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INFERENCE IN MARKOV CHAINS HAVING
STOCHASTIC ENTRY AND EXIT

BY

GEORGE T. DUNCAN
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JANUARY, 1971

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Inference in Markov Chains Having Stochastic Entry and Exit

GEORGE T. DUNCAN and LIZBIE G. LIN

A model suitable for statistical inference in Markov chains is considered featuring Poisson entry and absorbing state exit. Maximum likelihood estimates are obtained and likelihood ratio tests derived. Necessary asymptotic theory is developed. Forecasted occupancy counts are presented. An application is made to the distribution of Ninth Federal Reserve District member banks with respect to the ratio of farm loans to total loans. Predictions given by the model are examined for goodness of fit and are found to be adequate.

1. INTRODUCTION

Considerable interest has been expressed concerning appropriate statistical methods for treating entry and exit problems in a Markov chain. Nevertheless, the fact that a variable number of entities are under observation over the several time periods is often ignored. But Conneman and Harrington [3] argue effectively, in the context of dynamic analysis in agricultural economics, that greater attention should be paid to the problems of exit and entry. A way of handling entry and exit is suggested by Adelman [1]:

"There is, however, one modification which must be made in the Markov process before we can use it profitably in our work - we must provide for entry into and departure from the industry. To do this, we add to our m size classes a large additional group which acts as a reservoir of potential entrants into the system. We then assign as the probability of moving from the zeroth group to, say, the j^{th} group a value just sufficient to make the average number of firms entering the j^{th} class per year correspond to the actual number of new firms started annually in the appropriate size range. Similarly, the failure of a firm will be represented as a movement into the zeroth class."

It is difficult to see how this approach will maintain the appropriate mean rate of entry into the various states. Nevertheless this method is common, as is illustrated by Hallberg [5]: "The fifth state was utilized to represent a pool from which entrants may come and to which exants may go." The basic methodological difficulty with this approach is the fact that the number of entities in this reservoir state can never be known. As recognized in [1]:

"The above definition of p_{ij}, however, still leaves the probabilities p_{0j} arbitrary. For, since, by the very nature of the case, no data on the number of businesses retaining the status of potential entrants could be collected, a_{00} could not be evaluated empirically. This deficiency was remedied by assuming that $\Sigma_{j=0}^{6} a_{0j} = 100{,}000$."

We propose a model which avoids this difficulty and articulates well with the work of Anderson and Goodman [2]. In our model entry is described by a Poisson distribution and exit by the use of an absorbing state. The model studied by Leysieffer [9] provides for a Poisson entry into a single state and an exit which occurs independently of the state presently occupied. It will be pointed out that this form of entry can be made equivalent to a Poisson entry into each of the states of the chain. In many cases, however, it will be useful to have the probability of exit dependent on the state presently occupied. The model of Section 2 provides for this by using an absorbing state as an exit. Methods of statistical inference are considered for an absorbing state model, a case which was not developed in [2]. The implications of a Poisson model for the description of the arrival of new entities in the system is explored. Maximum likelihood estimates are obtained in Section 3. Necessary asymptotic theory is developed in Section 4. Various interesting likelihood ratio tests are derived in Section 5.

The problem of forecasting the occupancy counts in the various states is taken up in Section 6. A formula for the expected occupancy counts at each time t is given and a limiting result obtained.

An empirical example is presented in Section 7 which is concerned with the distribution of Ninth Federal Reserve District member banks in terms of their ratios of farm loans to total loans. The data were taken from December Call Reports submitted by each bank during the time period 1954-1969. The possible loan ratios are categorized so that five loan ratio states and an absorbing state constitute the states of the Markov chain.

2. THE MODEL

The chain is observed at fixed times, $t= 0,1,\dots,T$, as a random number of entities move independently of one another through states labeled $i= 0,1,\dots,m$. The randomness in the number of entities will be due to two factors: one, the entities may have entered the system at different times, and, two, there is the possibility in each time period of an entity leaving the system. We first consider the probabilistic behavior of an entity which is presently in the system and then later consider how it got there. Initially, each of the entities is assumed to make transitions between states according to the same probability law. We designate states $1,\dots,m$ as the active portion of the system under consideration. State 0 is then outside the active portion of the system and a movement to this state will indicate an exit from this active portion. We treat state 0 as absorbing, i.e., once an entity has left the set of states $1,\dots,m$ it has zero probability of ever returning.

The sample space of a single entity has as elements a vector specifying the time that the entity entered the system in one of the states $1,\ldots,m$, the state initially occupied and the sequence of states subsequently occupied. We shall call the time of entry, t^*, and the sequence of states $(i(t^*),\ldots,i(T))$. Entry at the time $t^* = 0$ is equivalent to being initially in the chain.

The situation can be described as follows: first one of $T+1$ spaces, S_{t^*}, $t^* = 0,1,\ldots,T$, is selected. The points in the space S_{t^*} will be vectors of dimension $T-t^*+1$ with first coordinate $i(t^*)$, one point corresponding to each of the possible histories an entity may have. Since an entity may enter the absorbing state at any of the times $t^*+1, t^*+2,\ldots,T$ or may not be absorbed at all, the number of points in the space is

$$m+m^2+\ldots+m^{T-t^*+1} = \frac{m(1-m^{T-t^*+1})}{1-m}.$$

Thus, the total number of histories that an entity might have is

$$\sum_{t^*=0}^{T} \frac{m(1-m^{T-t^*+1})}{1-m} = \frac{m}{1-m}\{T+1-\frac{m}{1-m}(1-m^{T+1})\}$$

$$= \frac{m}{m-1}\{\frac{m}{m-1}(m^{T+1}-1)-T-1\}.$$

Note that we have omitted a finite number of histories which would otherwise have *a priori* probability zero.

We define a random variable, $X(t)$, on the sample space of an entity such that $X(t) = i$ when the entity occupies state i at time t. Let the (nonstationary) transition probabilities be denoted by

$$p_{ij}(t) = P(X(t) = j|X(t-1) = i). \tag{2.1}$$

The chain is said to have stationary or constant transition probabilities if $p_{ij}(t) = p_{ij}$ for $t = 1,\ldots,T$. Since state 0 is taken to be absorbing we have

$$p_{00}(t) = 1 \text{ and } p_{0j}(t) = 0 \qquad t = 1,\ldots,T;\ j = 1,\ldots,m. \tag{2.2}$$

In general, given that the time of entry, t^*, and state of entry, $i(t^*)$, are fixed, the sequence of states $(i(t^*),\ldots,i(T))$ has probability $p_{i(t^*)\ldots i(T)}$. If the chain has constant transition probabilities and is of first order, the sequence of states $(i(t^*),\ldots,i(T))$ has probability $p_{i(t^*)i(t^*+1)}\cdots p_{i(T-1)i(T)}$. (Note that if some $i(t) = 0$, $t \geq t^*$, then all subsequent $i(t')$, $t' = t+1,\ldots,T$, are also zero.)

We have the random occupancy counts $n_i(t)$, denoting the number of entities occupying the i^{th} state at time t. Thus, in particular, we have the initial counts, $n_i(0)$, which we assume to be multinomially distributed with sample size

$$n = \sum_{i=1}^{m} n_i(0) \tag{2.3}$$

and i^{th} occupancy probability η_i. Note that $n_0(0)$ is necessarily zero and hence $\eta_0 = 0$.

We denote by $e_i(t)$ the random number of entities which enter state i, $i = 1,\ldots,m$, at time t, $t = 1,\ldots,T$, from outside the system. This stochastic entry is described by a Poisson probability model. It will be assumed that entries are made independently into each state and over time; the mean rate of entry is ξ_i, possibly different for each state i but constant over time. Thus

$$P(\tilde{e}_i(t) = e_i(t)) = \frac{\xi_i^{e_i(t)}\exp[-\xi_i]}{e_i(t)!} \quad \text{where } e_i(t) = 0,1,2,\ldots;\ \xi_i \geq 0. \tag{2.4}$$

Note that the parameter ξ_i may be zero indicating that with probability one there will be no additional entries into state i.

The following assumptions would give rise to this Poisson probability model:

(1) Entities observed to be in the system for the first time at time t actually entered at some time in $(t-1, t)$;

(2) The number of entities entering during the non-overlapping time intervals $(a,a+h)$ and $(b,b+h)$ are independent and identically distributed random variables;

(3) If the probability that an entity enters in $(a,a+h)$ is $p(h)$, we have $\frac{p(h)}{h} \to \lambda$, a constant, as $h \to 0$;

(4) The probability of two or more arrivals in the time interval $(a,a+h)$ approaches 0 as $h \to 0$.

This form of entry can be viewed as equivalent to that of Leysieffer [9] if his single state of entry is entered at time t^*-1 and left with probability one at time t^*. An "artificial" transient state would then have been introduced whose function would be to receive a Poisson distributed random number of entries, X, and then to distribute them in a multinomial fashion to each of the active states of the chain during the next time period. Now the mean of X will be $\xi=\xi_1+\ldots+\xi_m$ and each $\tilde{e}_i$ will remain Poisson distributed since

$$P(\tilde{e}_i = e_i) = \sum_{x=0}^{\infty} P(\tilde{e}_i = e_i \mid X = x)\, P(X=x) \tag{2.4}$$

which in turn equals

$$\sum_{x=e_i}^{\infty} \binom{x}{e_i} p_i^{e_i} (1-p_i)^{x-e_i} \xi^x \exp(-\xi)/x!$$

$$= \frac{1}{e_i!}\, p_i^{e_i} (1-p_i)^{-e_i} \sum_{x=e_i}^{\infty} [(1-p_i)\xi]^x \exp(-\xi)/(x-e_i)! \tag{2.5}$$

and this then equals

$$\frac{1}{e_i!} p_i^{e_i}(1-p_i)^{-e_i}[(1-p_i)\xi]^{e_i}\exp(-\xi) \sum_{\nu=0}^{\infty} [(1-p_i)\xi]^{\nu}/\nu!$$

$$=(p_i\xi)^{e_i}\exp(-p_i\xi)/e_i! \qquad (2.6)$$

3. ESTIMATION

Consider a typical sequence of states $(i(t^*),\ldots,i(T))$. The number of entities which follow this particular sequence, i.e., enter the system for the first time at time t^* in state $i(t^*)$, etc., is denoted by $n_{i(t^*)\ldots i(T)}$.

A basic statistic of interest is the number of entities which make a transition from state g to state h at time t $(t = 1,\ldots,T)$. We write this as $n_{gh}(t)$. Then

$$n_{gh}(t) = \Sigma\, n_{i(t^*)\ldots i(T)} \quad , \qquad (3.1)$$

where the sum is over all sequences such that $i(t-1) = g$, $i(t) = h$, $i(t^*) \leq i(t-1)$.

The quantity n_{gh} is defined as the total number of entities making a transition from state g to state h, i.e.,

$$n_{gh} = \sum_{t=1}^{T} n_{gh}(t). \qquad (3.2)$$

We now develop in detail the probability relations for the first order Markov chain whose transition probabilities are possibly not constant over time. First, we condition on the number of entries from outside the system into each of the m states, $1,\ldots,m$, and during each of the t time periods, $1,\ldots,T$. Thus we are holding fixed, $\tilde{e}_i(t) = e_i(t)$ for $i = 1,\ldots,m$ and $t = 1,\ldots T$. Then the conditional probability of observing the

$$\sum_{i=1}^{m} \sum_{t=0}^{T} e_i(t) = \Sigma\, n_{i(t^*)\ldots i(T)} \tag{3.3}$$

sequences of the form $(i(t^*),\ldots,i(T))$ is

$$\prod \left[p_{i(t^*)i(t^*+1)}(t^*+1)\ldots p_{i(T-1)i(T)}(T) \right]^{n_{i(t^*)\ldots i(T)}} \tag{3.4}$$

where the sum in (3.3) and the product in (3.4) are overall values of the indices with $t^* = 0,1,\ldots,T-1$. Now (3.4) can be written as

$$\prod_{t^*=0}^{T} \left[\prod_{i(t^*),i(t^*+1)} \{p_{i(t^*)i(t^*+1)}(t^*+1)\}^{n_{i(t^*)\ldots i(T)}} \right] \ldots$$
$$\left[\prod_{i(T-1),i(T)} \{p_{i(T-1)i(T)}(T)\}^{n_{i(t^*)\ldots i(T)}} \right] \tag{3.5}$$
$$= \left[\prod_{i(0),i(1)} \{p_{i(0)i(1)}(1)\}^{n_{i(0)i(1)}(1)} \right] \ldots$$
$$\left[\prod_{i(T-1),i(T)} \{p_{i(T-1)i(T)}(T)\}^{n_{i(T-1)i(T)}(T)} \right] = \prod_{t=1}^{T} \prod_{g,h} p_{gh}(t)^{n_{gh}(t)} .$$

Now this result is formally the same as the one Anderson and Goodman [2] obtain when the number of entities moving through the chain remains constant. Therefore defining

$$n_i(t-1) = \sum_{j=1}^{m} n_{ij}(t) , \tag{3.6}$$

the conditional distribution of the $n_{ij}(t)$ is given by

$$\prod_{t=1}^{T}\left\{\prod_{i=1}^{m}\left[\frac{n_i(t-1)!}{\prod_{j=1}^{m} n_{ij}(t)!}\prod_{j=1}^{m} p_{ij}(t)^{n_{ij}(t)}\right]\right\}. \tag{3.7}$$

The joint distribution of the $e_i(t)$ and the $n_{ij}(t)$ is obtained from (3.7) by multiplying by the marginal distribution of the $e_i(t)$.

Then to obtain the likelihood function we multiply the conditional likelihood given the initial counts $n_i(0)(i = 1,\ldots,m)$ and the entry counts $e_i(t)(i = 1,\ldots,m;\ t = 1,\ldots,T)$ by these respective marginal likelihoods. The likelihood function is given by

$$L = n!\prod_{i=1}^{m}\frac{\eta_i^{n_i(0)}}{n_i(0)!}\prod_{t=1}^{T}\frac{\xi_i^{e_i(t)}}{e_i(t)!}\exp(-\xi_i)\cdot\prod_{t=1}^{T}\left\{\prod_{i=1}^{m}\left[\prod_{j=1}^{m} p_{ij}(t)^{n_{ij}(t)}\right]\right\}. \tag{3.8}$$

By the factorization criterion the sufficient statistics for the parameters η_i $(i = 1,\ldots,m)$, ξ_i $(i = 1,\ldots,m)$, and $p_{ij}(t)$ $(i,j = 0,1,\ldots,m;$ $t = 1,\ldots,T)$ are then n, $n_i(0)$ $(i = 1,\ldots,m)$, $e_i(t)$ $(i = 1,\ldots,m)$, and $n_{ij}(t)$ $(i,j = 0,1,\ldots,m;\ t = 1,\ldots,T)$.

Noting the way L factors, the maximum likelihood estimates of the parameters are immediately obtained as the usual multinomial and Poisson estimates.

Thus,

$$\hat{\eta}_i = \frac{n_i(0)}{n} \qquad (i = 1,\ldots,m); \tag{3.9}$$

$$\hat{\xi}_i = \sum_{t=1}^{T} e_i(t)/_T \quad (i = 1,\ldots,m); \tag{3.10}$$

$$\hat{p}_{ij}(t) = \frac{n_{ij}(t)}{n_i(t-1)} = \frac{n_{ij}(t)}{\sum_{k=0}^{m} n_{ik}(t)} \quad \begin{array}{l}(i = 1,\ldots,m;\ j = 0,\ldots,m;\\ t = 1,\ldots,T);\end{array} \tag{3.11}$$

$$\hat{p}_{00}(t) = 1 \text{ and } \hat{p}_{0j}(t) = 0 \quad \begin{pmatrix} j = 1,\ldots,m \\ t = 1,\ldots,T \end{pmatrix} . \tag{3.12}$$

(Note that (3.12) follows from the fact that state 0 is a priori absorbing.)

If it is assumed that the transition probabilities are stationary, we obtain instead of (3.11),

$$\hat{p}_{ij} = \frac{\sum_{t=1}^{T} n_{ij}(t)}{\sum_{t=1}^{T}\sum_{k=0}^{m} n_{ik}(t)} \quad \begin{pmatrix} i = 1,\ldots,m \\ j = 0,1,\ldots,m \\ t = 1,\ldots,T \end{pmatrix} . \tag{3.13}$$

4. ASYMPTOTIC BEHAVIOR

In developing asymptotic results, we let $\xi_i \to \infty$ and $n \to \infty$ in such a way that

$$\frac{\xi_i}{n} \to c_i \ , \ 0 < c_i < \infty. \tag{4.1}$$

Consider a certain time t^*, $1 \le t^* \le T$, and a state $i(t^*)$. Then we note that, conditional on a fixed value of the Poisson distributed $e_i(t^*)$, the set of random variables $n_{i(t^*)i(t^*+1)\ldots i(T)}$ has a multinomial distribution with sample size $e_i(t^*)$ and parameters $p_{i(t^*)i(t^*+1)}\cdots p_{i(T-1)i(T)}$. Thus, conditionally, they are asymptotically normal when properly scaled as $e_i(t^*)$ increases by the multivariate central limit theorem. Just as in [2], the $\{n_{i(0)i(1)\ldots i(T)}\}$ are also asymptotically normal random variables with sample size $n_i(0)$ and parameters $p_{i(0)i(1)\ldots i(T)}$. Since the $n_{ij}(t)$ are fixed linear combinations of these multinomial variables, they are also asymptotically normally distributed.

Now considering the fact that the $e_i(t)$ contribute new entities to the chain throughout its observed length, it is of interest to consider the unconditional behavior of the $n_{i(t^*)\ldots i(T)}$. We note, as in Section 2, that if a category count has a multinomial distribution conditional on a fixed total sample size and the sample size is Poisson distributed, then the unconditional distribution of the category count is Poisson. Thus unconditionally we have the asymptotic result that as $\xi_i \to \infty$, the Poisson variates (again when properly scaled), $n_{i(t^*)\ldots i(T)}$, are normally distributed. As in the conditional case, the $n_{ij}(t)$ are then asymptotically normally distributed.

We now establish the asymptotic normality and consistency of the transition probability estimates, $p_{ij}(t)$. We have that

$$\sqrt{n}\ \hat{p}_{ij}(t) = \frac{n_{ij}(t) / \sqrt{n}}{n_i(t-1) / n} \qquad \left(\begin{array}{l} i = 1,\ldots,m \\ j = 0,1,\ldots,m \\ t = 1,\ldots,T. \end{array} \right) . \tag{4.2}$$

Now if we define $n_{k\ell;ij}(t)$ to be the number of entities which enter the system at time ℓ in state k and make a transition from state i to state j at time t, we have

$$n_i(t-1) = \sum_{k=1}^{m} \sum_{\ell=0}^{T-1} \sum_{j=0}^{m} n_{k\ell;ij}(t) \quad i = 0,1,\ldots,m \quad t = 1,\ldots,T. \tag{4.3}$$

Now since, conditionally on the entry count $e_k(\ell)$, the $n_{k\ell;ij}(t)$ are multinomial random variables with sample size $e_k(\ell)$, the conditional expectation is given by

$$E(n_{k\ell;ij}(t) \mid e_k(\ell)) = e_k(\ell) p_{ki}{}^{(t-\ell-1)} p_{ij} , \tag{4.4}$$

where $p_{rs}^{(m)}$ denotes an m-step transition probability from state r to state s. The unconditional expectation is then expressed as

$$E(n_{k\ell;ij}(t)) = \begin{cases} \xi_k p_{ki}^{(t-\ell-1)} p_{ij}, & \ell \geq 1 \\ \eta_k p_{ki}^{(t-1)} p_{ij}, & \ell = 0 \end{cases} . \tag{4.5}$$

Then with (4.1) in mind we can rewrite (4.3) as

$$\frac{n_i(t-1)}{n} = \sum_{k=1}^{m} \sum_{j=0}^{m} \frac{n_{k0;ij}(t)}{n} + \sum_{k=1}^{m} \sum_{j=0}^{m} \sum_{\ell=1}^{T-1} \frac{n_{k\ell;ij}(t) c_k}{\xi_k} \left(\frac{1}{n} \frac{\xi_k}{c_k}\right). \tag{4.6}$$

Taking the limit in probability as $n \to \infty$ we then obtain

$$p \lim_{n\to\infty} \frac{n_i(t-1)}{n} = \sum_{k=1}^{m} \sum_{j=0}^{m} \eta_k p_{ki}^{(t-1)} p_{ij} + \sum_{k=1}^{m} \sum_{\ell=1}^{T-1} \sum_{j=0}^{m} \frac{c_k \xi_k p_{ki}^{(t-\ell-1)} p_{ij}}{\xi_k}. \tag{4.7}$$

Since $\sum_j p_{ij} = 1$ for each fixed i, (4.7) simplifies to

$$p \lim_{n\to\infty} \frac{n_i(t-1)}{n} = \sum_{k=1}^{m} \eta_k p_{ki}^{(t-1)} + \sum_{k=1}^{m} \sum_{\ell=1}^{T-1} c_k p_{ki}^{(t-\ell-1)}, \tag{4.8}$$

which is a nonzero constant. But the numerator of the right hand side of (4.2) converges to a normal random variable and then by the Mann-Wald theory $\sqrt{n}\, \hat{p}_{ij}(t)$ is asymptotically normal as $n \to \infty$.

We now show that our estimates $\hat{p}_{ij}(t)$ are consistent.

Write

$$p_{ij}(t) = \frac{n_{ij}(t)/n}{\sum_{k'=0}^{m} n_{ik'}(t)/n} = \frac{\sum_{k=1}^{m} \sum_{\ell=0}^{T-1} n_{k\ell;ij}(t)/n}{\sum_{k=1}^{m} \sum_{\ell=0}^{T-1} \sum_{k'=0}^{m} n_{k\ell;ik'}(t)/n} . \tag{4.9}$$

But the probability limit of $\hat{p}_{ij}(t)$ is then

$$\frac{\sum_{k=1}^{m} \eta_k \, p_{ki}^{(t-1)} \, p_{ij} + \sum_{k=1}^{m} \sum_{\ell=1}^{T-1} \xi_k p_{ki}^{(t-\ell-1)} p_{ij}}{\sum_{k=1}^{m} \sum_{k'=0}^{m} \eta_k p_{ki}^{(t-1)} p_{ik'} + \sum_{k=1}^{m} \sum_{\ell=1}^{T-1} \sum_{k'=0}^{m} \xi_k p_{ki}^{(t-\ell-1)} p_{ik'}} \tag{4.10}$$

$$= \frac{\left\{ \sum_{k=1}^{m} \eta_k p_{ki}^{(t-1)} + \sum_{k=1}^{m} \sum_{\ell=1}^{T-1} \xi_k p_{ki}^{(t-\ell-1)} \right\} p_{ij}}{\left\{ \sum_{k=1}^{m} \eta_k p_{ki}^{(t-1)} + \sum_{k=1}^{m} \sum_{\ell=1}^{T-1} \xi_k p_{ki}^{(t-\ell-1)} \right\}} = p_{ij} \; ,$$

where we have factored p_{ij} out of the numerator and used the fact that $\sum_{k'=0}^{m} p_{ik'} = 1$ in the denominator.

5. HYPOTHESIS TESTS

5.1 Likelihood Ratio

For statistical hypotesis testing on the transition probabilities, the $p_{ij}(t)$'s, we will make use of the likelihood ratio criterion. In the tests considered the parameters will be assumed to span a space Ω; under the null hypothesis they will span a subspace of Ω called ω. Define λ to be the ratio of the supremum of the likelihood function under ω to the supremum of the likelihood function under Ω. Then with the asymptotic results of section 4 we have that $-2 \log \lambda$ is asymptotically distributed as χ^2 with degrees of freedom given by the dimension of Ω minus the dimension of ω.

5.2 Test for Stationarity in the Absorbing State Case

Assume the chain is first order and consider the null hypothesis

$$H : p_{ij}(t) = p_{ij} \qquad t = 1,\ldots,T.$$

We are then testing the hypothesis that transition probabilities are constant over time against the alternative that the transition probabilities vary with time.

Let 0 be an absorbing state. Then for $t = 1,\ldots,T$,

$$p_{00}(t) = 1 = p_{00} \text{ and } p_{0j}(t) = 0 = p_{0j} \quad j = 1,\ldots,m.$$

If we omit irrelevant constants the likelihood function maximized under the null hypothesis is then

$$\prod_{t=1}^{T} \prod_{i=1}^{m} \prod_{j=0}^{m} \hat{p}_{ij}^{\,n_{ij}(t)} = \prod_{i=1}^{m} \prod_{j=0}^{m} \hat{p}_{ij}^{\,n_{ij}}, \tag{5.1}$$

while under the union of null and alternative hypotheses the maximized likelihood is

$$\prod_{t=1}^{T} \prod_{i=1}^{m} \prod_{j=0}^{m} \left[p_{ij}(t)\right]^{n_{ij}(t)} \quad . \tag{5.2}$$

The likelihood ratio is then

$$\lambda = \prod_{t=1}^{T} \prod_{i=1}^{m} \prod_{j=0}^{m} \left[\frac{\hat{p}_{ij}}{\hat{p}_{ij}(t)}\right]^{n_{ij}(t)} \quad . \tag{5.3}$$

We have $-2 \log \lambda$ asymptotically distributed as χ^2 with $Tm^2 - m^2 = m^2(T-1)$ degrees of freedom. Therefore an approximate size α test rejects when

$$-2 \log \lambda > \chi^2_{1-\alpha,}\ (m^2(T-1)). \tag{5.4}$$

5.3 Simultaneous Test of Order and Stationarity With State 0 Absorbing

Consider the following null and alternative hypotheses:

H: first order and stationary

K: second order and nonstationary

The likelihood ratio for this test is then

$$\lambda = \frac{\prod_{j=0}^{m} \prod_{k=0}^{m} \hat{p}_{jk}^{n_{jk}}}{\prod_{t=2}^{T} \prod_{i=0}^{m} \prod_{j=0}^{m} \prod_{k=0}^{m} \hat{p}_{ijk}(t)^{n_{ijk}(t)}} \quad . \tag{5.5}$$

where the two-step transition probabilities are defined by

$$p_{ijk}(t) = P(X(t) = k \mid X(t-1) = j,\ X(t-2) = i). \tag{5.6}$$

The maximum likelihood estimates are given by

$$\hat{p}_{ijk}(t) = \frac{n_{ijk}(t)}{n_{ij}(t-1)} \quad \text{and} \tag{5.7}$$

$$\hat{p}_{jk} = \sum_{t=2}^{T} n_{jk}(t) \Big/ \sum_{t=1}^{T-1} n_j(t) \tag{5.8}$$

while $\hat{p}_{i00} = 1$ $(i = 0,1,\ldots,m)$, $\hat{p}_{0jk} = 0$ $(j = 1,\ldots,m;\ k = 0,1,\ldots,m)$, $\hat{p}_{i0k} = 0$ $(i = 0,1,\ldots,m;\ k = 1,\ldots,m)$. Now $-2 \log \lambda$ is asymptotically χ^2 with degrees of freedom $(T-1)\ [m^2(m+1)-m^2] - m^2 = (T-1)m^3 - m^2$.

5.4 Test That Later Entries Follow the Same Probability Law as the Original Entities

It is possible that the transition probabilities governing the behavior of the original entities in the chain may be different from that of the late entries. This can be tested in the following manner. Suppose that the chain is first order with constant transition probabilities. Let

$$H:\ p^0_{ij} = p^*_{ij} \quad i,j = 1,\ldots,m$$

where the p^0_{ij}'s are the transition probabilities of the original entities and p^*_{ij}'s are the transition probabilities of the newcomers. Then

$$\lambda = \frac{\prod_{i,j=0}^{m} \hat{p}_{ij}^{\,n_{ij}}}{\prod_{i,j=0}^{m} \hat{p}_{ij}^{0\; n_{ij}^{0}} p_{ij}^{*\; n_{ij}^{*}}} \tag{5.9}$$

where $n_{ij} = n_{ij}^{0} + n_{ij}^{*}$; $-2 \log \lambda$ is asymptotically χ^2 with $2\,[(m+1)^2 - (m+1)] - [(m+1)^2 - (m+1)] = m^2 + m$ degrees of freedom in the non-absorbing state case and $2[(m+1)m - m] - [(m+1)m - m] = m^2$ degrees of freedom when state 0 is absorbing.

6. FORECASTING

Let (a_t, f_t') be the row vector of dimension (m+1) which gives the number of entities occupying each of the m+1 possible states at time t (t=0,1,2,...T). For forecasting purposes the expected value of (a_t, f_t') is of interest, and in our model

$$E\,(a_t, f_t') = (0, f_0')\,P^t + (0, \xi')\sum_{j=0}^{t-1} P^j\,. \tag{6.1}$$

where $\xi' = (\xi_1 \ldots, \xi_m)$.

Now with state 0 known to be absorbing, P can be written in partitioned form as

$$P = \begin{bmatrix} 1 & 0 \\ r & Q \end{bmatrix}\,. \tag{6.2}$$

Using this partitioned form we note that for $t \geq 1$

$$P^t = \begin{bmatrix} 1 & 0 \\ (\sum_{k=0}^{t-1} Q^k)r & Q^t \end{bmatrix} \tag{6.3}$$

and since $P^0 = I$,

$$\sum_{j=0}^{t-1} P^j = \begin{bmatrix} t & 0 \\ \{I+\sum_{j=0}^{t-2}(t-1-j)Q^2\}r & \sum_{j=0}^{t-1} Q^j \end{bmatrix} . \tag{6.4}$$

Thus substituting in (6.1) we obtain

$$E(a_t, f_t') = (\{f_0'(\sum_{k=0}^{t-1} Q^k) + \xi'(I+\sum_{j=0}^{t-2}(t-1-j)Q^j)\}r, f_0'Q^t + \xi'\sum_{j=0}^{t-1} Q^j). \tag{6.5}$$

Substituting the sample estimates of Q and ξ in equation (6.5) gives a forecast for time t of the number of entities absorbed (i.e., leaving the system), a_t, and the occupancy numbers, f_t'.

We might consider the asymptotic result as $t \to \infty$. Here we note that if state 0 is the only non-transient state, then I-Q has an inverse [8, Theorem 3.2.1]. Therefore $Q^t \to 0$ as $t \to \infty$ and

$$\lim_{t \to \infty} E(a_t, f_t') = (+ \infty, \xi' (I-Q)^{-1}). \tag{6.6}$$

7. AN EMPIRICAL EXAMPLE

7.1 The Problem -- Changes in the Distribution of Ninth Federal Reserve District Member Banks in Terms of Their Ratios of Farm Loans to Total Loans

During the past few decades, numerous revolutionary changes in the agricultural industry have stimulated the use of large capital investments by farmers as a means of modern farm production. As a result, there was a tremendous demand for farm credit in rural areas, and agricultural loans became one of the major credit operations of rural commercial banks. During the past twenty years, the volume of both loans to farmers and total loans in commercial banks grew substantially. Agricultural loans outstanding at country member banks of the Ninth Federal Reserve District, which is comprised of the four states of Minnesota, Montana, North Dakota, and South Dakota, plus Upper Michigan and Northwestern Wisconsin, were almost tripled every ten years. Besides the seven-county Twin City metropolitan area, the region is dominated by agricultural businesses which include dairying, commercial grain, livestock and ranching. In 1969, nine-tenths of district member banks made some loans to farmers, and about half of these banks had farm loan ratios of at least 21 percent of their total loans. Farm loans outstanding at district country member banks rose from $78 million in 1948 to $203 million in 1958 and up to $620 million in 1969. Meanwhile, total loans at these banks grew from $518 million 21 years ago to $1,341 million in 1958 and up to $3,832 million in 1969. Despite the growing loan volumes in the district, the ratio of farm loans to total loans at country member banks as a whole was very stable, about 16 percent per year. However, the historical changes of farm loan ratios in individual banks reflect a somewhat different picture. In the past 16 years, ratios of farm loans to total loans in each member bank not only changed frequently but also moved from one farm loan ratio class to another (classes are defined in Table 1).

Table 1. DISTRIBUTION OF MEMBER BANKS BY RATIOS OF FARM LOANS TO TOTAL LOANS

NINTH FEDERAL RESERVE DISTRICT

States Year	S_0 Exit (cumulated)	S_1 0%	S_2 >0-<21%	S_3 21-<41%	S_4 41-<61%	S_5 61% & Over	Total S_0- S_5	Existing Total S_1- S_5
1954	-	32	218	132	64	27	473	473
1955	1	35	208	141	59	30	474	473
1956	4	35	212	139	60	27	477	473
1957	5	32	211	127	73	31	479	474
1958	5	33	222	121	73	27	481	476
1959	6	37	202	109	79	50	483	477
1960	9	34	200	117	85	38	483	474
1961	10	37	197	113	78	51	486	476
1962	18	34	207	99	87	43	488	470
1963	18	37	206	101	88	53	503	485
1964	19	38	213	105	80	58	513	494
1965	21	42	207	94	81	71	516	495
1966	25	42	206	94	86	66	519	494
1967	28	45	203	96	86	63	521	493
1968	32	44	206	107	80	53	522	490
1969	38	50	205	109	77	49	528	490

Table 1 shows the number of member banks in different farm loan ratio classes during the 16-year period. The number of member banks which had no farm loans (S_1) increased substantially from 1954 to 1969 partly because of the many new banks in the district's Standard Metropolitan Statistical Areas. The number of banks which made smaller proportions of their loans to farmers (those in S_2 and S_3) decreased during this time span, while the number in the higher ratio classes (S_4 and S_5) rose substantially until the last two years. Several factors may have contributed to reducing the number of banks in the latter categories in 1968 and 1969: higher growth rates in non-agricultural loans than in farm loans, faster farmer loan payments, delayed capital investments by farmers due to high cost of borrowings, tight monetary policy resulting in bank loan contractions, etc. But regardless of causes it is important to be able to estimate the extent to which banks shift among loan ratio classes. Also forecasts of the distribution of banks in these loan ratio classes provide essential policy guidance. A Markov chain analysis based on the theory of the previous sections provides a framework for such an investigation.

The application of the Markov process model in economic and social studies has gained wide attention among researchers in recent years. Although various journal articles related to this subject appeared in the fifties [1,2,4,7,10,11], it was not until the middle and late sixties that numerous research papers on this subject began to be published in the fields of economics, agricultural economics, psychology, and medicine. In agricultural economics, the pioneers in this area were Judge and Swanson [6] in 1961. Later on, most of the studies were concentrated on the estimation of transitive probabilities of existing firms and projections of firms in a particular agricultural industry. In 1969, Conneman and Harrington introduced the importance of new entries of firms in a dairy industry [3]. That paper, however, only worked with experimental data without a supporting theoretical framework. A month later, Hallberg developed a procedure of projecting agricultural firms by nonstationary transition probabilities as a function of exogenous variables [5].

It is hoped that this paper would offer both a theoretical framework of stochastic exit and entry in a Markov chain as stated in the first six sections, and an empirical example for testing various assumptions as well as projecting bank distributions, as presented in this last section.

7.2 The Model and the Data

It is empirically true that a bank's farm loan operation, measured by its ratio of farm loans to total loans, varies each year. It is also reasonable to believe that a bank's farm loan operation is a continual phenomenon to the extent that the farm loan operation of a bank in one year is likely to influence its farm loan operation in the following year. This is due to the fact that a farmer's indebtedness in one year is likely to affect the farmer's indebtedness in the succeeding year. This idea of the year-to-year dependence in a bank's farm loan operation suggests the application of a Markov chain model. Once a bank's mobility is estimated, it is possible, using the derived model to project that bank's future movements from one stage to another.

The empirical data in this study were taken from December Call Reports submitted by each member bank in the Ninth Federal Reserve District in the time period 1954-1969. Ranges of a bank's ratio of farm loans to net total loans were used to define states of the Markov chain. Then, using this method of definition, five loan ratio states and an absorbing state were determined:

1) S_0 (exit):	An absorbing state for banks leaving the Federal Reserve System, for example, by withdrawing memberships, liquidating bank properties, and merging with other banks.
2) S_1 (0%):	A state which includes banks with no farm loans, such as some banks in the metropolitan area.
3) S_2 (>0-<21%):	A state which includes banks that make some but less than 21 percent of their loans to farmers.
4) S_3 (21-<41%):	A state which includes banks that make 21 to less than 41 percent of their loans to farmers.
5) S_4 (41-<61%):	A state which includes banks that make 41 to less than 61 percent of their loans to farmers.
6) S_5 (61% and over):	A state which includes banks that make 61 percent or more of their loans to farmers.

The data from 1954 to 1967 were used in estimating the transition probabilities rather than 1954 to 1969 in order to test how well the transition matrix of 1954-1967 could predict the bank distribution in 1968 and 1969. Likelihood ratio tests were made to detect whether or not the transition probabilities of those two years were different from those of 1954-1967.

The Markov chain in this case, thus, is an absorbing one with states $i = 0,1,2,3,4,5$ (where 0 is the absorbing state) observed at fixed times $t = 0,1,2,\ldots,13$ (where 0 is the starting year 1954 and $T = 13$ is the year 1967). Table 1 presents a historical view of the distribution of banks in these various states as well as the accumulated number of banks which left the Federal Reserve System in the Ninth District since 1955.

7.3 Estimation of the Transition Matrix and the Rate of Bank Entries

For a first order stationary absorbing Markov chain, the relevant statistic is the number of banks moving from state i to state j at any time t during the 14-year period. This is

$$n_{ij} = \sum_{t=1}^{13} n_{ij}(t) \text{ for } i = 1,\ldots,5;\ j = 0,1,\ldots,5.$$

The absorbing state model is appropriate since no banks came back to the System after they had left. Therefore, $n_{0j} = 0$ for $j = 1,2,\ldots,5$. The frequency matrix of banks' movement in the district since 1954 until 1967 is a matrix $N = [n_{ij}]$:

$$N = \begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{cccccc} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \left[\begin{array}{cccccc} 28 & 0 & 0 & 0 & 0 & 0 \\ 2 & 413 & 49 & 1 & 2 & 1 \\ 10 & 50 & 2507 & 134 & 6 & 2 \\ 8 & 2 & 113 & 1181 & 178 & 10 \\ 4 & 1 & 5 & 127 & 738 & 118 \\ 4 & 1 & 2 & 5 & 87 & 473 \end{array}\right] \end{array} \quad \begin{array}{c} \sum_{t=1}^{13}\sum_{j=0}^{5} n_{ij}(t) \\ 28 \\ 468 \\ 2709 \\ 1492 \\ 993 \\ \underline{572} \\ 6262 \end{array}$$

The estimated transition matrix $\hat{P} = [\hat{p}_{ij}]$ where, for example, $\hat{p}_{11} = \frac{413}{468} = .8825$ according to equation (3.13) is:

$$\hat{P} = \begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{cccccc} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .0043 & .8825 & .1047 & .0021 & .0043 & .0021 \\ .0037 & .0185 & .9254 & .0495 & .0022 & .0007 \\ .0054 & .0013 & .0757 & .7916 & .1193 & .0067 \\ .0040 & .0010 & .0050 & .1280 & .7432 & .1188 \\ .0070 & .0018 & .0035 & .0087 & .1521 & .8269 \end{bmatrix} \end{array} \quad \begin{array}{c} \sum_{j=0}^{5} \hat{p}_{ij} \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{array}$$

This transition matrix provides substantial dynamic information on bank movements among states. The diagonal elements in the matrix indicate probabilities of banks remaining in their own particular states. For example, p_{11} indicates that 88 percent of the banks in S_1 will remain in S_1, that is, will continue to make no farm loans in the following year. The high values of diagonal probabilities in the above matrix imply a strong tendency for banks to stay in their original states. However, the degree of immobility of banks in each state is different. For instance, the high values of p_{11} and p_{22} indicate strong tendencies for banks in states S_1 and S_2 to remain in their respective states. The first column of the transition matrix also reveals the probability of a bank in any state leaving the system next year. Banks with the largest farm loan ratios are most likely to leave the system than those banks in the states with smaller loan ratios, but their probabilities of leaving are all less than 1 percent. A general picture of bank mobility in terms of the number of banks moving could also be obtained from the matrix N.

During the 14-year period, 28 member banks left the system, while 5312 observations (one bank in a particular year is treated as one observation) or about 85 percent of the sample remained in their own states during a two-year period. There were 922 observations or 15 percent of the sample moving among states in this time span.

As stated in the model, entry of new member banks is allowed into state i at any arbitrary time t*. The pattern of these new entries in the Ninth Federal Reserve District is shown in Table 2.

Table 2. NUMBER OF NEW MEMBER BANKS WHICH ENTERED EACH STATE ($e_i(t)$)

Year	S_1	S_2	S_3	S_4	S_5	Total
1955		1				1
1956	1	2				3
1957	1	1				2
1958	1	1				2
1959	1	1				2
1960						0
1961	2	1				3
1962	1		1			2
1963	2	6	2	3	2	15
1964	4	4	2			10
1965		1	1	1		3
1966	1		1		1	3
1967			1		1	2
1968	1					1
1969	2	1	2	1		6

The estimated mean rate of entry $\hat{\xi}_i$ during the first 14-year period, thus, is computed according to equation (3.10):

States	S_1	S_2	S_3	S_4	S_5	Total
ξ_i	1.1	1.4	.6	.3	.3	3.7

The 1961 liberalization of chartering and regulatory policies for national banks by James Saxon, the Comptroller of the Currency, had encouraged the entry of and the conversion of banks to national banks in 1963 and 1964. There were twelve and eight new national banks in 1963 and 1964 respectively. This fact makes the assumption of constant mean rate of entry quite untenable. It would be realistic in this situation to regress $\xi(t)$ on the relevant exogenous variables. If this is done with the policy change above, the following estimated mean rates of entry would be obtained:

	S_1	S_2	S_3	S_4	S_5	Total
$\xi_i(1) = \ldots = \xi_i(8)$	.9	.9	.1	0	0	1.9
$\xi_i(9) = \ldots = \xi_i(13)$	1.4	2.2	1.4	.8	.8	6.6

7.4 Testing of Hypotheses

In economic analysis, it is essential to test statistically some of the basic assumptions underlying the model in use. The hypothesis tests developed in Section 5 can be used to examine several assumptions about the model.

a) Test of stationarity for first order Markov chain: (see section 5.2).

The values of $n_{ij}(t)$'s and $p_{ij}(t)$'s are listed in Appendix Tables 1 and 2 respectively. The 13-year average p_{ij}'s and n_{ij}'s are shown in Section 7.3. According to equation (5.3),

$$\log \lambda = \sum_{i=1}^{5} \sum_{j=0}^{5} n_{ij} \log \hat{p}_{ij} - \sum_{t=1}^{13} \sum_{i=1}^{5} \sum_{j=0}^{5} n_{ij}(t) \log \hat{p}_{ij}(t)$$

$= - 3253.08 - (-3068.69) = - 184.399$ then,

$- 2 \log \lambda = 368.798$ with $(5)^2(13-1) = 300$ degrees of freedom. The χ^2 table shows that the calculated value exceeds the table value of 341.395 at 5 percent significance level, but is fairly close to the table value 366.844 at .5 percent significance level.

b) Test of both order and stationarity: (see section 5.3). According to equation (5.5), $\log \lambda = - 393.64$ and $- 2 \log \lambda = 787.28$ with $(13-1)\ (5)^3 - (5)^2 = 1475$ degrees of freedom. This value is much less than the table value for rejection at 5 percent significance level for the corresponding degrees of freedom.

c) Test of transition probabilities for the new entries: (see section 5.4).
According to equation (5.9), $\log \lambda = - 3253.08-[-3141.6 + (-102.571)] = -5.91$, and $- 2 \log \lambda = 11.82$ which is smaller than the χ^2 table value with $(5)^2 = 25$ degrees of freedom at the 5 percent significance level. This test leads to a conclusion that the new entries follow the same probabilistic law as the ones originally in the system.

Based on the results of the above tests, the small variation in $\hat{p}_{ij}(t)$ over t, the large sample size and the desire for a simple mathematical model, the system is well represented as a first order and stationary Markov chain.

7.5 Forecasting

If the chain is of first order and stationary, forecasting of its future behavior could be made as stated in Section 6. To predict future bank distributions in this example, f_0' represents the row vector of the bank distribution in the starting year, 1954, i.e., $f_0' = (32, 218, 132, 64, 27)$, and $\hat{P}$ is the estimated 13-year average transition matrix as shown in Section 7.3. The estimated annual rate of new bank entries is $\hat{\xi}' = (1.1, 1.4, 0.6, 0.3, 0.3)$ if a constant mean rate of entry is assumed. The procedure given in Section 6 is used to obtain projected bank distributions (A) for each successive year since 1954; they are shown by a dotted line in the figure for the period 1955-69 and in the upper half of Table 3 for years 1970-75.

Table 3. PREDICTED BANK DISTRIBUTIONS (A) and (B), 1970-75
NINTH FEDERAL RESERVE DISTRICT

Year	S_0 Exit (cumulated)	S_1 0%	S_2 >0- < 21%	S_3 21-<41%	S_4 41-<61%	S_5 61% & Over	Existing Total S_1-S_5
	(A) With A Constant Mean Rate of Entry						
1970	35	43	200	106	87	62	498
1971	37	43	200	106	87	63	499
1972	40	43	199	106	88	64	500
1973	42	43	199	106	89	64	501
1974	44	43	199	107	89	65	503
1975	46	43	199	107	90	65	504
	(B) With Varied Mean Rates of Entry						
1970	35	44	202	108	88	64	506
1971	37	44	203	109	90	65	511
1972	39	44	203	110	91	66	514
1973	42	45	204	111	92	67	519
1974	44	45	205	112	93	68	523
1975	47	45	206	113	95	69	528

Using two mean rates of entry, as estimated in the latter part of Section 7.3, the predicted bank distributions (B) have been plotted with a broken line in the figure for the period 1955-69 and listed in the lower half of Table 3 for years 1970-75. The first estimated annual rate of entry $\hat{\xi}' = (0.9, 0.9, 0.1, 0, 0)$ was used in projecting bank distributions from 1955 to 1962, while the second entry rate of $\hat{\xi}' = (1.4, 2.2, 1.4, 0.8, 0.8)$ was used in projecting bank distributions from 1963 to 1975.

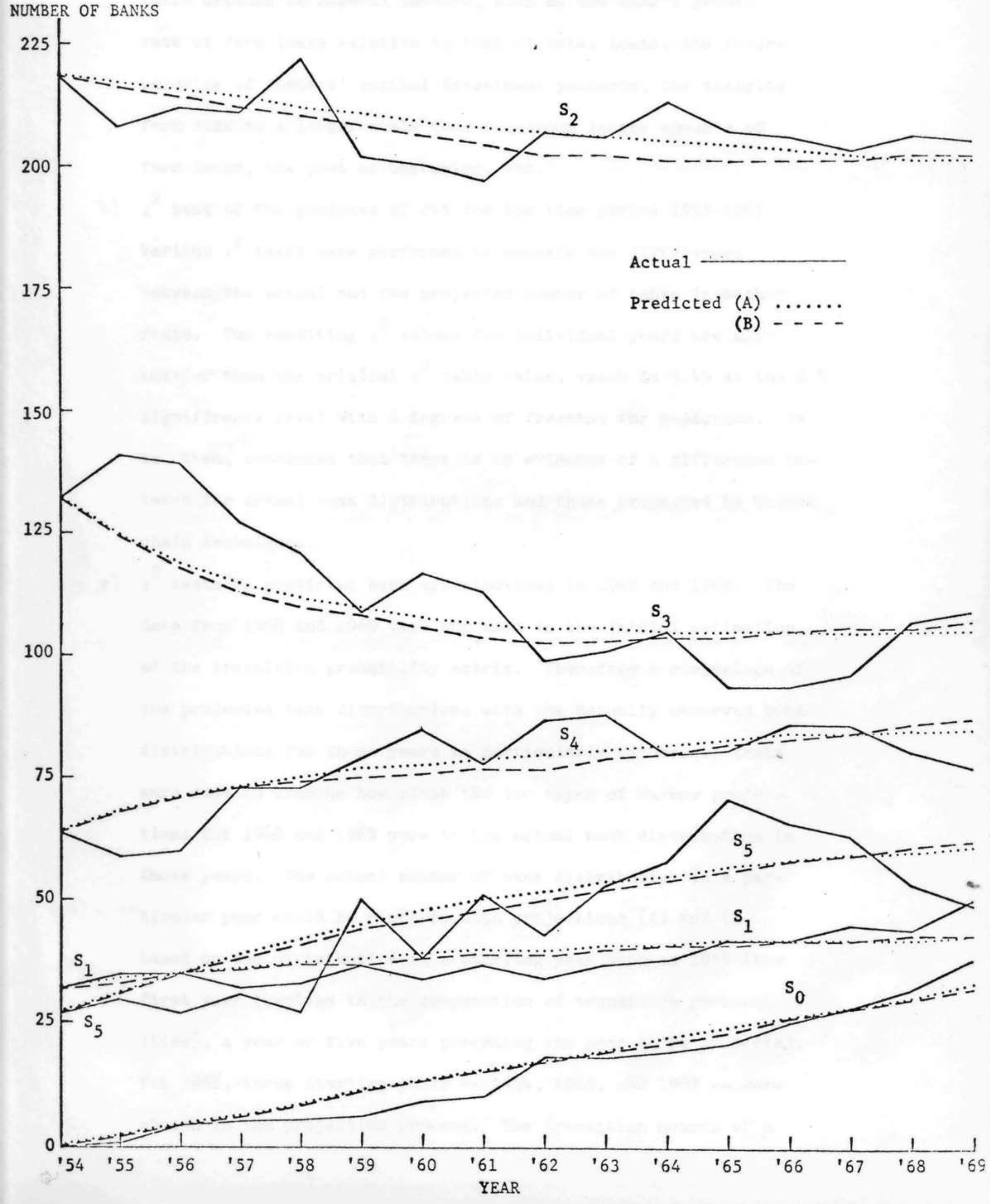
Although the projected numbers of bank distributions (A) and (B) are slightly different, the projected percentage distributions of banks from these two processes are very close. The overall percentage distribution of banks in the early 1970's will be less than 10 percent in S_1, about 40 percent in S_2, 20 percent in S_3, less than 20 percent in S_4, and slightly over 10 percent in S_5.

7.6 The Goodness of Fit

How good are these Markov chain projections?

a) Graphic comparison: The figure gives a general view of the difference between the actual and the projected number of banks in each state. Both projections (A) and (B) not only provide smooth trend lines for each state, but most of the time also closely follow the actual number of banks in these states during the past 16 years. Does the fact that the actual number of banks in S_4 and S_5 (as shown in the figure) has declined in the past few years suggest that the downward trends in these two categories will continue in the future? This question is difficult to answer, first, because 1969 was a very unusual year since tight monetary policy prevailed; and second, because the value of a bank's farm loan

ACTUAL AND PREDICTED NUMBER OF MEMBER BANKS IN FARM LOAN RATIO CLASSES, NINTH FEDERAL RESERVE DISTRICT



ratio depends on several factors, such as the bank's growth rate of farm loans relative to that of total loans, the future behavior of farmers' capital investment patterns, the changing farm size to a larger scale thus requiring larger amounts of farm loans, the cost of borrowing, etc.

b) χ^2 test of the goodness of fit for the time period 1955-1967: Various χ^2 tests were performed to measure the differences between the actual and the projected number of banks in each state. The resulting χ^2 values for individual years are all smaller than the critical χ^2 table value, which is 9.49 at the 0.5 significance level with 4 degrees of freedom, for rejection. It is, then, concluded that there is no evidence of a difference between the actual bank distributions and those projected by Markov chain techniques.

c) χ^2 tests on predicted bank distributions in 1968 and 1969: The data from 1968 and 1969 were not used in the initial estimation of the transition probability matrix. Therefore a comparison of the projected bank distributions with the actually observed bank distributions for those years is particularly valuable. Tests were made to examine how close the two types of Markov projections for 1968 and 1969 were to the actual bank distribution in those years. The actual number of bank distribution in a particular year could be compared with projections (A) and (B) based on the distribution in a starting year such as 1954 (the first year involved in the computation of transition probabilities), a year or five years preceding the year to be projected. For 1968, three starting years -- 1954, 1963, and 1967 -- were chosen in the projection process. The transition matrix of a

13-year average was used in all three cases. The resulting projections (A) and (B) and the actual bank distributions in 1968 as well as their respective χ^2 test values are shown in the upper half of Table 4. All projections have χ^2 values less than the critical table value $\chi^2_{.95,4} = 9.49$ for rejection. Thus, any one of them is a good predictor for bank distributions in 1968.

Three starting years -- 1954, 1964, and 1968 -- were used in the projection for 1969. The two types of predicted number of banks in each class and their corresponding χ^2 values are shown in the lower half of Table 4. These χ^2 values are also smaller than the critical value for rejection.

Table 4. ACTUAL AND PROJECTED BANK DISTRIBUTIONS FOR 1968 AND 1969 USING 13-YEAR AVERAGE $\hat{P}$

NINTH FEDERAL RESERVE DISTRICT

States		S_0	S_1	S_2	S_3	S_4	S_5	Total Existing	χ^2-value
1968 Actual		32	44	206	107	80	53	490	--
1954 base	A	30	42	201	105	85	60	494	1.50
	B	30	43	202	106	86	61	498	1.71
1963 base	A	29	40	200	105	86	61	492	2.40
	B	29	42	204	109	89	63	507	2.96
1967 base	A	30	45	202	98	86	63	494	3.07
	B	30	45	203	99	86	64	497	3.16
1969 Actual		38	50	205	109	77	49	490	--
1954	A	33	43	201	105	86	61	496	5.43
	B	32	43	202	107	87	62	501	6.22
1964	A	30	41	206	106	86	62	501	7.87
	B	30	43	210	110	89	64	516	8.53
1968	A	34	44	205	106	81	55	491	2.53
	B	34	44	206	107	82	55	494	2.29

d) Test for the stationarity of the transition probabilities in 1968 and 1969:

i) Test of 1968 transition probabilities: Since the data for 1968 were available, the frequency distribution in each state and a transition matrix $[p_{ij}(1968)]$ were calculated (see 4th column from the end in both Appendix Tables 1 and 2). It is of interest to test if the 1968 p_{ij}'s follow the same probabilistic law as the one with the 13-year average p_{ij}'s. The null hypothesis under testing, then, is

$$H_0: \; p_{ij}(1968) = p_{ij} \qquad \text{vs.}$$

$$H_a: \; p_{ij}(1968) \neq p_{ij} \, .$$

If n_{ij}' and $\hat{p}_{ij}'$ represent the pooled 14-year (1954-1968) number count of banks and the average transition probability respectively (see second column from the end in both Appendix Tables 1 and 2), the likelihood ratio test can be written as follows:

$$\lambda = \prod_{i=1}^{5} \prod_{j=0}^{5} \left[\frac{\prod_{t=1}^{14} \hat{p}_{ij}'}{\prod_{t=1}^{13} \hat{p}_{ij} \; \hat{p}_{ij} \; (1968)} \right]^{n_{ij}(t)}$$

$-2 \log \lambda$ is asymptotically χ^2 with $m^2 = 25$ degrees of freedom. The computed value of $-2 \log \lambda$ is 33.9 which is less than the χ^2 value of 37.7 for rejection. It is, then, concluded that the new 1968 transition matrix is not different from the one with 13-year averages at a .05 level of significance.

ii) Test of 1969 transition probabilities: By the same procedure, the stationarity of transition probabilities in 1969 could be tested, with T = 15 and the pooled 15-year n_{ij}'s and $\hat{p}_{ij}$'s (see last columns in Appendix Tables 1 and 2) for the numerators, and pooled 14-year n_{ij}'s, $\hat{p}_{ij}$'s, $n_{ij}(1969)$, $\hat{p}_{ij}(1969)$ for the denominators in the above equation. The frequency distribution and transition probabilities of 1969 banks are listed in the third columns from the end of the Appendix Tables 1 and 2. The computed value of $-2 \log \lambda$ is 24.77 which is also less than the critical value of χ^2 for rejection, 37.7. Therefore, it is also concluded that the transition matrix from 1968 to 1969 is not different from the one with 14-year averages.

8. CONCLUSION

The model proposed in Section 2 for a Markov chain having stochastic entry and exit provides a general tool for predicting the behavior of complex systems. Due to the completeness of historical data available on banks in the Ninth Federal Reserve District, it was possible to apply and test the theoretical model on these empirical data. The results shown in Section 7 reveal that (1) the movements of district member banks could be effectively described as a Markov chain with stochastic entry and exit; (2) the transition probabilities of these banks could be treated as first order and stationary; (3) the transition probabilities of newly entered banks follow the same probabilistic law as the original banks in the system; (4) projections based on 1954 provide estimates yielding relatively low chi-square values. It is, then, hoped that the methods illustrated here will suggest interesting new paths for future researchers into banking system behavior. Most importantly, it is hoped that the model presented in this study will provide some insight into the ways of handling statistical problems involving entry and exit.

APPENDIX

Appendix Table 1. NUMBER OF MEMBER BANKS IN FIRST ORDER MARKOV CHAIN STATES

NINTH FEDERAL RESERVE DISTRICT

($n_{00}(t)$ = NUMBER IN COLUMN 1, TABLE 1, $n_{0j}(t) = 0$ for $j = 1, \ldots, 5$)

Years n_{ij}	1954-55	1955-56	1956-57	1957-58	1958-59	1959-60	1960-61	1961-62	1962-63	1963-64	1964-65	1965-66	1966-67	1967-68	1968-69	14-year total (1954-68)	15-year total (1954-69)
n_{10}	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	3	3
n_{11}	27	29	30	30	31	29	31	31	30	32	35	38	40	40	40	453	493
n_{12}	5	6	4	2	2	6	2	5	4	5	2	4	2	4	3	53	56
n_{13}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
n_{14}	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	2	2
n_{15}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	2
n_{20}	0	0	0	0	0	1	1	2	0	0	1	3	2	1	2	11	13
n_{21}	8	5	1	2	4	4	4	2	5	1	7	3	4	3	8	53	61
n_{22}	192	192	197	199	197	187	183	187	191	199	196	192	195	192	190	2699	2889
n_{23}	17	11	12	10	19	10	12	6	11	6	9	6	5	7	6	141	147
n_{24}	1	0	1	0	2	0	0	0	0	0	0	2	0	0	0	6	6
n_{25}	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	2	2
n_{30}	1	0	0	0	0	1	0	3	0	1	0	1	1	1	2	9	11
n_{31}	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	2	2
n_{32}	10	11	9	19	2	7	11	14	5	5	4	10	6	10	11	123	134
n_{33}	109	117	106	98	86	92	93	82	77	86	81	76	78	83	87	1264	1351
n_{34}	12	13	22	10	28	9	11	14	17	8	19	6	9	2	7	180	187
n_{35}	0	0	2	0	4	0	2	0	0	0	1	1	0	0	0	10	10
n_{40}	0	2	0	0	1	1	0	0	0	0	0	0	0	1	1	5	6
n_{41}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
n_{42}	0	1	0	0	0	0	0	1	0	0	3	0	0	0	0	5	5
n_{43}	15	10	9	12	4	12	8	10	11	11	3	11	11	17	12	144	156
n_{44}	39	41	43	55	47	63	64	61	65	66	59	66	69	65	58	803	861
n_{45}	10	5	8	6	21	2	13	6	11	11	15	4	6	3	9	121	130
n_{50}	0	1	0	0	0	0	0	2	0	0	1	0	0	0	1	4	5
n_{51}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1
n_{52}	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	2	2
n_{53}	0	1	0	1	0	2	0	0	0	0	0	0	1	0	2	5	7
n_{54}	7	6	7	8	2	12	3	12	3	6	1	12	8	13	11	100	111
n_{55}	20	22	20	21	25	36	35	37	40	47	55	59	56	50	39	523	562

Appendix Table 2. FIRST ORDER TRANSITION PROBABILITIES FOR MEMBER BANKS IN THE NINTH FEDERAL RESERVE DISTRICT

($p_{00}(t) = p_{00} = 1, p_{0j}(t) = p_{0j} = 0$ for $j = 1, \ldots, 5$)

Years p_{ij}	1954–55	1955–56	1956–57	1957–58	1958–59	1959–60	1960–61	1961–62	1962–63	1963–64	1964–65	1965–66	1966–67	1967–68	1968–69	14-year average (1954–68)	15-year average (1954–69)
p_{10}	0	0	.0286	0	0	0	0	.0270	0	0	0	0	0	.0222	0	.0058	.0054
p_{11}	.8438	.8286	.8571	.9375	.9394	.7838	.9118	.8378	.8823	.8649	.9211	.9048	.9524	.8889	.9091	.8830	.8851
p_{12}	.1562	.1714	.1143	.0625	.0606	.1622	.0588	.1352	.1177	.1351	.0526	.0952	.0476	.0889	.0682	.1033	.1005
p_{13}	0	0	0	0	0	.0270	0	0	0	0	0	0	0	0	0	.0020	.0018
p_{14}	0	0	0	0	0	.0270	0	0	0	0	.0263	0	0	0	0	.0039	.0036
p_{15}	0	0	0	0	0	0	.0294	0	0	0	0	0	0	0	.0227	.0020	.0036
p_{20}	0	0	0	0	0	.0050	.0050	.0102	0	0	.0047	.0145	.0097	.0049	.0097	.0038	.0042
p_{21}	.0367	.0240	.0047	.0095	.0180	.0198	.0200	.0101	.0242	.0049	.0329	.0145	.0194	.0148	.0388	.0182	.0196
p_{22}	.8807	.9231	.9293	.9431	.8874	.9257	.9150	.9492	.9227	.9660	.9202	.9275	.9466	.9458	.9224	.9269	.9266
p_{23}	.0780	.0529	.0566	.0474	.0856	.0495	.0600	.0305	.0531	.0291	.0422	.0290	.0243	.0345	.0291	.0484	.0471
p_{24}	.0046	0	.0047	0	.0090	0	0	0	0	0	0	.0097	0	0	0	.0020	.0019
p_{25}	0	0	.0047	0	0	0	0	0	0	0	0	.0048	0	0	0	.0007	.0006
p_{30}	.0076	0	0	0	0	.0092	0	.0265	0	.0099	0	.0106	.0107	.0104	.0187	.0057	.0065
p_{31}	0	0	0	0	.0083	0	0	0	0	.0099	0	0	0	0	0	.0013	.0012
p_{32}	.0757	.0780	.0647	.1496	.0165	.0642	.0940	.1239	.0505	.0495	.0381	.1064	.0638	.1042	.1028	.0774	.0791
p_{33}	.8258	.8298	.7626	.7747	.7107	.8440	.7949	.7257	.7778	.8515	.7714	.8085	.8298	.8646	.8131	.7960	.7970
p_{34}	.0909	.0922	.1583	.0757	.2314	.0826	.0940	.1239	.1717	.0792	.1810	.0638	.0957	.0208	.0654	.1133	.1103
p_{35}	0	0	.0144	0	.0331	0	.0171	0	0	0	.0095	.0107	0	0	0	.0063	.0059
p_{40}	0	.0339	0	0	.0137	.0126	0	0	0	0	0	0	0	.0116	.0125	.0046	.0052
p_{41}	0	0	0	0	0	.0127	0	0	0	0	0	0	0	0	0	.0009	.0009
p_{42}	0	.0170	0	0	0	0	0	.0128	0	0	.0375	0	0	0	0	.0046	.0043
p_{43}	.2344	.1695	.1500	.1644	.0548	.1519	.0941	.1282	.1264	.1250	.0375	.1358	.1279	.1977	.1500	.1335	.1346
p_{44}	.6094	.6949	.7167	.7534	.6438	.7975	.7529	.7821	.7471	.7500	.7375	.8148	.8023	.7558	.7250	.7442	.7429
p_{45}	.1562	.0847	.1333	.0822	.2877	.0253	.1530	.0769	.1265	.1250	.1875	.0494	.0698	.0349	.1125	.1122	.1121
p_{50}	0	.0333	0	0	0	0	0	.0392	0	0	.0172	0	0	0	.0189	.0063	.0073
p_{51}	0	0	0	0	0	0	0	0	0	0	0	0	.0151	0	0	.0016	.0014
p_{52}	0	0	0	.0323	0	0	0	0	0	0	.0172	0	0	0	0	.0031	.0029
p_{53}	0	.0333	0	.0323	0	.0400	0	0	0	0	0	0	.0152	0	.0377	.0079	.0102
p_{54}	.2593	.2000	.2593	.2580	.0741	.2400	.0790	.2353	.0698	.1132	.0173	.1690	.1212	.2063	.2075	.1575	.1613
p_{55}	.7407	.7334	.7407	.6774	.9259	.7200	.9210	.7255	.9302	.8868	.9483	.8310	.8485	.7937	.7359	.8236	.8169

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