t+1}) - b_{t,h}^{t+1}$.

Proof

The sequential budget constraints for consumer t, h are:

$$\begin{aligned} b_{t,h}^t + m_{t,h} &= \frac{B_t q_{t,h}^t}{Q_t} \\ b_{t,h}^{t+1} - m_{t,h} &= \frac{B_{t+1} q_{t,h}^{t+1}}{Q_{t+1}} \end{aligned}$$

where $B_t = \sum_{h \in (G_{t-1} \cup G_t)} b_h^t$ and $Q_t = \sum_{h \in (G_{t-1} \cup G_t)} q_h^t$. Working with the period t constraint we have:

$$\begin{aligned} b_{t,h}^t Q_t &= B_t q_{t,h}^t - m_{t,h} Q_t \\ &= [b_{t,h}^t + \hat{B}_t] q_{t,h}^t - m_{t,h} Q_t \\ \Rightarrow b_{t,h}^t [q_{t,h}^t + \hat{Q}_t] &= [b_{t,h}^t + \hat{B}_t] q_{t,h}^t - m_{t,h} Q_t \\ \Rightarrow b_{t,h}^t [q_{t,h}^t + \hat{Q}_t - q_{t,h}^t] &= \hat{B}_t q_{t,h}^t - m_{t,h} Q_t \\ \Rightarrow b_{t,h}^t \hat{Q}_t &= \hat{B}_t q_{t,h}^t - m_{t,h} Q_t \\ \Rightarrow b_{t,h}^t &= \frac{\hat{B}_t q_{t,h}^t - m_{t,h} Q_t}{\hat{Q}_t} \end{aligned}$$

and

$$\begin{aligned} b_{t,h}^t + m_{t,h} &= \frac{\hat{B}_t q_{t,h}^t - m_{t,h} Q_t + m_{t,h} \hat{Q}_t}{\hat{Q}_t} \\ &= \frac{[\hat{B}_t - m_{t,h}] q_{t,h}^t}{\hat{Q}_t} \end{aligned}$$

Similarly, we have

$$b_{t,h}^{t+1} - m_{t,h} = \frac{[\tilde{B}_{t+1} + m_{t,h}] q_{t,h}^{t+1}}{\tilde{Q}_{t+1}}.$$

This leads us to the following:

$$\begin{aligned} B_t &= \hat{B}_t + b_{t,h}^t \\ &= \hat{B}_t + \frac{[\hat{B}_t - m_{t,h}] q_{t,h}^t}{\hat{Q}_t} \\ &= \frac{[\hat{B}_t - m_{t,h}] Q_t}{\hat{Q}_t} \\ \text{and } B_{t+1} &= \frac{[\tilde{B}_{t+1} + m_{t,h}] Q_{t+1}}{\tilde{Q}_{t+1}} \end{aligned}$$

Using this we have

$$\begin{aligned} \frac{Q_t}{B_t} &= \frac{\hat{Q}_t}{[\hat{B}_t - m_{t,h}]} \\ \Rightarrow \frac{Q_t}{B_t} b_{t,h}^t &= \frac{\hat{Q}_t}{[\hat{B}_t - m_{t,h}]} \times \frac{\hat{B}_t q_{t,h}^t - m_{t,h} Q_t}{\hat{Q}_t} \end{aligned}$$

$$\begin{aligned}
&= \frac{\hat{B}_t q_{t,h}^t - m_{t,h} Q_t}{[\hat{B}_t - m_{t,h}]} \\
\Rightarrow -q_{t,h}^t + \frac{Q_t}{\hat{B}_t} b_{t,h}^t &= \frac{-\hat{Q}_t m_{t,h}}{\hat{B}_t - m_{t,h}}
\end{aligned}$$

Similarly,

$$-q_{t,h}^{t+1} + \frac{Q_{t+1}}{\tilde{B}_{t+1}} b_{t,h}^{t+1} = \frac{\tilde{Q}_{t+1} m_{t,h}}{\tilde{B}_{t+1} + m_{t,h}}$$

The allocations of consumer t, h in both the time periods are given as:

$$\begin{aligned}
x_{t,h}^t &= \omega_{t,h}^t - q_{t,h}^t + \frac{Q_t b_{t,h}^t}{\hat{B}_t} \\
x_{t,h}^{t+1} &= \omega_{t,h}^{t+1} - q_{t,h}^{t+1} + \frac{Q_{t+1} b_{t,h}^{t+1}}{\tilde{B}_{t+1}}
\end{aligned}$$

Using the result above this can be re-written as:

$$x_{t,h}^t = \omega_{t,h}^t - \frac{\hat{Q}_t m_{t,h}}{\hat{B}_t - m_{t,h}} \quad (10)$$

$$x_{t,h}^{t+1} = \omega_{t,h}^{t+1} + \frac{\tilde{Q}_{t+1} m_{t,h}}{\tilde{B}_{t+1} + m_{t,h}} \quad (11)$$

The optimization problem for agent t, h (under perfect foresight which is equivalent to the Nash assumption) is

$$\text{Max } u_{t,h} \left(\omega_{t,h}^t - \frac{\hat{Q}_t m_{t,h}}{\hat{B}_t - m_{t,h}}, \omega_{t,h}^{t+1} + \frac{\tilde{Q}_{t+1} m_{t,h}}{\tilde{B}_{t+1} + m_{t,h}} \right)$$

The first order condition (which are sufficient for interior equilibria) are given by

$$-\frac{\partial u_{t,h}}{\partial x_{t,h}^t} \cdot \left(\frac{\hat{Q}_t}{(\hat{B}_t - m_{t,h})} + \frac{\hat{Q}_t m_{t,h}}{(\hat{B}_t - m_{t,h})^2} \right) + \frac{\partial u_{t,h}}{\partial x_{t,h}^{t+1}} \cdot \left(\frac{\tilde{Q}_{t+1}}{(\tilde{B}_{t+1} + m_{t,h})} + \frac{\tilde{Q}_{t+1} m_{t,h}}{(\tilde{B}_{t+1} + m_{t,h})^2} \right) = 0$$

which on rearranging gives us

$$-\frac{\partial u_{t,h}}{\partial x_{t,h}^t} \cdot \left(\frac{\hat{Q}_t \hat{B}_t}{(\hat{B}_t - m_{t,h})^2} \right) + \frac{\partial u_{t,h}}{\partial x_{t,h}^{t+1}} \cdot \left(\frac{\tilde{Q}_{t+1} \tilde{B}_{t+1}}{(\tilde{B}_{t+1} + m_{t,h})^2} \right) = 0 \quad (12)$$

Using the characterization of the interior equilibrium we study existence of stationary equilibria. In a stationary environment zero-bids and zero-offers is always a trivial stationary equilibrium. The more interesting issue is of the stationary equilibria where there is always trade at each date. Even in this case an interior stationary equilibrium with zero-net trade can exist. We focus on interior stationary equilibrium with non-zero net trades. We will have non-zero net trades if $m_{t,h} \neq 0$. First we study the case where there is a single consumer in each generation. The consumers are identical except for the date of birth. This will give us the basic idea of how to extend the result to a general environment. In addition we impose Inada conditions where the marginal utility of zero consumption is infinite. To economize on the notation, in a stationary equilibrium, let

$b_t^t = b_1$, $b_t^{t+1} = b_2$, $q_t^t = q_1$, $q_t^{t+1} = q_2$, and $\omega_t^t = \omega_1$, $\omega_t^{t+1} = \omega_2$. Also denote $\frac{\partial u_t}{\partial x_t^t} = u_1$ and $\frac{\partial u_t}{\partial x_t^{t+1}} = u_2$.

Proposition 4.2

Consider an economy with a single consumer in each generation, and a single commodity at each date. A stationary equilibrium exists with non-zero net trade at each date if the economy is of the Samuelson type, i.e.,

$$\frac{\partial u_t(\omega_t^t, \omega_t^t)/\partial x_t^t}{\partial u_t(\omega_t^t, \omega_t^t)/\partial x_t^{t+1}} < 1.$$

Proof

The first order conditions that would characterize an interior stationary equilibrium are:

$$-u_1 \left(\omega_1 - \frac{q_2 \bar{m}}{b_2 - \bar{m}} \right) \frac{q_2 b_2}{(b_2 - \bar{m})^2} + u_2 \left(\omega_2 + \frac{q_1 \bar{m}}{b_1 + \bar{m}} \right) \frac{q_1 b_1}{(b_1 + \bar{m})^2} = 0 \quad (13)$$

From the budget constraints:

$$b_1 + \bar{m} = \frac{q_1}{q_1 + q_2} (b_1 + b_2)$$

$$b_2 - \bar{m} = \frac{q_2}{q_1 + q_2} (b_1 + b_2)$$

we obtain $b_2 - \bar{m} = \frac{(b_1 + \bar{m})q_2}{q_1}$. Substituting into the first order conditions and re-arranging we obtain:

$$-u_1 \left(\omega_1 - \frac{q_1 \bar{m}}{b_1 + \bar{m}} \right) (q_1 b_2) + u_2 \left(\omega_2 + \frac{q_1 \bar{m}}{b_1 + \bar{m}} \right) (q_2 b_1) = 0 \quad (14)$$

We show that non-zero solutions exist to this equation.

(i) *Sell-all equilibria*: In this case set $q_1 = \omega_1$, $q_2 = \omega_2$. Let the first-order conditions, i.e., equation (13), be denoted as a function $h(b_1, b_2)$. Then we have:

$$h(b_1, b_2) = -u_1 \left(\omega_1 - \frac{\omega_1 \bar{m}}{b_1 + \bar{m}} \right) (\omega_1 b_2) + u_2 \left(\omega_2 + \frac{\omega_1 \bar{m}}{b_1 + \bar{m}} \right) (\omega_2 b_1) = 0$$

and $b_2 = \bar{m} + \frac{\omega_1}{\omega_2}(b_1 + \bar{m})$.

First, consider $b_1 \rightarrow 0$, then $b_2 \rightarrow \bar{m}(1 + \frac{\omega_1}{\omega_2})$. It can be easily seen that as $\omega_1 - \frac{\omega_1 \bar{m}}{b_1 + \bar{m}} \rightarrow 0$ we have $-u_1(\cdot) \rightarrow -\infty$. As the other limits are finite, we have $\lim_{b_1 \rightarrow 0} h(b_1, b_2) = -\infty$.

Next, next consider $b_1 \rightarrow \infty$, then $b_2 \rightarrow \infty$. In this case, the first half of the first order condition tends to $-\infty$ while the second half tends to $+\infty$. We thus need to examine the relationship between $\frac{u_1(\omega_1)}{u_2(\omega_2)}$ and

$$\lim_{b_1 \rightarrow \infty} \frac{\omega_2}{\omega_1} \frac{b_1}{(\bar{m} + \frac{\omega_2}{\omega_1}(b_1 + \bar{m}))}.$$

Let $k = \frac{\omega_2}{\omega_1}$, we have, $h(b_1, b_2) > 0 \Leftrightarrow \frac{u_1(\omega_1)}{u_2(\omega_2)} < \lim_{b_1 \rightarrow \infty} \frac{k b_1}{\bar{m} + k(b_1 + \bar{m})}$. However, as $\lim_{b_1 \rightarrow \infty} \frac{k b_1}{\bar{m} + k(b_1 + \bar{m})} = \frac{\infty}{\infty}$, applying L'Hôpital's rule we have:

$$\lim_{b_1 \rightarrow \infty} \frac{kb_1}{\bar{m} + k(b_1 + \bar{m})} = \frac{k}{k}.$$

Thus, $h(b_1, b_2) > 0$ as $b_1 \rightarrow \infty$ if and only if

$$\frac{\partial u_t(\omega_t^t, \omega_t^t)/\partial x_t^t}{\partial u_t(\omega_t^t, \omega_t^t)/\partial x_t^{t+1}} < 1.$$

As the function $h(b_1, b_2)$ is a continuous function of b_1 for $b_1 \in (0, \infty)$ there exists a zero of the function. At this b_1^* , we will also have $b_2^* > 0$ as desired.

(ii) *General offers*: We consider general $(q_1, q_2) \gg 0$. If we look at the first order condition, and examine the limit as $b_1 \rightarrow 0$, the second term of the first order condition is equal to zero, while the first term becomes:

$$-u_1(\omega_1 - q_1, \omega_2 + q_1)q_1\bar{m}(1 + \tilde{k}) < 0$$

where $\tilde{k} = q_1/q_2$. Thus, $\lim_{b_1 \rightarrow 0} h(b_1, b_2, q_1, q_2) < 0$. If we look at the limit as $b_1 \rightarrow \infty$, $h(b_1, b_2, q_1, q_2) > 0$ requires,

$$\frac{\partial u_t(\omega_t^t, \omega_t^t)/\partial x_t^t}{\partial u_t(\omega_t^t, \omega_t^t)/\partial x_t^{t+1}} < 1$$

as before. Thus, we again get the existence of a stationary equilibrium.

The results indicate that there may be multiple steady-states.

5 The log-linear economy

In this section we study properties of the equilibria in a particular economy, one where each consumer has identical log-linear preferences. First, we characterize interior equilibrium in terms of the first order conditions. Using this characterization we study stationary equilibria. An in-depth study of non-stationary equilibria under different forecasting rules is the subject of a sequel paper. For stationary equilibria, we consider the ‘sell-all’ equilibrium, as well as general offers. We can find stationary equilibria in either case. This indicates an indeterminacy as we can arbitrarily fix the offers and the savings and still get a stationary equilibrium. However, for if we replicate agents in each period then the interior stationary Nash equilibria converge to the Walrasian equilibria of the overlapping generations model with the same type of agents. The non-autarky stationary Walrasian equilibrium is unique in this economy. The results we obtain are in accord with the results for market games with a finite set of markets and a finite set of consumers, in the sense that as the number of traders becomes large the Nash equilibria approach Walrasian equilibria. This result should generalize to general economies other than the specific example we consider.

5.1 Example 1

Consider an economy with a single agent in each generation and a single consumption good in each period. Each generation is identical and has pref-

ferences and endowments given by $u_t(x_t^t, x_t^{t+1}) = \alpha \ln x_t^t + (1 - \alpha) \ln x_t^{t+1}$, and $(\omega_t^t, \omega_t^{t+1}) = (\omega_1, \omega_2)$. As we are interested in stationary equilibria, denote $(x_t^t, x_t^{t+1}) = (x_1, x_2)$. In this economy we have $\frac{\partial u}{\partial x_1} = \frac{\alpha}{x_1}$ and $\frac{\partial u}{\partial x_2} = \frac{1 - \alpha}{x_2}$. In addition from the budget constraints we have $b_2 - m = \frac{(b_1 + m)q_1}{q_2}$. This can be used to simplify the allocation rule to yield $x_1 = \frac{\omega_1(b_1 + m) - q_1 m}{b_1 + m}$ and $x_2 = \frac{\omega_2(b_1 + m) + q_1 m}{b_1 + m}$. Using these facts, the first order conditions (Equation 14) for a stationary interior equilibrium can be written as:

$$-\alpha \left[\frac{\omega_1(b_1 + m) - q_1 m}{b_1 + m} \right]^{-1} q_1 \left[m + \frac{(b_1 + m)q_1}{q_2} \right] + (1 - \alpha) \left[\frac{\omega_2(b_1 + m) + q_1 m}{b_1 + m} \right]^{-1} q_2 b_1 = 0 \quad (15)$$

Straightforward manipulations yield:

$$-\frac{\alpha(q_1 m + (b_1 + m)q_2)}{(b_1 + m)\omega_1 - q_1 m} + \frac{(1 - \alpha)q_2 b_1}{(b_1 + m)\omega_2 + q_1 m} = 0 \quad (16)$$

As we take limits as $b_1 \rightarrow 0$, we have $h(b_1, b_2) < 0$, and as $b_1 \rightarrow \infty$, we have $\lim_{b_1 \rightarrow \infty} h(b_1, b_2) = \frac{-\alpha q_2}{\omega_1} + \frac{(1 - \alpha)q_2}{\omega_1} > 0$ if and only if $1 > \frac{\alpha/\omega_1}{(1 - \alpha)/\omega_2}$ which is the same condition as in Proposition 3.2.

To solve for the equilibrium bids we specialize the example further and set $\alpha = 1/2$. In this case equation (16) can be rewritten as:

$$[q_1 m + (b_1 + m)q_2][(b_1 + m)\omega_2 + q_1 m] = q_2 b_1 [(b_1 + m)\omega_1 - q_1 m]$$

Now set $m = 1$ (this will indicate there is an indeterminacy as the results hold in a neighbourhood of 1). Then the above expression can be simplified into a quadratic equation in b_1 , given by $A(b_1)^2 + Bb_1 + C = 0$, where we have:

$$\begin{aligned} A &= q_2(\omega_2 - \omega_1) \\ B &= q_2(\omega_2 - \omega_1 + 2q_1) + \omega_2(q_1 + q_2) \\ C &= (q_1 + q_2)(q_1 + \omega_2) \end{aligned}$$

(i) *Sell-all equilibria* In this case $q_1 = \omega_1$ and $q_2 = \omega_2$. If we solve for the roots we see that $B^2 - 4AC = 4r^2\omega_1\omega_2 > 0$, where $r = \omega_1 + \omega_2$. Thus, we do have real valued bids. There will be one positive root if we have $\omega_1 > 1/2$ given that we already have $\omega_1 > \omega_2$.⁵

(ii) In the general case we have

$$B^2 - 4AC = (q_2)^2(\Delta + 2q_1)^2 + (4q_1q_2\omega_2Q - 4q_1q_2Q\Delta) + (4(\omega_2)^2Q^2 - 2q_2\omega_2Q\Delta),$$

where $\Delta = \omega_2 - \omega_1$ and $Q = q_1 + q_2$. This can be shown to be positive for all $(\omega_1, \omega_2) \geq (q_1, q_2) \gg (0, 0)$.

⁵This restriction is probably an artifact of arbitrarily fixing $m = 1$.

5.2 Example 2

In this example we consider an economy with a stationary population, and a single consumption good in each period. The generations are identical and each generation consists of n consumers each of whom has identical preferences and endowments given by: $u_t(x_t^t, x_t^{t+1}) = \alpha \ln x_t^t + (1 - \alpha) \ln x_t^{t+1}$, and $(\omega_t^t, \omega_t^{t+1}) = (\omega_1, \omega_2)$. We drop the subscript indexing consumer h .

In this economy we have, $Q_t^t = nq_t^t$, $Q_t^{t+1} = nq_t^{t+1}$, $B_t^t = nb_t^t$, and $B_t^{t+1} = nb_t^{t+1}$. Then $\hat{Q}_t = Q_t - q_t^t = nq_t^{t+1} - (n-1)q_t^t$. Similarly, $\hat{B}_t = nb_t^{t+1} - (n-1)b_t^t$, $\tilde{Q}_{t+1} = nq_{t+1}^{t+1} - (n-1)q_t^{t+1}$, and $\tilde{B}_{t+1} = nb_{t+1}^{t+1} - (n-1)b_t^{t+1}$.

Using this the first order conditions for an interior equilibrium (equation 12) can be rewritten as:

$$-\frac{\alpha(\hat{B}_t - m_t)}{(\hat{B}_t - m_t)\omega_1 - \hat{Q}_t m_t} \left(\frac{\hat{Q}_t \hat{B}_t}{(\hat{B}_t - m_t)^2} \right) + \frac{(1 - \alpha)(\tilde{B}_{t+1} + m_t)}{(\tilde{B}_{t+1} + m_t)\omega_2 + \tilde{Q}_t m_t} \left(\frac{\tilde{Q}_{t+1} \tilde{B}_{t+1}}{(\tilde{B}_{t+1} + m_t)^2} \right) = 0$$

This simplifies to:

$$\frac{(1 - \alpha)\tilde{Q}_{t+1}\tilde{B}_{t+1}}{(\tilde{B}_{t+1} + m_t)(\tilde{B}_{t+1} + m_t)\omega_2 + \tilde{Q}_t m_t} = \frac{\alpha\hat{Q}_t\hat{B}_t}{(\hat{B}_t - m_t)(\hat{B}_t - m_t)\omega_1 - \hat{Q}_t m_t}$$

Manipulations of the budget constraints in each period for a consumer yield:

$$\begin{aligned}
q_i^t(\hat{B}_t - m) &= \hat{Q}_t(b_i^t + m) \\
q_i^{t+1}(\tilde{B}_{t+1} + m) &= \tilde{Q}_{t+1}(b_i^{t+1} - m)
\end{aligned}$$

Add the first expression across all young consumers except consumer h , and the second expression for all old consumers to obtain:

$$\begin{aligned}
\hat{Q}_i^t(\hat{B}_t - m) &= \hat{Q}_t(\hat{B}_t + (n-1)m) \\
Q_i^{t+1}(\tilde{B}_{t+1} + m) &= \tilde{Q}_{t+1}(B_t^{t+1} - nm)
\end{aligned}$$

where $\hat{Q}_i^t = \sum_{i \neq h \in G_t} q_{i,i}^t$, $\hat{B}_t = \sum_{i \neq h \in G_t} b_{i,i}^t$, $Q_i^{t+1} = \sum_{h \in G_t} q_{i,h}^{t+1}$, $B_t^{t+1} = \sum_{h \in G_t} b_{i,h}^{t+1}$.

Leading the first expression one period and adding these two equations we get:

$$\frac{Q_i^{t+1}}{\tilde{Q}_{t+1}}(\tilde{B}_{t+1} + m) + \frac{\hat{Q}_{t+1}^{t+1}}{\hat{Q}_{t+1}}(\hat{B}_{t+1} - m) = (\hat{B}_{t+1} - m)$$

This simplifies to:

$$(\tilde{B}_{t+1} + m) = \frac{\tilde{Q}_{t+1}}{\hat{Q}_{t+1}}(\hat{B}_{t+1} - m).$$

Once this is substituted into the first order condition, the first order condition simplifies to:

$$\frac{(1-\alpha)\hat{Q}_{t+1}[\tilde{Q}_{t+1}\hat{B}_{t+1} - (\tilde{Q}_{t+1} + \hat{Q}_{t+1})m]}{\tilde{Q}_{t+1}(\hat{B}_{t+1} - m)[(\hat{B}_{t+1} - m)\omega_t^{t+1} + \hat{Q}_{t+1}m]} = \frac{\alpha\hat{Q}_t\hat{B}_t}{(\hat{B}_t - m)[(\hat{B}_t - m)\omega_t^t - \hat{Q}_{t+1}m]} \quad (17)$$

Imposing the steady-state assumption gives:

$$\frac{(1-\alpha)[\tilde{Q}\hat{B} - (\tilde{Q} + \hat{Q})m]}{(\hat{B} - m)\omega_2 + \hat{Q}m} = \frac{\alpha\tilde{Q}\hat{Q}}{(\hat{B} - m)\omega_1 - \hat{Q}m}$$

In the case of $\alpha = 1/2$, the above equation can be further simplified to yield the following quadratic equation

$$\tilde{Q}(\omega_1 - \omega_2)\hat{B}^2 - m[(\tilde{Q} + \hat{Q})\omega_1 + \tilde{Q}(\omega_1 + \hat{Q}) + \tilde{Q}(\hat{Q} - \omega_2)]\hat{B} + m^2(\omega_1 + \hat{Q})(\tilde{Q} + \hat{Q}) = 0.$$

The roots \hat{B} are given by

$$\hat{B} = \frac{m[\zeta\omega_1 + \omega_1 - \omega_2 + 2\hat{Q}]}{2(\omega_1 - \omega_2)} \left(1 \pm \sqrt{1 - \frac{4(\omega_1 - \omega_2)\zeta(\omega_1 + \hat{Q})}{(\zeta\omega_1 + \omega_1 - \omega_2 + 2\hat{Q})^2}} \right)$$

where $\zeta = (\hat{Q} + \tilde{Q})/\tilde{Q}$. In addition, we have $\hat{B} = (n-1)b_1 + nb_2$ and $\hat{Q} = (n-1)q_1 + nq_2$. Using this we can simplify the expression for ζ as

$$\zeta = 1 + \frac{n(q_1 + q_2) - q_1}{n(q_1 + q_2) - q_2}.$$

Note that $\lim_{n \rightarrow \infty} \zeta = 2$. Taking limits as the number of consumers in each generation goes to ∞ we have:

$$\begin{aligned}
b_1^* + b_2^* &= \lim_{n \rightarrow \infty} \frac{\hat{B}}{n} \\
&= \frac{m}{\omega_1 - \omega_2} \lim_{n \rightarrow \infty} \frac{k\omega_1 + \omega_1 - \omega_2 + 2(nq_2 + (n-1)q_1)}{n} \\
&= \frac{2m(q_1 + q_2)}{\omega_1 - \omega_2}
\end{aligned}$$

Note that the expression in the large bracket giving the roots of \hat{B} goes to 0 as we take limits. Hence, we have

$$p^* = \frac{b_1^* + b_2^*}{q_1 + q_2} = \frac{2m}{\omega_1 - \omega_2}$$

which are the competitive equilibrium prices in the stationary monetary equilibrium of the overlapping generations economy with the same type of agents.

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