WP-25

Interest Rates and Prices in the Long Run:

A Study of the Gibson Paradox

by

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August 1971

Note: I would like to thank Phillip Cagan, John Hause, Edmund Phelps, Neil Wallace, and especially Christopher Sims for helpful comments on earlier versions of this paper. Thomas Turner contributed valuable research assistance. Part of the work on this paper was completed during my tenure as a visiting professor at the Federal Reserve Bank of Minneapolis.

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0. Introduction

This is a study of the relationship between commodity price inflation and interest rates. One of the chief avenues through which inflation has been posited to affect interest rates is through the effect of actual inflation on anticipated inflation, which is then taken to help determine the "nominal" rate of interest. For this reason, the manner in which price anticipations are formed is a topic that cannot help but occupy an important role in a study such as this.

Although many papers have been written on the topic,[1/] no single explanation of the relationship between inflation and interest commands wide acceptance. As proof of this statement, it is sufficient to note that the name that Keynes gave to that relationship -- the Gibson paradox -- has stuck. Keynes's claim was that over long periods of time in the United States, England, and other countries, interest rates had been highly correlated with the aggregate level of commodity prices. Keynes [20] named this empirical regularity the Gibson paradox[2/] since it seemed to contradict the prediction of classical monetary theory that the interest rate is independent of the price level. According to classical doctrines, the interest rate is determined by the "real" factors of productivity and thrift which impinge on the market for loanable funds, while the price level is determined primarily by the money supply, as described by the quantity-of-money theory of prices. The relationship between interest rates and prices which Keynes had detected was a paradox for classical monetary theory because it seemed to constitute a disconfirmation of one of the important predictions of that theory.

The most famous explanation of the paradox is due to Irving Fisher [5]. Fisher built upon the classical proposition that during a process of fully anticipated inflation, nominal rates of interest on assets whose returns are fixed in monetary value will rise so that the relationship among real yields on assets will not be affected. The rise in nominal interest rates on such assets is enforced by investors' unwillingness to hold those assets at any lower nominal rates of yield. Thus, ignoring uncertainty, the nominal rate of interest on assets with returns fixed in money terms, r, is taken to be the sum of the (real) rate of return on assets with returns fixed in real terms, ρ, and the anticipated rate of inflation, π:

$$r_t = \rho_t + \pi_t \tag{0.1}$$

Here the subscripts denote the period to which each variable refers. Relation (1) states that the real rates of return on all assets are equal regardless of whether their streams of returns are fixed in money or in real terms.

Fisher combined relation (1) with an equation designed to explain the formation of the public's anticipated rate of inflation, π_t. Anticipating the work of Cagan [3] and Friedman [8] by about twenty-five years, Fisher posited that people formed expectations by taking a weighted sum of current and past actual rates of inflation, i.e.

$$\pi_t = \sum_{i=0}^{m} v_i \frac{\Delta p_{t-i}}{p_{t-i-1}}, \tag{0.2}$$

where p_t is the price level at time t, and the v_i's and m are parameters. Substituting (2) into (1) produces Fisher's fundamental equation,

$$r_t = \rho_t + \sum_{i=0}^{m} v_i \frac{\Delta p_{t-i}}{p_{t-i-1}}. \tag{0.3}$$

In order to implement (3) empirically, Fisher made the crucial assumption that ρ is statistically independent of the second term in equation (3), that is, that

$$\rho_t = \alpha + \varepsilon_t$$

where α is a constant and where ε_t is a random disturbance distributed independently of $\Delta p_{t-i}/p_{t-i-1}$, $i = 0, \ldots\ldots, m$. Here α might be interpreted as the long-run real equilibrium rate of interest. Substituting the above equality into equation (3) yields

$$(0.4) \qquad r_t = \alpha + \sum_{i=0}^{m} v_i \frac{\Delta p_{t-i}}{p_{t-i-1}} + \varepsilon_t,$$

which is the equation that Fisher implemented in his empirical work. Using the distributed lag estimator which he had developed[3/], Fisher estimated equation (4) for both long-term and short-term interest rates and for several countries. In each case, he estimated that m was very large and that the coefficients v_i were positive, declining slowly with increases in i. In fact, the estimated v_i were distributed over such a long period of time that the estimated expected rate of inflation π_t resembles the level of prices much more closely than it resembles the current rate of inflation. Together with equation (1), very long lags in the formation of expectations described by equation (2) provide an explanation of the high correlation between interest rates and the price level. Thus, Fisher's econometric results seemed to explain the Gibson paradox.

However, it is possible to argue that Fisher's explanation of the Gibson paradox is really only a redefinition of it. Fisher's explanation raises the question of why people's anticipations of inflation are apparently so slow to adjust. As Cagan [2] has argued, the mean adjustment lags estimated by Fisher and others are so incredibly long, ranging from ten to thirty years or longer, that they seem highly implausible on the maintained hypothesis that they are the result of lags in the formation of expectations.

The force of that argument undoubtedly explains many economist's reluctance to accept Fisher's conclusions. Some authors[4/] have seemingly argued that Fisher's econometric techniques are at fault, suggesting that his results are the consequence of estimating equation (4) improperly. Yet that is exceedingly unlikely, since Fisher was a very clever statistician; in fact, using the distributed lag estimator that he invented avoids many of the pitfalls associated with the method proposed by L. M. Koyck [23] three decades later. Nevertheless, to dispel any doubts on the matter and to extend Fisher's results in a couple of directions, it seems worthwhile to present some estimates of Fisher's equation using an alternative distributed lag estimator. Those estimates are presented in section I; not surprisingly, for the most part they confirm Fisher's empirical results, long lags and all.

In section II we consider the merits of Cagan's contention that the distributed lags estimated by Fisher and others are implausibly long. That requires giving the notion of plausibility some operational content, which we try to accomplish by utilizing John F. Muth's [28] suggestion that the expectations of the market can fruitfully be hypothesized to be the same as the optimal forecasts of statistical theory. In this section, we adopt (quite restrictive) assumptions that are compatible with equation (2) being the correct form of model for forming "optimal" expectations. We then synthesize the distributed lag weights that would characterize optimal forecasts within that class of models. These synthetic distributed lags are very much "shorter" than those estimated by Fisher and others, and thus tend to confirm the doubts expressed by Cagan; but the extent of confirmation is much more limited than a casual glance at the shape of the "optimal" distributed lags would suggest.

In Section III we attempt to characterize the relationship between inflation and the interest rate with more precision, using cross-spectral methods to determine whether a model that, like Fisher's, incorporates only a one-way direction of influence from inflation to interest rates is sufficient to explain the data. Our results suggest that it is necessary to take into account a second direction of influence, with influence flowing from interest rates to the rate of inflation. This in turn implies that it is too much to hope that a single-equation model like equation (4) can adequately explain the relationship between inflation and interest rates. It also raises doubts about the adequacy of equation (2) as a model for the formation of expectations.

In section IV we try to illustrate how an empirical relationship like that discovered in section III might emerge. Such a relationship is predicted by a very simple model of the economy. Consequently, that model can be used to formulate an explanation of the Gibson paradox that is an alternative to Fisher's. In section IV we also discuss how much information about the relationship between interest rates and anticipated inflation can be gleaned from data on prices and interest rates alone.

Our conclusions are stated in section V.

I. Estimates of Fisher's Equation: 1870-1940

In order to implement Fisher's equation empirically, it is convenient to assume that the distributed lag weights in equation (1.4) trail off geometrically,

$$v_i = \gamma \lambda^i \qquad i \geq 0, \qquad |\lambda| < 1.$$

so that (0.4) becomes

$$(1.1) \qquad r_t = \alpha + \gamma \sum_{i=0}^{\infty} \lambda^i \frac{\Delta p(t-i)}{p(t-i-1)} + \epsilon_t .$$

As is well known, this specification is equivalent to assuming that the anticipated rate of inflation is formed via the "adaptive expectations" scheme

$$(1.2) \qquad \pi_t - \pi_{t-1} = \gamma \frac{\Delta p(t)}{p(t-1)} - (1-\lambda)\, \pi_{t-1}$$

which is equivalent with

$$(1.2') \qquad \pi_t = \gamma \sum_{i=0}^{\infty} \lambda^i \frac{\Delta p(t-i)}{p(t-i-1)}$$

Replacing (0.2) with the special form (1.2') yields equation (1.1).

In addition to its tractability for purposes of estimation, equation (1.1) has the advantage that the geometrical form may often be a "sensible" one to impose on the distribution of lag weights where the source of the distributed lag is an expectations generating mechanism. By sensible we mean that for an interesting class of stochastic processes that might be used effectively to approximate the evolution of the actual rate of inflation, it will be rational or optimal (in the least-squares sense) for market participants to forecast the rate of inflation by the scheme described by (2). This aspect of specification (2) will be exploited below in helping to determine the plausibility of our estimates.

Equation (1) was estimated by following a suggestion of Klein [16], which involves noting that the distributed lag in (1) can be rewritten as

$$\gamma \sum_{i=0}^{t-1} \lambda^i \frac{\Delta p(t-i)}{p(t-i-1)} + \gamma \sum_{i=t}^{\infty} \lambda^i \frac{\Delta p(t-i)}{p(t-i-1)} \quad .$$

The second sum equals

$$\lambda^t \gamma \sum_{i=0}^{\infty} \lambda^i \frac{\Delta p(0-i)}{p(0-i-1)}$$

or

$$\lambda^t \eta_0$$

where η_0 equals $\gamma \sum_{i=0}^{\infty} \lambda^i \Delta p(0-i)/p(0-i-1)$ and represents the systematic part of the initial condition of the difference equation. Klein's procedure involves estimating η_0 simultaneously with the other parameters of (1) by non-linear estimation techniques. In this case we applied the simple search procedure that Hildreth and Lu [16] have suggested.[5/]

For annual data for the U.S. spanning the period 1870-1940, and for various sub-periods within those years, we have estimated equation (1) for both long-term and short-term interest rates. Including rates on instruments with different terms to maturity seems an interesting thing to do since the horizon over which the forecasts of inflation are cast ought to be different. The speeds of adjustment of expectations of inflation are likely to vary with respect to horizon and hence across bond maturities. As the short-term rate we used the yield on commercial paper while Macaulay's [24] unadjusted yield index for railroad bonds was taken as the long-term rate. The commercial paper rate was used rather than the call loan rate because of the latters extraordinary behavior during stock market panics. Although a retail price index or GNP deflator would have been preferable, our desire to extend the period of observation back to 1870 limited our choice of a price index to a wholesale commodity price index. All data are annual averages of monthly data. The interest rate data are in percentage points while the rate of inflation is measured in percent.

Our estimates are reported in Tables 1 and 2. Without exception, the distributed lags are exceeding "long," the decay parameter λ being estimated as being close to unity in each case. Not infrequently, the least-squares estimate of λ is .99, the upper bound for the values of λ over which we searched. These large values of λ imply that the data are approximately described by

$$r_t = \alpha + \gamma \sum_{i=0}^{\infty} \Delta p(t-i)/p(t-i-1) + \epsilon_t,$$

or, since $\Delta p(t-i)/p(t-i-1) \approx \log_e p(t-i)/p(t-i-1)$,

$$r_t = \alpha + \gamma \sum_{i=0}^{\infty} \log_e \frac{p(t-i)}{p(t-i-1)} + \epsilon_t$$

or

$$r_t = \alpha + \gamma \log_e p(t) + \epsilon_t \ , \tag{3}$$

Macaulay's Unadjusted Railroad Yields Regressed
Against Current and Past Rates of Inflation

TABLE 1.1 $r_t = \alpha + \gamma \cdot \sum_{i=0}^{t-1} \lambda^i \frac{\Delta p(t-i)}{p(t-i-1)} + \lambda^t \eta_0$

Period	α	γ	λ	η_0	R_A^2	d.w.
1870-1929	3.4746 (.6673)	2.0735 (.1436)	.95	4.8934 (.1780)	.9277	.6627
1885-1929	.9426 (.5594)	1.7888 (.1907)	.99	3.9755 (.6557)	.6903	.7880
1895-1929	1.8536 (1.0181)	1.6451 (.2642)	.99	2.1240 (1.0691)	.7254	.8226
1900-1929	4.2129 (.1929)	1.5562 (.2306)	.95	-.4853 (.2440)	.8027	1.1306
1905-1929	4.2991 (.2017)	1.5318 (.2555)	.94	-.3070 (.2685)	.7340	1.1075
1880-1929	.7427 (.3710)	1.8409 (.1473)	.99	4.3576 (.4399)	.7603	.8378
1870-1936	4.2281 (0.678)	1.1223 (.2324)	.91	4.3334 (.2462)	.8283	.4518
1880-1936	2.2929 (.4783)	1.3964 (.2231)	.99	2.5746 (.5758)	.3994	.6171
1885-1936	3.1833 (.6097)	1.1305 (.2497)	.99	1.3984 (.7271)	.3278	.6315
1895-1936	4.6003 (.7976)	.9375 (.2664)	.99	-.7013 (.8546)	.4391	.7257
1900-1936	4.7380 (.1760)	.8937 (.2721)	.95	-1.0321 (.2716)	.4736	.8123

Estimated Standard errors are in parentheses

R_A^2 denotes adjusted R^2

d.w. denotes Durbin-Watson Statistic

Commercial Paper Rate Regressed Against
Current and Past Rates of Inflation

Table 1.2 $r_t = \alpha + \gamma \sum_{i=0}^{t-1} \lambda^i \frac{\Delta p(t-i)}{p(t-i-1)} + \lambda^t \eta_0$

Period	α	γ	λ	η_0	R_A^2	d.w.
1870-1929	4.3936 (.1620)	1.8558 (.5192)	.91	3.8305 (.5573)	.4347	1.6415
1885-1929	2.4255 (1.0139)	1.9993 (.6874)	.98	3.0489 (1.3658)	.1297	1.7793
1895-1929	2.6813 (1.1270)	2.3168 (.8716)	.97	1.8564 (1.3165)	.1652	1.4693
1900-1929	-3.2778 (3.5576)	2.4431 (.8672)	.99	8.0365 (3.6978)	.1722	1.2837
1905-1929	-4.7784 (4.2820)	2.5149 (.9288)	.99	9.6580 (4.4583)	.1821	1.2692
1870-1940	-3.4029 (.8910)	2.5110 (.5519)	.99	11.5720 (1.2899)	.5364	.8647
1885-1900	-8.2668 (1.5197)	4.0318 (.6791)	.99	15.0554 (1.8251)	.5467	.9866
1895-1940	-12.0294 (1.6328)	4.5561 (.6146)	.99	17.1390 (1.7594)	.6779	1.0280
1900-1940	-12.5334 (1.6143)	4.2502 (.6127)	.99	17.9610 (1.7462)	.7277	.9316
1905-1940	-13.9551 (1.7269)	3.9458 (.6346)	.99	19.2015 (1.9041)	.7525	.9888
1880-1929	1.0692 (1.316)	1.7201 (.5224)	.99	4.3564 (1.5604)	.1575	1.8035

Estimated standard errors are in parentheses

R_A^2 denotes adjusted R^2

d.w. denotes Durbin-Watson Statistic.

where p(t) is measured in the units of the "base-period" price, $p_{-\infty}$. Basically, therefore, the estimates simply recover the correlation between interest rates and the price level (more precisely, its logarithm) that characterizes these data, thus confirming the existence of the Gibson paradox.

The sum of the distributed lag coefficients $\gamma \sum_{i=0}^{\infty} \lambda^i$, which equals $\gamma/(1-\lambda)$, is easily calculated. The sum varies a good deal from period to period, ranging from about ten to one-hundred and eighty. The extent of this range is not surprising, since for large values of λ the sum of weights is very sensitive to small changes in γ and λ. To calculate the corresponding sum of weights in the expectations generator (2), it is necessary to divide the above sums by one-hundred, since we have measured interest rates in percentage points while recording inflation in percent.

These estimates corroborate the main outlines of Fisher's findings. For the long-run data, an extensive set of estimates can be found in Fisher's _Theory of Interest_. More recent studies using alternative distributed lag estimators that are somewhat more flexible than those used here also confirm Fisher's findings.[6] The cross-spectral calculations reported in section 3 also confirm the existence of very long lags in the interest-inflation relationship.

Thus, the Gibson paradox does infest the long-run data. While Fisher's explanation of that paradox formally "works," the implied lags in forming expectations do seem extraordinarily long. In the next section, we attempt to examine those lags in the light of their extraordinary length.

2. The Plausibility of Long Adjustment Lags

Ever since empirical results like those above were first reported by Irving Fisher, they have been challenged on the grounds that they seemed to imply implausibly long lags in the formation of expectations of inflation.[7/] The appeal of that challenge probably explains why economists in general retain strong reservations about Fisher's work in this area. Yet the argument that Fisher's results are implausible has never been made with sufficient care and force to displace Fisher's doctrine as the most widely cited explanation of the Gibson paradox. The reason seems to be that the criterion of plausibility has usually been very vague and has failed to provide the basis for a really compelling attack on Fisher's results. In this section, we try to remedy this defect in the argument by providing a fairly careful statement of what distributed lag estimates constitute implausible results.

By hypothesis, the distributed lag function in Fisher's equation emerges because people form expectations about future rates of inflation by in effect calculating a weighted sum of current and past actual rates of inflation. Fisher and his followers have maintained that by estimating equation (0.4) we recover the v_i's of equation (0.2), the weights used in forming expectations. To say that the estimates of the weights obtained by that procedure are implausible apparently means that they do not resemble the weights that really characterize the process by which people seem to form expectations about future rates of inflation. To substantiate that claim, extraneous information about the v_i's is obviously required. That information is exceedingly difficult to come by given the extreme paucity of data on expectations of inflation over the period we are studying. If such data were available in sufficient quantity, they could be used to estimate equation (0.2) directly and so establish a basis for a very straightforward check on the sensibility of Fisher's estimates of equation (0.4).

In the absence of data directly measuring expectations, there is an alternative source of information about the v_i's which it seems worthwhile to exploit. That information can be obtained at the cost of invoking John F. Muth's [28] argument that the expectations of the market can be hypothesized to be the optimal forecasts of statistical and economic theory. If the market's expectations were to deviate markedly from that standard, extraordinary profits would accrue to those who utilize the available information more efficiently than the market.

In order to implement Muth's argument it is necessary to posit a class of statistical models that are assumed to be capable of describing the evolution of the actual rate of inflation. Once such a class of models is specified, the model that, within this class, produces "optimal" forecasts can be identified. Those optimal forecasts will be used as the yardstick against which we will judge the "plausibility" of the expectations implied by Fisher's estimates.

In this section, we implement the above strategy by considering a class of very simple statistical models for the rate of inflation. In particular, we shall confine our search for optimal forecasts to those produced by members of the class of autoregressive models[8/]

$$x_{t+1} = \sum_{i=0}^{\infty} v_i \, x_{t-i} + u_{t+1} \tag{2.1}$$

where x_t is the rate of inflation and where u_{t+1} is an independently, identically distributed random variable with mean zero and variance σ_u^2.[9/] Our justification for considering this class of models is that they imply that the optimal forecast over each horizon will be formed by taking a weighted sum of current and past rates of inflation. The equations generating optimal forecasts are compatible in form with the expectations generator posited by Fisher and others. Thus, while equation (1) defines a very naive class of models for explaining the rate of inflation, its virtue is its consistency with the econometric practices of Fisher and others.

An alternative, somewhat deeper rationalization for employing this class of models can be produced by appealing to Wold's theorem which states that any covariance stationary stochastic process can be represented as the sum of two mutually uncorrelated processes

$$x_t = \eta_t + z_t$$

where η_t is a deterministic process, predictable with zero mean square error given a sufficient number of its own past values, and z_t is a one-sided moving sum of "white noise," i.e.

$$z_t = \sum_{j=0}^{\infty} c_j u_{t-j}$$

where u_t is an independently and identically distributed random variable with mean zero and finite variance. It is generally believed that economic time series exhibit no important deterministic (i.e. strictly periodic) components of variation, so that η_t is in effect zero. Then Wold's representation becomes simply

(2.2) $$x_t = \sum_{j=0}^{\infty} c_j u_{t-j} ,$$

which is the moving-average representation for x_t. In many cases, a series with such a moving average representation can also be represented as an autoregressive process of the form of equation (1).[10] In what follows, we shall find it convenient to assume that the inflation rate possesses both a moving average and an autoregressive representation.[11] The autoregressive representation will be most useful in writing down the form of the optimal predictors that are to be compared with Fisher's estimates, while the moving average representation will frequently be of help in suggesting economical parameterizations of the process that we are trying to model. We will find it useful to have at our disposal a formula relating the coefficients of the autoregressive and moving average forms. To derive this formula, it will be convenient to use the lag operator L defined by

$$L^n x_t = x_{t-n} .$$

For many operations, L can be manipulated like any other algebraic symbol. We shall utilize polynomials in the lag operator, such as

$$B(L) = b_0 + b_1 L + b_2 L^2 + \ldots + b_n L^n .$$

The polynomial B(L) is often called a lag-generating function.

Using the lag operator notation, we move equation (2) forward one period and write

$$x_{t+1} = c_0 u_{t+1} + \sum_{j=0}^{\infty} c_{j+1} u_{t-j} \tag{2.3}$$

$$= c_0 u_{t+1} + C(L) u_t ,$$

where $C(L) = \sum_{j=0}^{\infty} c_{j+1} L^j$. At time t, the least squares forecast of x at time t+1 is found by setting u_{t+1} equal to its expected value of zero:

$$E(x_{t+1} \mid u_t, u_{t-1}, \ldots) = C(L) u_t . \tag{2.4}$$

Using the autoregressive representation (1), the least squares forecast of x at time (t+1) is achieved by setting u_{t+1} equal to its expected value of zero:

$$E(x_{t+1} \mid x_t, x_{t-1}, \ldots) = \sum_{i=0}^{\infty} v_i x_{t-i} \tag{2.5}$$

$$= V(L) x_t$$

where $V(L) = \sum_{i=0}^{\infty} v_i L^i$. Expressing x_t in terms of its moving-average representation, (5) becomes

$$E(x_{t+1} \mid x_t, x_{t-1}, \ldots) = V(L) [c_o + LC(L)] u_t . \tag{2.6}$$

Being based on equivalent representations, (4) and (6) must yield identical forecasts, which implies that C(L) and V(L) are related by

(2.7) $$C(L) = V(L)[c_o + LC(L)].$$

If we normalize by dividing both sides of (7) by c_o, which we assume does not equal zero, we have

(2.8) $$C^*(L) = V(L)\ [1 + LC^*(L)]$$

where $C^*(L) = \sum_{j=0}^{\infty} \frac{c_{j+1}}{c_o} L^j$.

Once V(L) is known, the generating functions for forming least squares forecasts over horizons of more than one period can be easily determined. For example, the two-period-forward forecast can be found by substituting the one-period-forward forecast $\hat{x}_{t+1}$ for x_{t+1} in the formula

$$\hat{x}_{t+2} = V(L)\ x_{t+1} .$$

Thus,

$$\hat{x}_{t+2} = [v_o V(L) + v_1 + v_2 L + v_3 L^2 + \ldots]\ x_t .$$

Similarly, the expectations generator for $\hat{x}_{t+j}$ can be calculated recursively by substituting $\hat{x}_{t+i}$ for x_{t+i}, all $i > 0$, in the expression

(2.9) $$x_{t+j} = V(L)\ x_{t+j-1} .$$

Through use of relations (8) and (9), information about the stochastic process generating the x_t's, i.e., information about V(L) or C(L), can be used to calculate the lag-generating function $V_j(L)$ that produces the optimal forecast over any horizon j:

$$\hat{x}_{t+j} = V_j(L)\ x_t .$$

It is these lag-generating functions that we propose to compare with those implied by estimates of Fisher's equation.

In order to implement our procedure, it is necessary to impose some additional restrictions on the process generating the actual rate of inflation. We will assume that the x-process is a mixed autoregressive, moving-average error process of low order. This restriction is compatible with the rest of our analysis since it implies via relations (8) and (9) that the lag-generating functions $V_j(L)$ will be members of Jorgenson's [19] class of rational distributed lag functions. That is a desirable result since for the most part the empirical results in section I and in other studies were obtained by estimating rational distributed lag functions. Thus, for example, suppose that the x-process takes the form of the second-order mixed autoregressive, moving-average error process

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + b_1 u_{t-1} + b_2 u_{t-2} + u_t \tag{2.10}$$

where u_t is a white noise and a_1, a_2, b_1, and b_2 are parameters. The process can be written

$$\begin{aligned} x_t &= \frac{b_1 L + b_2 L^2}{1-a_1 L-a_2 L^2} u_t + \frac{1}{1-a_1 L-a_2 L^2} u_t \\ &= \frac{(b_1+a_1)L + (b_2+a_2)L^2}{1-a_1 L-a_2 L^2} u_t + u_t \ . \end{aligned}$$

For this process C(L) is thus given by

$$C(L) = \frac{(b_1 + a_1) + (b_2 + a_2)L}{1-a_1L-a_2L^2} .$$

Using relation (8) it follows that V(L), the lag-generating function for the optimal one-period forward forecast of inflation, is given by [12/]

$$V(L) = \frac{(b_1 + a_1) + (b_2 + a_2)L}{1 + b_1L + b_2L^2} . \qquad (2.11)$$

Here V(L) is a (first-order numerator, second-order denominator) rational distributed lag generating function. Specializing this somewhat, suppose $b_2 = a_2 = 0$, so that (10) becomes a first-order mixed autoregressive, moving-average-error process. Then V(L) is

$$V(L) = \frac{b_1 + a_1}{1 + b_1 L} = (b_1 + a_1) \sum_{i=0}^{\infty} (-b_1)^i L^i ,$$

which with a_1 equaling one and b_1 being negative is the lag-generating function that Cagan and Nerlove used to model the formation of expectations, and which was used extensively in section I.

Our strategy is thus to estimate the parameters of a mixed autoregressive, moving-average-error process like (10) for wholesale commodity price inflation in the U.S., and then use relations (8) and (9) to calculate $V_j(L)$ over various horizons. The mixed autoregressive, moving average error processes were estimated by the least-squares search procedure described by Jenkins and Watts [18]. For example, to estimate (10) given x_t, t=1, ... , T, it was assumed that $\hat{u}_0 = \hat{u}_{-1} = \hat{u}_{-2} = 0$. For predetermined values of a_1, a_2, b_1, and b_2, equation (10) was then solved repeatedly for the residuals $\hat{u}_t$, t=1, ... , T associated with these particular parameter values. The associated sum of squared residuals $\sum_{t=1}^{T} \hat{u}_t^2$ was then calculated. By calculating the sum of squared residuals repeatedly, conducting a search over the (a_1, a_2, b_1, b_2)-space for the parameter values which yield the minimum sum of squared residuals, the least squares estimates of the parameters were found. Our final estimates emerged from a search over a grid whose width was .01 for each parameter.

Table 1 records the results of estimating first-order autoregressive, first-order moving average error processes for data on the rate of inflation of wholesale commodity prices in the United States, the same data used in the regressions reported in section I. Use of the first order autoregressive, moving-average

Table 2.1

Estimated Parameters of Mixed
Autoregressive Moving Average Error Process

Period	a_1	b_1	SSR/N
1870-1929	.29	-.05	.008497
1870-1970	.22	.06	.008108
1880-1929	-.49	.75	.009089
1880-1970	-.45	.75	.008529
1880-1914	.27	-.21	.002822

SSR denotes sum of squared residuals

N = number of observations minus four

Table 2.2 (a_2 constrained to equal - b_2)

Estimated Parameters of Mixed
Autoregressive Moving Average Error Process

Period	a_1	a_2	b_1	b_2	SSR/N
1870-1929	.76	-.60	-.49	.60	.008144
1870-1940	.88	-.61	-.63	.61	.008542

SSR denotes sum of squared residuals

N = number of observations minus six

Table 2.3

Estimated Parameters of Mixed
Autoregressive Moving Average Error Process

Period	a_1	a_2	b_1	b_2	SSR/N
1870-1929	.11	-.56	-.24	1.05	.007431
1870-1940	.15	-.60	-.24	1.06	.007484

SSR denotes sum of squared residuals

N = number of observations minus seven

error model implies that at time t the optimal forecast over horizon j will be given by

$$\hat{x}_{t+j} = \frac{(a_1 + b_1)\, a_1^{\,j-1}}{1 + b_1 L} x_t \tag{2.12}$$

where a_1 and b_1 are the parameters that appear in (10). The estimates in Table 1 imply that within this class of models the distributed lag weights associated with the optimal forecasts decline in absolute value swiftly with increases in lag, since the decay parameter b_1 is estimated to be much less than unity. It is interesting to note that the lag distributions implied by the estimates for three of the time periods studied display oscillating weights rather than smoothly changing ones like those generally assumed in empirical estimation.

A 99 percent confidence region for the estimates for the period 1870 through 1940 is shown in figure 1.[13/] The region is banana shaped and includes the parameter estimates obtained for the other sub-periods. Notice that the region includes the origin, which means that at the 99 percent confidence

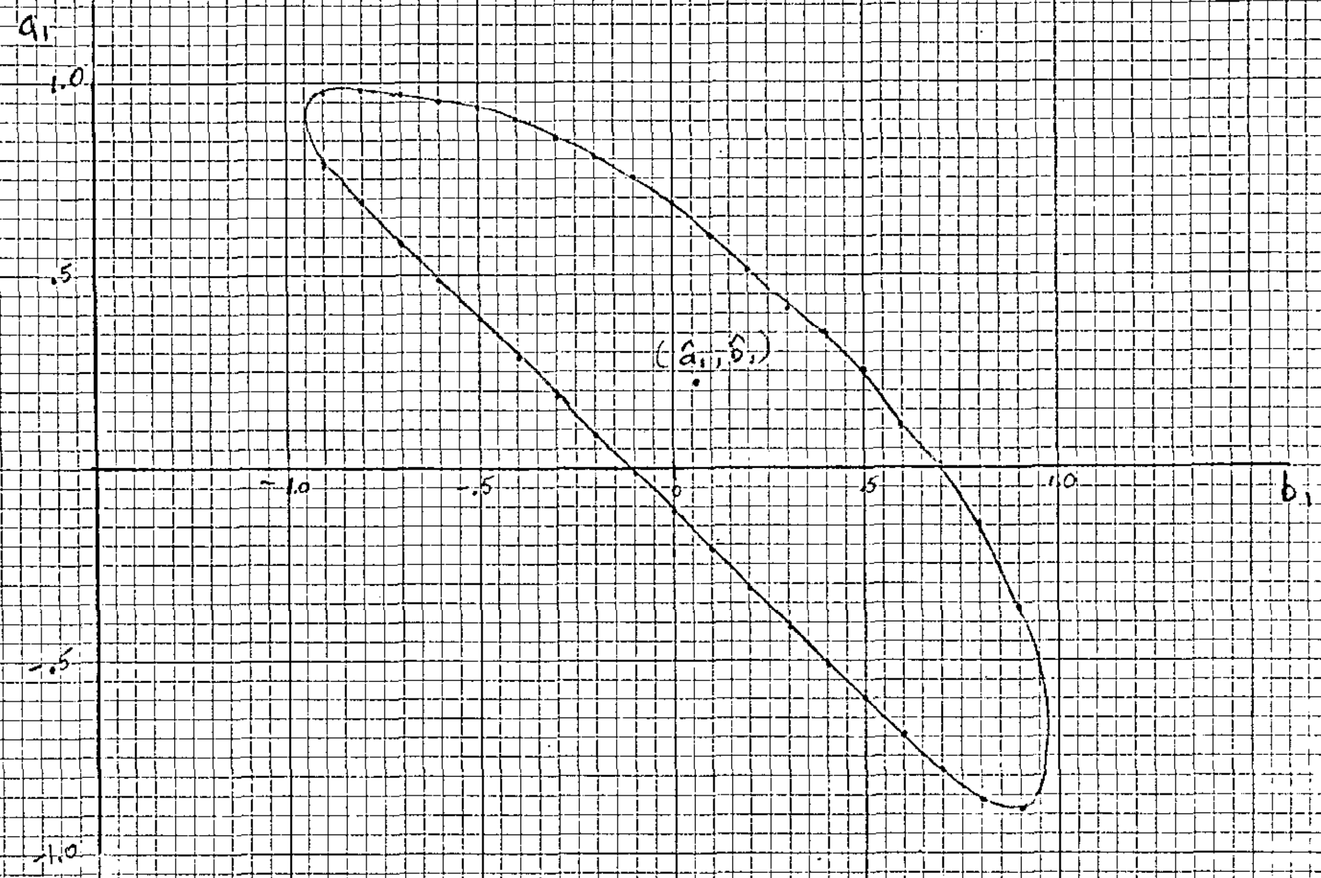


Figure 2.1

99 percent confidence region around $(\hat{a}_1, \hat{b}_1)$ 1870–1940.

level we cannot reject the hypothesis that the inflation rate behaved like white noise.

For variables that closely resemble white noise, we would expect to find an elongated confidence region resembling the one shown in figure 1. For suppose x_t evolves according to

$$x_t = u_t \tag{2.13}$$

where u_t is white noise, i.e. an independently and identically distributed random variable with variance σ_u^2. Such an x process can be represented equally well by any first-order mixed autoregressive moving average error process

$$x_t = a_1 x_{t-1} + b_1 u_{t-1} + u_t$$

that has the property $a_1 = - b_1$; for using the lag operator notation, we can rewrite the above equation as

$$x_t = \frac{1 + b_1 L}{1 - a_1 L} \; u_t .$$

If $b_1 = - a_1$, the numerator and the denominator in the above expression cancel, leaving us with $x_t = u_t$. Thus if we fit a first order mixed autoregressive, moving-average error process, we will find that the sum-of-squares surface has a valley extending along the line $a_1 = - b_1$. For such a process, parameter values implying widely different mean lags in (11) may differ very little in their explanatory power. That is because the mean lag in (12), which for $a_1 \neq b_1$ is given by

$$\frac{- b_1}{1 + b_1} \quad ,$$

depends only on the value of b_1. Inspection of figure 1 shows that parameter values implying very long mean lags are contained in the confidence region around the least squares estimates. This fact has the consequence that on the basis of their very long mean lags alone, it is not possible to reject the estimates in section 1 as implying very poor forecasts of inflation. Very long mean lags in (12) are compatible with "nearly optimal" forecasting provided that $a_1 \approx - b_1$, which implies that the sum of the weights in (12) will be small, since the sum of weights in (12) is given by

$$\frac{(a_1 + b_1)\ a_1^{\,j-1}}{1 + b_1}$$

It will be recalled that in section I we estimated lag distributions with sums of weights on the order of from one-tenth to two. With a value of b_1 of -.99, a sum of weights of two requires a value of a_1 of 1.01, if the sum of weights is interpreted as applying to a one-period forecast. Such calculations establish that the estimates reported in section I represent (a_1, b_1) pairs that lie close to and sometimes within, the upper left hand corner of the confidence region depicted in figure 1. On the other hand, the slope of the sum-of-squares surface is very steep near the upper left hand corner of the confidence region, so that, for example, the sum of squares increases quickly as a_1 increases above unity where b_1 is slightly less than unity.

These results suggest that ruling on the plausibility of the very long distributed lags estimated in section I is a rather delicate matter. The configuration of the distributed lag weights associated with the best-fitting first order mixed autoregressive, moving-average error process is very different than the patterns estimated in section I. Yet there are lag distributions with very long lags that would not have generated forecasts significantly worse than those produced by the optimal forecasting schemes. These very long lag distributions closely resemble those estimated in section I. Thus, while the forecasts produced with the aid of the lag distributions of section I are not fully optimal, neither are they "very ridiculous." We therefore conclude that unless one is prepared to lean very heavily on the hypothesis that forecasts are optimal, placing great emphasis on even relatively minor differences in means of squared errors of forecasts associated with alternative predictors, it is difficult to sustain the charge that estimates of Fisher's equations are implausible on the basis of the results presented above.[14/]

The mean lags associated with the j-period-forward forecasts formed via the set of geometrical distributed lag functions (12) all equal $-b_1/(1+b_1)$ and thus are independent of the horizon of forecast j. Milton Friedman [9], among others, has contended that it is often reasonable to expect the mean lag to be positively associated with the horizon over which expectations are cast. It thus seems desirable to adopt more general parameterizations of the x-process that permit such a dependence between mean lag and horizon to emerge as a consequence of optimal forecasting. One very simple such parameterization is the second-order mixed autoregressive, moving-average error process

$$x_t = a_1 x_{t-1} - b_2 x_{t-2} + b_1 u_{t-1} + b_2 u_{t-2} + u_t ,$$

which implies, via relation (11) that the distributed lag generating function appropriate for forming one-period forward forecasts is given by

$$V(L) = \frac{(b_1 + a_1)}{1 + b_1 L + b_2 L^2} .$$

It has been shown elsewhere (see Sargent and Wallace [31]) that for such a process the mean lag in the lag generating function for forecasts j periods forward will be positively associated with horizon j if and only if $b_2 < 0$ where $b_1 + a_1 > 0$.

Table 2 reports least-squares estimates of the second order mixed auto-regressive, moving-average error process where a_2 is constrained to equal $- b_2$, as is required in the above parameterizations. As for the first order processes, the estimates imply lag-generating functions for the optimal forecasts that are quite short. In addition, since b_2 is estimated to be positive, the mean lag of those lag - generating functions diminishes rather than increases as the horizon of forecast increases.

Table 3 reports least-squares estimates of the second order mixed auto-regressive, moving-average error process where a_2 and b_2 are not constrained to be equal. Once again, these estimates imply that the optimal one-period forward forecasts of inflation are not characterized by long lag distributions. For example, the estimates based on data for the period from 1870 to 1929 imply that the lag generating function to be used in calculating one-period forecasts is given by

$$V(L) = \frac{- .13 + .49L}{1 - .24L + 1.05L^2} .$$

Expanding this polynomial in L shows that in addition to following an oscillatory pattern, the weights decline fairly swiftly with increases in lag.

In summary, our estimates of the lag distributions that would be optimal for forecasting inflation are all very much shorter than those that are obtained by estimating Fisher's equation directly. On the face of it, this finding would seem to confirm Cagan's argument that estimates of Fisher's equation are characterized by lags that are implausibly long. However, the estimates we have made are surrounded by large confidence regions, regions that include implied lag distributions with very long mean lags. For this reason, our estimates do not seriously diminish the credibility of estimates of expectations produced by estimating Fisher's equation directly. The optimal expectations implied by our estimates differ from those obtained by estimating Fisher's equation chiefly in their behavior at very low frequencies. With the relatively short time series at our disposal it is difficult to say much about those low-frequency components, making it unwise to stress differences at those frequencies.[15/]

III. Cross-Spectral Analysis of Interest and Inflation

The model used in the preceding two sections and in many other studies of the Gibson paradox to analyze the relationship between interest and inflation can be written

$$(3.1)\quad r_t = a(L)\ r_t + b(L)\ x_t + \epsilon_t$$

$$(3.2)\quad x_t = d(L)\ x_t + u_t$$

where u_t and ϵ_t are mutually independent white noises and a(L), b(L), and c(L) are one-sided polynomials in the lag operator L, i.e.

$$a(L) = \sum_{j=1}^{\infty} a_j L^j$$

$$b(L) = \sum_{j=0}^{\infty} b_j L^j$$

$$d(L) = \sum_{j=1}^{\infty} d_j L^j$$

Equation (1) can be written in the distributed lag form

$$r_t = \frac{b(L)}{1 - a(L)}\ x_t + \frac{1}{1 - a(L)}\ \epsilon_t ,$$

which is the form estimated in section 1. The experiments involving optimal forecasting that we reported in section 2 entailed determining whether the relationship between $b(L)/[1-a(L)]$ and $d(L)$ was consistent with interpreting $[b(L)/(1-a(L))]\ x_t$ as the market's forecast of inflation over some horizon.

The class of models summarized by (1) and (2) are distinguished by the fact that they admit only one direction of influence, one which flows from inflation to the interest rate. Thus, while a large value of u_t will increase the rate of inflation, thereby influencing subsequent rates of interest, the rate of inflation is posited to be independent of ϵ_t. The model asserts that there occurs no feedback from the interest rate to current or subsequent rates of inflation. In that sense, the rate of inflation is assumed to be exogenous. That is a specification that would undoubtedly seem unduly restrictive to most economists, particularly those with monetarist inclinations. In this section, we use techniques that permit and almost invite us to subject that specification to an empirical test. The techniques fall within the field of cross-spectral analysis and were developed by Akaike [1], Granger [12,13], and Sims [32].

The model described by equations (1) and (2) can be regarded as a special case of the system

$$(3.3)\qquad \begin{bmatrix} 1 - a(L) & -b(L) \\ - c(L) & 1-d(L) \end{bmatrix} \begin{bmatrix} r_t \\ x_t \end{bmatrix} = \begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix}$$

where $c(L) = \sum_{j=0}^{\infty} c_j L^j$, and where u_t and ϵ_t are again mutually independent white noises. The system formed by equations (1) and (2) emerges when c(L) equals zero. Now (3) is a quite general representation of the two series r_t and x_t, being the *autoregressive representation of the vector process* $\begin{bmatrix} r_t \\ x_t \end{bmatrix}$. *For we know that*

<u>any</u> two covariance stationary, indeterministic variables, say r_t and x_t, can be represented by pairs of one-sided moving averages of the same mutually independent white noises:

$$(3.4) \qquad \begin{bmatrix} r_t \\ x_t \end{bmatrix} = \begin{bmatrix} \alpha(L) & \beta(L) \\ \gamma(L) & \delta(L) \end{bmatrix} \begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix}$$

where $\alpha(L)$, $\beta(L)$, $\gamma(L)$, and $\delta(L)$ are one-sided polynomials in the lag operator L. The capacity of (4) to represent such a $\begin{bmatrix} r \\ x \end{bmatrix}$ process emerges from a generalization of the theorem of Wold that we referred to in section 2. Equations (3) and (4) are equivalent representations of the processes, one being easily derived from knowledge of the other.[16/]

Sims [32] has proved a very useful theorem that permits a straightforward test for the existence of feedback. Supposing as we now are that 1-d(L) is not zero, Sims's theorem states that r_t can be expressed as a one-sided distributed lag of x_t with a residual w(s) that is independent of x_t for all t, if and only if c(L) = 0, i.e. if and only if there is no feedback from interest rates to subsequent rates of inflation.[17/] The theorem implies that we can test for the presence of feedback from x to r by estimating a two-sided distributed lag of r on x,

$$(3.5) \qquad r_t = \sum_{j=-m_1}^{m_2} h_j x_{t-j} + v_t$$

where m_1 and m_2 are positive parameters, the h_j's are the estimated distributed lag parameters and v_t is a statistical residual. The absence of such feedback implies that $h_j = 0$ for $j < 0$. If future values of x_t make a significant contribution to explaining r_t in (5), we must reject the hypothesis that there is no feedback, i.e. that $c(L) = 0$.

While two-sided distributed lag functions such as (5) have been estimated only rarely in the time domain, they underlie cross-spectral methods. Cross-spectral analysis involves estimating the parameters of (5), but most studies using cross-spectral analysis summarize the parameters by reporting only various statistics for the frequency domain, statistics whose implications for the presence of feedback are generally very difficult to read. Since those statistics do provide an interesting way to look at some information, however, we shall present some frequency-domain statistics as well as estimates of the h_j's in (5). Another reason for using the cross-spectral method is that it has some computational advantages in estimating relationships like (5), in which we can expect very long distributed lags.

Cross-spectral calculations proceed in terms of the Fourier transform of (5):

$$(3.6)\quad R(w) = H(w)\ X(w) + V(w)$$

where $R(w) = \sum_t r_t e^{-iwt}$, $H(w) = \sum_j h_j e^{-iwj}$, $V(w) = \sum_t v_t e^{-iwt}$ and $X(w) = \sum_t x_t e^{-iwt}$.

The function $H(w)$ is called the transfer function and is characterized by its squared amplitude or "gain" and its phase. The spectral density functions of r, x, and v are given by

$$f_r(w) = E\left|R(w)\right|^2 ,$$

$$f_x(w) = E\left|X(w)\right|^2 ,$$

$$f_v(w) = E\left|V(w)\right|^2 .$$

From (6), we know that $f_x(w)$ and $f_r(w)$ are related by

$$f_r(w) = |H(w)|^2 f_x(w) + f_v(w)$$

where $|H(w)|^2$ is the squared amplitude of the transfer function. The cross-spectrum between r and x is given by

$$\begin{aligned} f_{rx}(w) &= E(X^*(w)\ R(w)) \\ &= H(w)\ E|X(w)|^2 \\ &= H(w)\ f_x(w) \quad , \end{aligned} \tag{3.7}$$

assuming that v and x are statistically independent, which implies that $E(X^*(w)\ V(w)) = 0$. Here the asterisk denotes complex conjugatation. It is usual to characterize the cross spectrum by its gain, $|H(w)|^2$, its phase,

$$\varphi(w) = \tan^{-1} \frac{\mathrm{Im}\ (f_{rx}(w))}{\mathrm{Re}\ (f_{rx}(w))} ,$$

and its coherence,

$$\mathrm{coh}(w) = \frac{|f_{rx}(w)|^2}{f_x(w)\ f_r(w)} ,$$

which indicates the proportion of the variance in one series that is explained by the other series at a given frequency band centered at w.

From (7) it is clear that

$$\begin{aligned} \varphi(w) &= \tan^{-1} \frac{\mathrm{Im}\ (H(w))}{\mathrm{Re}\ (H(w))} \\ &= \tan^{-1} \frac{\sum_{j=-m_1}^{m_2} h_j \sin w_j}{\sum_{j=-m_1}^{m_2} h_j \cos w_j} . \end{aligned}$$

As Howrey and Klein [17] have observed, differentiating the above expression with respect to w and evaluating the result at w equals zero yields

$$(3.8) \qquad \varphi'(0) = \frac{\sum_{j=-m_1}^{m_2} j\, h_j}{\Sigma\, h_j},$$

which is the meanlag of h(L).

Solving (7) for H(w) yields

$$(3.9) \qquad H(w) = \frac{f_{rx}(w)}{f_x(w)},$$

which suggests recovering the h_j's of (5) by taking the inverse Fourier transform of estimates of the right side of (9):

$$(3.10) \qquad \hat{h}_j = (2n)^{-1} \sum_{k=-n+1}^{n} \frac{\hat{f}_{rx}(w_k)}{\hat{f}_x(w_k)}\, e^{iw_k j}, \quad w_k = \frac{\pi k}{n},$$

where the hats denote estimated magnitudes and where
n is the maximal lag used in the covariograms and
cross-covariograms that are used in calculating the spectral densities and the cross spectrum. The estimator in (10) is Hannan's "inefficient estimator." Utilizing this estimator has computational advantages in a study such as this one in which the distributed lags seem to be very long.

Figures 1 and 2 and Tables 1 and 2 report estimates of the spectral densities and cross spectra between the rate of inflation of wholesale commodity prices and both the commercial paper rate and Macaulay's unadjusted yield index over the period 1880-1929. The data are annual averages of monthly data. The

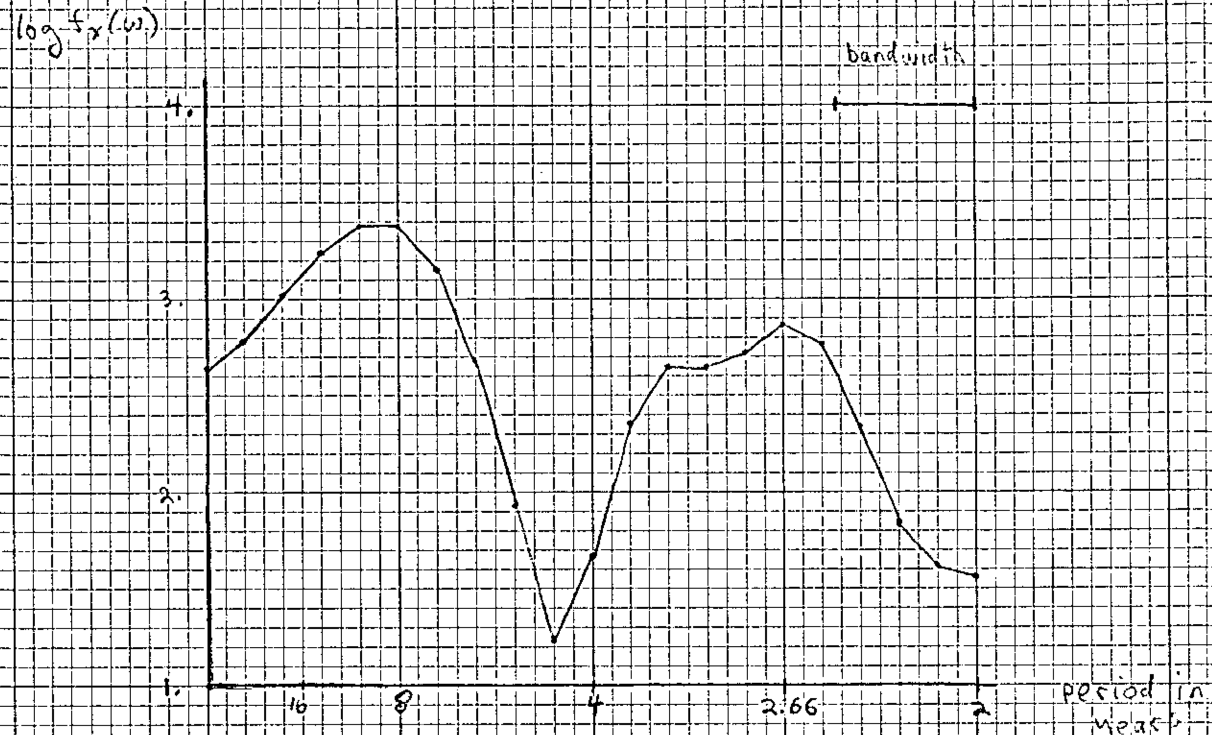
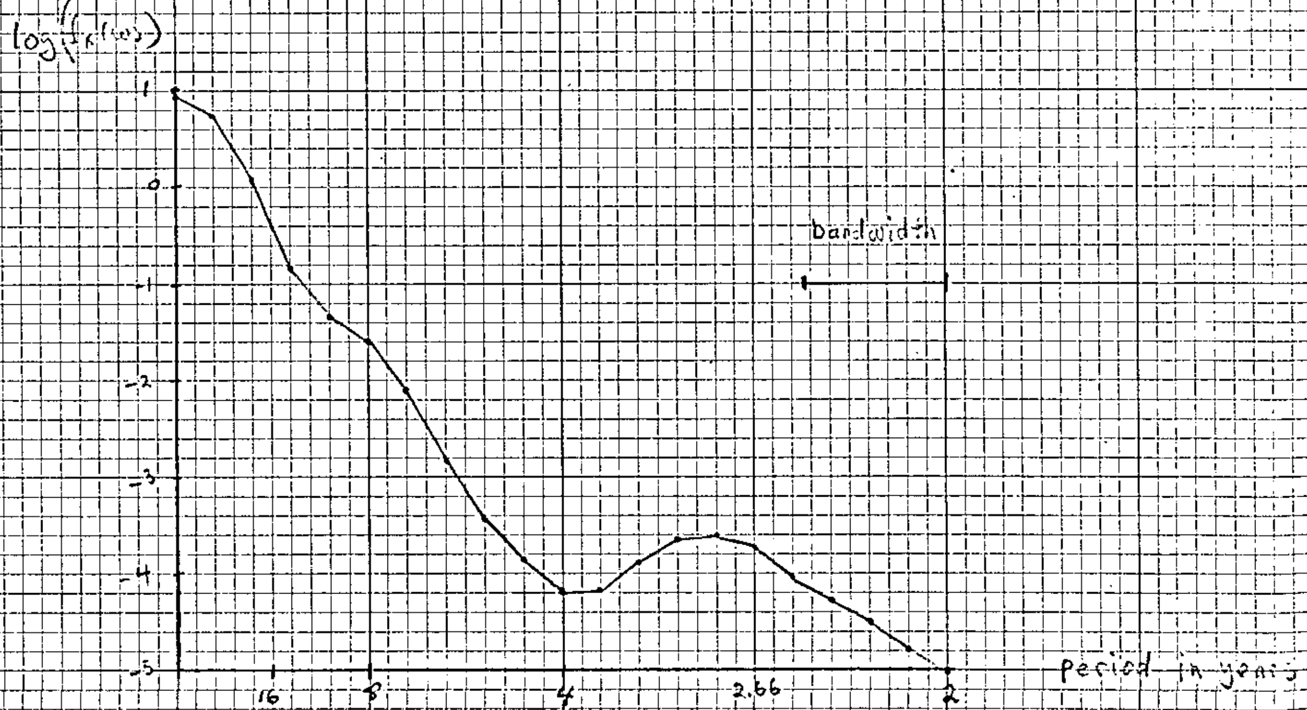


figure 1a

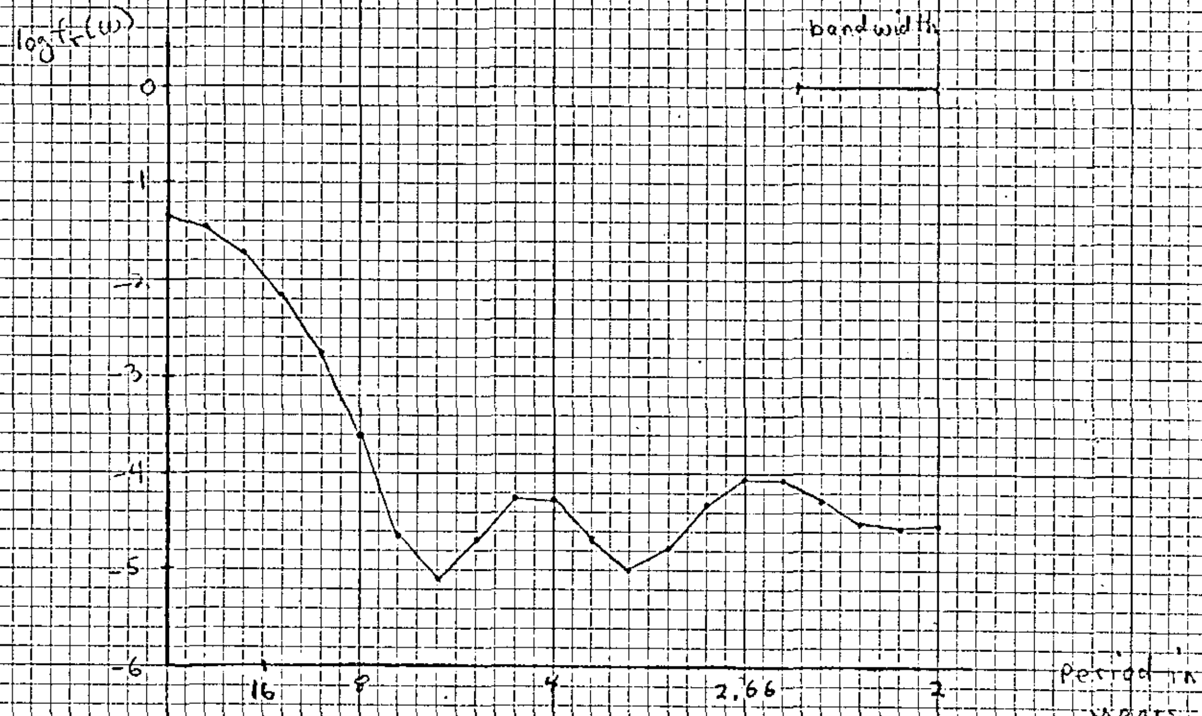
Inflation, Wholesale price index (1880-1929)



Wholesale price Index 1880-1929

figure 1b

figure 2a
Commercial Paper rate 1880-1929



Macaulay's unadjusted yield index (1880-1929)

figure 2b

Table 3.1

Cross Spectrum between Inflation

and Macaulay's unadjusted Yield Index

Period (years)	Coherence	Phase (radians)*		Gain
∞		0.0		
40	.429	-1.32	(.38)	.0792
20	.674	-1.45		.0767
13.3	.691	-1.44		.0551
10.	.648	-1.42		.0368
8.	.571	-1.55		.0229
6.67	.491	1.25		.0142
5.71	.206	.72		.0092
5.0	.040	- .34		.0073
4.44	.080	- .55		.0183
4.0	.158	- .22		.0205
3.64	.205	- .30		.0136
3.33	.143	- .59		.0082
3.08	.011	.07		.0026
2.86	.313	- .97		.0165
2.67	.606	-1.36		.0244
2.50	.621	1.50		.0257
2.35	.341	1.19		.0211
2.22	.051	.36		.0093
2.11	.034	.88		.0082
2.0	.120	.00		.0162

$$\frac{\text{cospec }(0)}{f_x(0)} = \frac{-.4328}{14.16} = -.0306$$

*negative phase indicates that interest rate "leads" inflation rate at frequency in question.

Table 3.2

Cross spectrum between inflation

and commercial paper rate

Period (years)	Coherence	Phase (radians)*	Gain
∞			
40	.413	-1.47 (.39)	.053
20	.679	-1.54	.078
13.3	.710	-1.45	.083
10.	.641	-1.13	.074
8.	.667	- .81	.070
6.67	.598	- .81	.065
5.71	.372	-1.25	.056
5.0	.251	1.26	.057
4.44	.209	1.32	.062
4.0	.490	-1.56	.089
3.64	.463	1.31	.085
3.33	.364	.99	.074
3.08	.284	.83	.062
2.86	.274	1.10	.053
2.67	.399	1.37	.054
2.5	.511	1.45	.056
2.34	.529	1.33	.062
2.22	.560	1.01	.067
2.1	.431	.82	.050
2.0	.144	.00	.025

$$\frac{\text{Cospec }(0)}{f_x(0)} = \frac{-.1175}{14.16} = -.0083$$

*negative phase indicates that interest rate "leads" inflation rate at frequency in question

Table 3.3

Regression of commercial paper rate on past and future

rates inflation (1880-1929)

$$r_t = \sum_{j=-20}^{20} h_j \frac{\Delta p_{t-j}}{p_{t-j-1}}$$

j	Coefficients on future rate of inflation	Coefficient on past rates of inflation
0		.0055
1	-.0504	.0267
2	.0034	.0004
3	.0033	.0179
4	.0043	.0121
5	-.0122	.0129
6	.0010	.0023
7	-.0030	-.0012
8	-.0078	-.0044
9	.0010	.0047
10	.0034	-.0004
11	.0026	.0011
12	-.0031	-.0010
13	.0009	.0026
14	.0003	.0004
15	.0023	.0014
16	-.0006	-.0003
17	.0002	.0016
18	.0003	.0005
19	.0017	.0008
20	-.0002	-.0002

Estimated standard error of coefficients equals .0106

Table 3.4

Regression of Macaulay's Yield Index on past and

Future rates of inflation (1880-1929)

$$r_t = \sum_{j=-20}^{20} h_j x_{t-j}$$

$\lvert j \rvert$	Coefficients on future rates of inflation	Coefficients on lagged rates of inflation
0		.0002
1	-.0100	.0041
2	-.0109	.0001
3	-.0182	.0108
4	-.0070	.0084
5	-.0057	.0046
6	-.0105	.0037
7	-.0015	.0029
8	-.0062	.0037
9	-.0031	.0013
10	-.0031	.0004
11	-.0021	-.0001
12	-.0022	.0014
13	-.0012	.0004
14	-.0012	-.0000
15	-.0015	-.0007
16	-.0002	.0003
17	-.0012	-.0003
18	-.0002	.0001
19	-.0008	-.0010
20	-.0001	-.0001

Estimated standard error of coefficients = .0062

Table 3.5

Regression of inflation rate on past and future

values of commercial paper rate (1880-1929)

$$x_t = \sum_{j=-20}^{20} k_j \, r_{t-j}$$

$\lvert j \rvert$	Coefficients on future commercial paper rates	Coefficients on lagged commercial paper rates
0		0.392
1	2.256	-5.152
2	0.468	0.072
3	3.013	-1.533
4	0.979	1.065
5	1.274	-0.821
6	0.240	0.039
7	-0.078	-0.598
8	-0.116	-1.000
9	0.680	-0.071
10	-0.024	0.372
11	0.647	0.224
12	-0.057	-0.153
13	0.563	-0.037
14	-0.042	-0.010
15	0.365	0.267
16	-0.031	-0.018
17	0.375	0.182
18	0.004	-0.002
19	0.276	0.319
20	-0.028	-0.028

Estimated standard error of coefficients equals 1.1124

Table 3.6

Regression of Inflation Rate on past and future values

of Macaulay's unadjusted yield index (1880-1929)

$$x_t = \sum_{j=-20}^{20} k_j \, r_{t-j}$$

$\lvert j \rvert$	Coefficients on future interest rates	Coefficients on past interest rates
0		2.56
1	5.49	-5.33
2	- .90	-4.17
3	4.94	-8.88
4	- .27	2.54
5	-3.39	3.48
6	.39	-1.55
7	.61	4.69
8	.36	-3.49
9	- .39	-3.62
10	.05	.19
11	- .89	1.36
12	- .37	.22
13	.26	.16
14	.15	- .26
15	- .48	-1.53
16	.04	- .15
17	- .26	.23
18	- .18	.27
19	- .51	- .45
20	.03	.03

Estimated standard error of coefficients - 3.59

spectral densities and cross spectra were calculated by the standard method of taking the Fourier transforms of the estimated covariance and cross-covariance functions. The spectra and cross spectra were calculated using a Parzen window, twenty being the maximal lag used in the estimates reported in the tables and the figures. However, we also carried out calculations with other maximal lags to make sure that our results are not unduly dependent on this particular choice of maximal lag. For our purposes here, twenty seems to be an adequate number of lags. Both x and r are measured in percentage points.

Notice that while the spectral density of the price level has the general form of what Granger calls the "typical spectral shape", with power decreasing fairly regularly with increasing frequency, the spectral density of the inflation rate does not have that shape, being much flatter. The wide peaks in the spectrum suggest that the inflation rate experienced weak "business cycles" with somewhat irregular periods centered at about eight and three years, periodicities which roughly correspond in length with NBER major and minor business cycles. The width of the peaks suggests that these "cycles" were quite variable in length. The fairly flat nature of this spectral density simply confirms our finding in section II, which we obtained by fitting the parameters of mixed autoregressive, moving average error processes, that the inflation rate fairly closely resembled white noise.[18/]

The spectral density of the commercial paper rate resembles that of the inflation rate, its two peaks being centered at slightly longer periodicities than are those of the inflation rate. On the other hand, the spectral density of Macaulay's yield index has a fairly typical shape having very great power at the lowest frequencies. From a comparison of the spectral densities of Macaulay's rate and the inflation rate it is apparent that if a fairly "white" variable like the inflation rate is to explain Macaulay's rate well via a one-sided distributed lag, that lag must be very long. Thus, consider the geometric distributed lag model that we fit in section I,

$$r_t = \frac{\beta^2}{1 - \lambda L} x_t + u_t , \qquad 0 < \lambda < 1 ,$$

where u_t is a statistical residual. The spectrum of r is related to the spectrum of x via the relationship

$$f_r(w) = \left| \frac{\beta}{1 - \lambda e^{-iw}} \right|^2 f_x(w) + f_u(w)$$

$$= \frac{\beta^2}{1+\lambda^2 - 2\lambda \cos w} f_x(w) + f_u(w) .$$

If $f_u(w)$ is flat or relatively small, it will require a large value of λ to transform the relatively flat spectrum $f_x(w)$ into the "typical" spectral shape assumed by $f_r(w)$. It requires a large λ to deliver a gain that decreases swiftly with increases in angular frequency w.

Statistics that summarize the cross spectrum between the inflation rate and each yield are reported in Tables 1 and 2. The phase statistics at low frequency are very large, implying that the interest rate is very much out of phase with (that is displays a large lead over or lag behind) the low frequency oscillations of the rate of inflation. To make this more clear, we can approximate the derivative of the phase at zero frequency, which by (8) equals the mean lag of (5), by

$$(3.12)\qquad \tilde{\Phi}'(0) \simeq \frac{\Phi\,(.157) - \Phi(0)}{.157 - 0}\ .$$

where .157 is the angular frequency corresponding to a periodicity of forty years.[19/] The phase at zero frequency is zero, whereas at the frequency band centered at a periodicity of forty years it is estimated to be -1.47 radians for the commercial paper rate and -1.32 radians for Macaulay's rate. These estimates have sizable approximate asymptotic standard errors of .39 and .38 radians, respectively.[20/] In addition to their high standard errors, it is known that the phase statistic may be biased for processes which, like the ones under study, are badly out of phase. Thus the following estimates of the mean lag in (5) must be interpreted somewhat cautiously.

Substituting -1.47 into the above expression for the mean lag delivers an estimate of 9.4 years. Since the phase statistic is negative, the indication is that the mean lag of the rate of inflation behind the interest rate is 9.4 years or, what is equivalent since we are using the phase statistic at the forty year periodicity to estimate the mean lag, that the interest rate lags behind the rate of inflation by approximately 31 years. For Macaulay's rate, the indication is that the mean lag of inflation behind interest is about 8.4 years. The estimated phase statistics thus are consistent with the very long mean lags estimated in section I using time domain methods together with one-sided distributed lags.

Next notice that for Macaulay's rate the gain of the interest rate over the rate of inflation behaves in the expected manner, generally decreasing with increasing frequency. This implies that the h's in (5) constitute a "colorful" filter that help transform the relatively "white" inflation rate into a variable that displays high power at low frequencies.

Using (7) and the definition of H(w) we have

$$H(0) = \sum_{j=-m_1}^{m_2} h_j = \frac{f_{rx}(0)}{f_x(0)} .$$

Substituting estimated quantities into the above expressions produces estimates of a sum of weights of -.0083 for the commercial paper rate and - .0306 for Macaulay's yield. These estimates are close to zero and thus are very different than those found in section I, which were on the order of +1. The differences between the above estimates and those obtained in section I must be traced to our here relaxing the assumption of a one-sided distributed lag function which we employed in section I.

Our estimates of equation (5), obtained by substituting our cross-spectral estimates into equation (10), are reported in Tables 3 and 4. The estimates of (5) for Macaulay's yield index depict a two-sided lag distribution in which the largest coefficient is on a future rate of inflation. The coefficients on the future rates are predominantly negative, and they outweigh the coefficients on the lagged rates of inflation, which for the most part are positive. It is those negative coefficients on future rates of inflation that explain why the sum of weights that we calculated above was negative. The large estimated standard error for the estimated coefficients suggests that these results ought to be interpreted cautiously.[21/] Nevertheless, the results pretty clearly force us to reject the hypothesis that there is no feedback from interest rates to the rate of inflation. Thus equations (1) and (2) constitute an unduly restrictive description of the relationship between inflation and Macaulay's yield index.

The estimate of equation (5) for the commercial paper rate, which is reported in Table 3.3, is also two sided. The lag distribution is characterized by a small positive weight on the current inflation rate, a large negative one on next period's inflation rate, and a sizable positive weight on inflation lagged once. The nature of these estimates is such that we should consider the possibility that, rather than being a symptom of feedback, the two-sided character of the lag distribution results from aggregation over time of data that are more adequately described by a one-sided lag distribution in a shorter time frame. In particular, Sims [33] has pointed out that if the true continuous-time relationship is

$$r(t) = a\frac{d}{dt}\; x(t)$$

then the discrete time distributed lag weights will be approximately

$$h_o = 0$$

$$h_j = \frac{a}{|j|}(-1)^{|j|} \qquad j = 1, 2, \cdots$$

$$h_j = -\frac{a}{|j|}(-1)^{|j|} \qquad j = -1, -2, \cdots$$

With a value of a of about -.035, this scheme would very approximately describe the distributed lag in Table 3.3. To investigate this possibility we have estimated (5) for quarterly data for the commercial paper rate over the period 1880-1929. The data are averages of monthly data. The estimates, which are reported in Table 3.3a were obtained using Hannan's inefficient estimator, eighty being the maximal lags in the covariograms and cross-covariograms used to construct the estimates. If the data are more properly described by a continuous-time relationship, then the estimates from the quarterly data should follow the same oscillatory pattern just described. They do not follow such a pattern, instead being fairly consistent with the estimates from the annual data. There is a string of sizable negative weights on future rates of inflation, and a string of sizable positive weights on lagged rates of inflation. We conclude from this that the two-sided character of the estimates from the annual data is not simply a result of inappropriate aggregation over time; instead, it indicates the presence of apparent feedback from the commercial paper rate to the rate of inflation.

Tables 5 and 6 report our estimates of the reverse distributed lag relationship

$$(3.13) \qquad x_t = \sum h_j r_{t-j} + U_t'$$

where u_t' is a statistical residual and m in this case equals twenty. It too appears two-sided, with predominantly negative coefficients on lagged interest rates but positive ones on future interest rates. The apparently two-sided nature of this relationship implies that the rate of inflation has some influence on subsequent rates of interest.

Table 3.3a

Regression of Commercial Paper Rate on Past and Future Values of Wholesale Commodity Price Inflation (Quarterly) 1880-1929

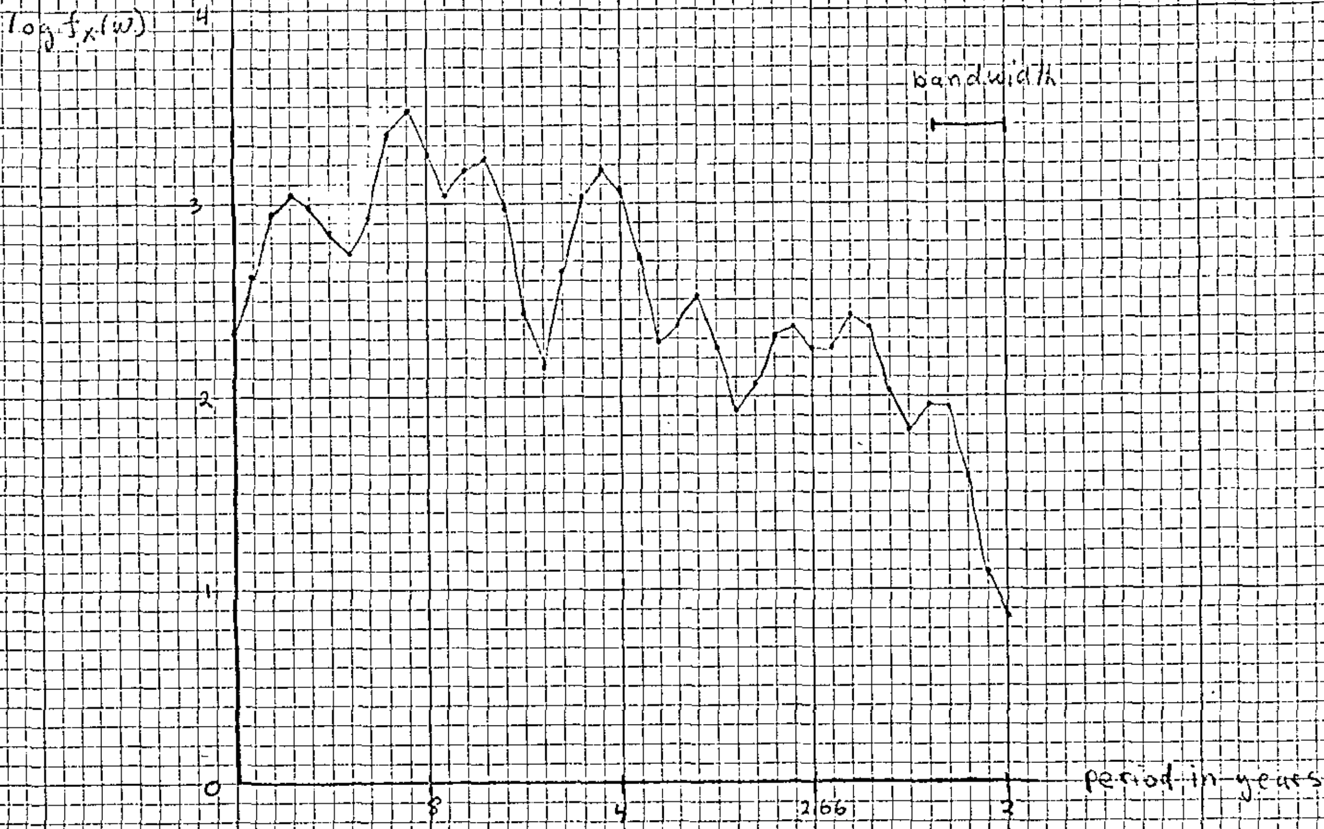
$$r_t = \sum_{j=-20}^{20} k_j x_{t-j}$$

$\lvert j \rvert$	Coefficients on Future Values of Inflation	Coefficients on Past Values of Inflation
0		-.0437
1	-.0485	-.0086
2	-.0559	.0320
3	-.0394	.0222
4	-.0308	.0237
5	-.0172	.0097
6	-.0111	.0219
7	.0059	.0154
8	-.0177	.0137
9	-.0036	.0082
10	-.0183	-.0086
11	-.0019	.0041
12	-.0070	.0200
13	.0113	-.0063
14	-.0048	.0141
15	-.0029	.0271
16	-.0049	.0058
17	-.0130	.0062
18	-.0077	.0105
19	-.0028	.0232
20	-.0089	.0094

Estimated standard error of estimated coefficients equals .0127

Figure 3 and tables 7, 8, and 9 report the results of a cross-spectral analysis of inflation in average wholesale prices in Great Britain and the yield on British consols over the period 1800-1938. The data are annual, the inflation rate being calculated from the Gayer-Schwartz-Rostow index from 1800 to 1846, and from the Saurbeck-Statist index from 1847-1938.22/ In this case, forty was the maximal lag for the covariances used in estimating the spectral densities and cross spectrum. Examinations of the behavior of the yield on consols were an important part of both Fisher's and Keynes's work on the Gibson paradox.

Figures 3a and 3b report the logs of the spectral densities of the inflation rate and the yield on consols, respectively. While the spectrum of the inflation rate is fairly flat, that of the interest rate has the "typical" spectral shape, with power generally decreasing with increases in angular frequency. This means that if inflation is to explain a sizable proportion of the variation in interest rates, the impulse-response function connecting the two variables must be a long distributed lag function.



Inflation, Great Britain (1800-1938)

figure 3.9

Yield on Consols, Great Britain (1800-193

Table 3.7

Cross Spectrum Between Inflation and Yield on Consols (1800-1938)

Period (years)	Coherence	Phase	Gain
∞			
80	.338	1.46 (.39)	.164
40.	.573	1.54	.147
26.67	.544	-1.52	.092
20.	.365	-1.35	.045
16.	.176	-1.19	.022
13.3	.206	-1.38	.024
11.4	.423	-1.24	.031
10.	.588	- .96	.028
8.89	.663	- .79	.023
8.0	.597	- .71	.018
7.27	.530	- .77	.014
6.67	.595	- .96	.013
5.71	.175	- .77	.008
5.33	.108	- .60	.010
5.0	.149	-1.10	.013
4.71	.181	-1.57	.010
4.44	.206	1.11	.008
4.21	.253	.54	.009
4.0	.370	.07	.013
3.81	.459	- .08	.017
3.64	.357	.15	.016
3.47	.311	.75	.013
3.33	.278	.92	.012
3.20	.216	1.02	.012
3.08	.257	1.29	.015
2.96	.363	1.32	.013
2.86	.497	1.01	.012
2.76	.576	.82	.013
2.67	.456	.91	.011
2.58	.271	1.33	.007
2.50	.124	-1.39	.004
2.42	.010	- .58	.001
2.35	.046	- .94	.003
2.29	.296	- .28	.010
2.22	.450	- .16	.012
2.16	.294	- .32	.008
2.11	.071	- .96	.004
2.05	.055	.69	.005
2.0	.082	- .00	.007

$$\frac{\text{cospec }(0)}{f_x(0)} = \frac{.3043}{10.3292} = .02946$$

*negative phase indicates that interest rate "leads" inflation rate at frequency in question.

Table 3.8

Regression of yield on consols on lagged and future rates of inflation (1800-1938)

$$r_t = \sum_{j=-20}^{20} h_j x_{t-j}$$

$\|j\|$	Coefficients on future rates of inflation	Coefficients on lagged rates of inflation
0		.0012
1	-.0098	.0042
2	-.0066	.0047
3	-.0043	.0102
4	-.0054	.0122
5	-.0051	.0072
6	-.0083	.0086
7	-.0063	.0094
8	-.0058	.0078
9	-.0061	.0075
10	-.0086	.0067
11	-.0057	.0094
12	-.0060	.0069
13	-.0048	.0071
14	-.0052	.0068
15	-.0047	.0054
16	-.0039	.0048
17	-.0032	.0027
18	-.0025	.0028
19	-.0020	.0022
20	-.0019	.0016
21	-.0022	.0013
22	-.0014	.0019
23	-.0011	.0013
24	-.0010	.0010
25	-.0013	.0016
26	-.0012	.0014
27	-.0010	.0010
28	-.0008	.0011
29	-.0007	.0008
30	-.0005	.0007

Estimated standard error of coefficients = .0053

Sum of estimated coefficients = .02946

Table 3.9

Regression of inflation rate on past and future

yields on consols (1800-1938)

$$x_t = \sum_{j=-30}^{30} k_j r_{t-j}$$

$\|j\|$	Coefficients on future consol yields	Coefficients on lagged consol yields
0		4.20
1	3.21	-17.50
2	- .99	- 1.92
3	7.13	1.91
4	6.23	2.94
5	-5.90	6.16
6	- .52	- 5.93
7	-2.90	.14
8	.52	- .89
9	2.91	.14
10	-3.72	- 3.97
11	4.72	3.57
12	-3.80	- .21
13	.25	2.07
14	1.62	- 2.29
15	- .57	- 2.15
16	.82	.96
17	-2.57	1.75
18	- .00	- 1.21
19	- .64	- .37
20	.03	.38
21	.75	- 1.58
22	- .13	.70
23	- .35	- .51
24	- .27	.44
25	- .32	- .23
26	.36	.02
27	- .33	- .67
28	.39	- .03
29	- .59	- .34
30	- .15	.29

Estimated standard error of coefficients $\doteq$ 3.45

Inspection of Table 3.7 establishes that interest and inflation are badly out of phase, and that the gain of the consol yield over inflation does generally decrease with increases in frequency, thus confirming the presence of the Gibson paradox, i.e. a very long lag between inflation and interest. Using (8) to estimate the mean lag of the impulse-response function, we have

$$\Phi'(0) \approx \frac{\Phi(.075) - \Phi(0)}{.075}$$

$$= \frac{1.46}{.075} = 18.6 \text{ years,}$$

which is an estimate of the mean lag of the yield on consols behind the rate of inflation.

Tables 3.8 and 3.9 report the results of estimating the regression of r on past and future x, and the regression of x on past and future r. The general pattern of the results resembles that for the U.S. data. Regressing r on x produces positive coefficients on past values of x, but negative ones on future values of x. Regressing x on r produces predominantly negative coefficients on past x, positive ones on future x. Unlike the results for the U.S. data, for the regression of r on x, the coefficients that are largest in absolute value are associated with lagged values of x. However, a substantial number of future values of x pick up sizable negative coefficients. The sum of the estimated coefficient in the regression of r on x is only .029, a result of the substantial negative coefficients on future x's. Like the results for the U.S. data, these estimates suggest that an explanation of the interest-inflation relationship that does not permit feedback from interest to inflation is probably unduly restrictive.

Another way to view our estimates of equation (5) is provided by asking what they imply about the relationship between the interest rate and the price level, that is, by asking what they imply about the lag distribution in the relationship

$$(3.14) \qquad r_t = \sum_{j=-\infty}^{\infty} f_j \log p_{t-j} + v_t'$$

$$= f(L) \log p_{t-j} + v_t'$$

where v_t' is a statistical residual and $f(L) = \sum_{j=-\infty}^{\infty} f_j L^j$.

Since x_t approximately equals $(I-L) \log p_t$, equation (5) can be written approximately as

$$(3.15) \qquad r_t = h(L)(I-L) \log p_t + v_t$$

$$= (I-L)h(L) \log p_t + v_t$$

Comparison of (14) and (15) establishes that the coefficients of $f(L)$ and $h(L)$ ought approximately to be related by

$$(3.16) \qquad f_j = h_j - h_{j-1}.$$

Tables 3.10, 3.11, and 3.12 report estimates of equation (3.14) for the commerical paper rate, Macaulay's unadjusted yield index, and the British consol yield, respectively. The estimates of the r - log p relationships are largely compatible with our estimates of the r-x relationships. For example, for the regression for the British consol yield, the large positive coefficient on current log p and the several sizable coefficients on lagged values of log p are compatible with the marked rise in the value of the coefficients in the r - x distributed lag coefficients as the lag of x behind r goes from -1 to +4. The negative values on the first two future values of log p imply that coefficients in the r - x relationship will decrease as the lag index goes from -3 to -1. The lag distribution in the r - log p relationship is characterized by a much shorter mean lag than is that for the r - x relationship. For example, using the estimated phase statistic at the 80-year periodicity (which equals -.099 radians with an approximate standard error of .167 radians) to estimate the mean lag for the r - log p relationship for the British consol yield, we obtain an estimated mean lag of the consol yield behind the logarithm of the price level of 1.3 years.

Notice that for both Macaulay's unadjusted yield and the commerical paper rate, the r - log p relationship has sizable, predominantly negative coefficients on future values of log p. This implies feedback from r to log p. For the British consol yield, however, the coefficients on future values of log p are relatively smaller, and are roughly compatible with the absence of feedback from r to log p (of course, we cannot rule out contemporaneous feedback).

The relationship between the r - x and the r - log p regressions shows again how our regressions confirm the presence of the "Gibson paradox" in the data we have studied. A very "short" distributed lag relationship between r and log p can easily appear to be a very long, and perhaps two-sided, distributed lag between r and x.

Table 3.10

Regression of Commerical Paper Rate on Past and Future Values of One Hundred Times the Logarithm of Wholesale Price Index

$$r_t = \sum_{j=-20}^{20} h_j \log p_{t-j}$$

|j|	Coefficients on Future Values of Log p	Coefficients on Past Values of Log p
0		.04357
1	-.03817	.01961
2	.00202	-.01865
3	-.00341	.01439
4	.00770	-.00549
5	-.00592	.00403
6	.00792	-.00670
7	.00793	-.00168
8	-.00601	.00472
9	-.00189	.00332
10	.00547	-.00084
11	.00542	.00580
12	-.00190	-.00033
13	.00114	.00388
14	.00089	-.00038
15	.00294	.00310
16	.00022	-.00041
17	.00304	.00248
18	.00035	-.00007
19	.00296	.00314
20	.00002	.00002

Estimated standard error of coefficients = .01237

Table 3.11

Regression of Macaulay's Yield on Past and Future Values of One Hundred Times the Logarithm of the Wholesale Price Index

$$r_t = \sum_{j=-20}^{20} h_j \log p_{t-j}$$

|j|	Coefficients on Future Values of Log p	Coefficients on Past Values of Log p
0		.00626
1	-.00374	.00609
2	.00390	-.00388
3	-.00712	.00578
4	-.00071	-.00417
5	..00153	-.00132
6	-.01017	.00022
7	.00585	-.00313
8	-.00113	.00008
9	-.00153	-.00217
10	-.00190	.00007
11	-.00124	-.00069
12	.00068	-.00026
13	-.00140	-.00096
14	-.00086	-.00037
15	-.00141	-.00150
16	-.00009	-.00019
17	-.00132	-.00166
18	-.00022	-.00020
19	-.00148	-.00168
20	-.00016	-.00016

Estimated standard error of coefficients = .00508

Table 3.12

Regression of Consol Yield on Past and Future Values of One Hundred Times the Logarithm of Wholesale Price Index

$$r_t = \sum_{j=-20}^{20} h_j \log p_{t-j}$$

$\lvert j \rvert$	Coefficients on Future Values of Log p	Coefficients on Past Values of Log p
0		.01145
1	-.00246	.00421
2	-.00148	.00137
3	.00172	.00596
4	.00060	.00247
5	.00285	-.00371
6	-.00165	.00201
7	.00023	.00170
8	.00087	-.00100
9	.00254	.00004
10	-.00235	-.00014
11	.00050	.00303
12	-.00075	-.00146
13	.00078	.00049
14	-.00024	.00010
15	-.00072	-.00061
16	-.00034	-.00028
17	-.00047	-.00147
18	-.00007	.00052
19	.00014	-.00025
20	.00041	-.00027

Estimated standard error of coefficients = .00195

It may be of interest to note that Irving Fisher himself was aware of the reverse influence that the interest rate seems to have exerted on the rate of inflation. In the Theory of Interest, Fisher wrote:

"As implied by what has just been said regarding banking policy, the relationships of P' and i [x and r, respectively, in our notation] are mutual. A change in i undoubtedly has an effect upon P' as well as the reverse. Our analyses have demonstrated that, in a decisive majority of instances, price changes precede changes in i. This does not mean that changes in the interest rate can never be used to forecast changes in prices and in business activity. In fact, an arbitrary increase in i at any time does tend to pull down the level of general commodity prices, while a decrease in i tends to increase P. This is a fact which has been quite well established and is made use of by central banks in formulating their banking and credit policies.

"The influence of changes in interest rates upon prices and business activity is made use of also by forecasting agencies in making their prognostications of business and price movements for the near future. The fact that i follows P', in most instances over secular and cyclical periods, is not inconsistent with the other fact that every increase or decrease in i exerts an influence upon P in the opposite direction. Within limits, a fall in the rate of interest may and often does produce a rise in prices and of business activity almost immediately. This effect may be continued for many months until increased prices again become dominant and pull the interest rate up again.

"In so far as the rate of interest is cause and the price movements are effect, the correspondence is <u>just the opposite</u> of that which occurs in so far as the price movements are cause and the interest movements effect." (pp. 443-44).

Fisher was thus clearly aware of the mutual influence that appears to characterize the relationship between interest and inflation. In addition, there are indications that he made some preliminary attempts to implement this insight in his formal statistical work. For example, in Charts 45 and 50 in the Theory of Interest, Fisher reported correlation coefficients between interest and lagged inflation for both positive and negative lags. For a fairly serially uncorrelated inflation rate, these correlograms ought to have patterns generally similar to those displayed by two-sided distributed lags. Fisher's charts do broadly resemble the distributed lags we have reported above, the correlations between interest and lagged inflation generally being positive, those between interest and subsequent inflation being negative. Moreover, Fisher seems to have experimented with regressions of inflation on lagged rates of interest, although he viewed these experiments as unsuccessful.[23/]

We conclude that equations (1) and (2) seem not adequately to account for the relationship between interest and inflation. Instead, while working within the class of bivariate models described by equation (3), it is necessary to take into account feedback from current rates of interest to subsequent rates of inflation. If one wished to continue working only with models of that class, the proper approach would be to attempt to estimate the parameters in all four "corners" of the matrix on the left of (3). That task might well be intractable, since the parameters of (3) will be estimable only under circumstances which will not generally obtain. Moreover, even if that system were to have been estimated, it would not be easy to interpret the results.

That is, few would be willing to interpret the second equation of (3) as a "structural" relationship explaining the evolution of the rate of inflation.

In fact, even the first equation of (3) would lose the relatively straightforward interpretation that it had when C(L) was specified to be zero. This is because it will no longer be "rational" to form expectations of inflation by looking at current and lagged rates of inflation alone, since current and past interest rates are of some help in predicting subsequent rates of inflation. If one wants to build rationality into the model, it will consequently be a more delicate job to interpret the first equation of (3).

Probably a more fruitful approach than trying to estimate and interpret (3) would follow from noting that what appears to be feedback between r and x within the context of model (3) may emerge because some omitted series are influencing both r and x. For example, in the context of model (3), feedback between x and r could emerge because the variable "aggregate demand" or "monetary growth" affects both r and x. Such influences are implied by most macroeconomic models. In the next section of this paper, we attempt to estimate the dividends that are likely to flow from this somewhat expanded approach to explaining the historical relationship between interest and inflation.

IV. Inflation and Interest in a Simple Macroeconomic Model

The results of the preceding section cast considerable doubt on the adequacy of Fisher's explanation of the Gibson paradox. That explanation of the relationship between inflation and interest hypothesizes that there is a single direction of influence, one flowing from inflation to interest. Our empirical results imply that such an hypothesis is overly simple because it neglects the influence that the interest rate seems to exert upon subsequent rates of inflation. In this section, we describe an approach that accommodates that apparent feedback from interest to inflation and also provides an alternative explanation for the Gibson paradox.

Our approach is to analyze the relationship between interest and inflation that emerges in a simple stochastic aggregative model. In its short-run behavior, the model is basically "Keynesian" in nature. However, price level adjustments and anticipations of inflation play important roles, especially in governing the evolution of the system over time, so that the model incorporates some of the key elements of monetarist doctrine. In the absence of stochastic terms the model will in the long run approach the neo-classical steady-state of Solow [40] and Swan [41]. In this model, the interest rate and the rate of inflation are jointly determined, together with output and its composition. Our purpose is to determine whether, given plausible parameter values, the model is capable of generating the kind of relationship between interest and prices that obtains in the actual historical data.[24]

Our analysis of the interest-inflation relationship will be based on a discrete-time, stochastic version of the model. However, to ease the exposition, we begin by describing a nonstochastic, continuous-time version of the model. The model is basically a simplified version of the "Keynesian" model described by Sargent and Wallace [31]. The model contains one good whose output Y is determined by the production function

$$Y = Y(K,N)$$

where K is the capital stock and N is employment. The production function is assumed to be linearly homogeneous in K and N, and to be characterized by positive though diminishing marginal products and positive cross-partial derivatives. We will generally take advantage of the linear homogeneity of the production function and write it in the intensive form

$$\frac{Y}{K} = Y(1,\frac{N}{K}) = y(\frac{N}{K})$$

or

$$(4.1) \qquad y = y(\lambda) \qquad , y'(\lambda) > 0,\ y''(\lambda) < 0$$

where $y = \frac{Y}{K}$ and $\lambda = \frac{N}{K}$.

Labor is assumed to be a perfectly variable factor of production, with firms choosing the employment level so that at each moment the marginal product of labor equals the real wage,

$$(4.2) \qquad \frac{w}{p} = y'(\lambda)$$

where w is the money wage rate and p is the price of the one good in the model.

While firms can adjust their employment levels at a point in time, it is assumed that capital is a fixed factor of production at any moment, there being no market in which firms can purchase or sell existing stocks of the one good in the model. The absence of this market is presumably due to the high transaction costs that might be assumed to be associated with trading existing stocks of capital. Alternatively, it might be attributed to the possibility that capital put in place is not a homogeneous good, being specialized to each firm. For such reasons, firms are not able instantaneously to obtain more capital when there is a gap between the marginal product of capital, $y(\lambda) - \lambda y'(\lambda)$, and the cost of capital, which equals the nominal interest rate on bonds, r, plus the constant rate of depreciation, δ, minus the public's anticipated rate of inflation π. Instead, firms respond to a gap between the marginal product and cost of capital by investing as if attempting to change the capital-labor ratio at a rate that depends directly on that gap:

$$\frac{(\dot{\frac{1}{\lambda}})}{\frac{1}{\lambda}} = I(y(\lambda) - \lambda y'(\lambda) - (r + \delta - \pi)) \quad , I' > 0 \quad , I(0) = 0$$

or

$$\text{(4.3)} \qquad \frac{I}{K} = \frac{\dot{N}}{N} + I(y(\lambda) - \lambda y'(\lambda) - (r + \delta - \pi)).$$

where dots above variables denote time derivatives and where $I(=\dot{K})$ is net investment. To simplify (3), we set $\dot{N}/N$ equal to its long-run equilibrium value n which equals $\dot{N}_s/N_s$, the proportionate rate of growth the labor supply. We assume that n is a constant. Thus, letting i equal I/K, (3) becomes

$$\text{(4.3')} \qquad i = n + I(y(\lambda) - \lambda y'(\lambda) - (r + \delta - \pi)),$$

which is the kind of Keynesian investment demand function that has been posited by Stein [35] and Tobin [36]. We assume that firms finance investment by issuing only equities.

Consumption per unit of capital, c, is assumed to depend linearly on output per unit capital minus taxes per unit capital, t, minus the depreciation rate, δ:

$$\text{(4.4)} \qquad c = z(y - t - \delta) \qquad 0 < z < 1$$

where z is a constant. Taxes per unit of capital, t, are assumed to be collected by the government in a fashion that makes them independent of income and relative prices at a point in time.

Equilibrium in the goods market requires that at each moment of time the supply of output per unit capital equals the demand,

$$\text{(4.5)} \qquad y = c + i + \delta + g$$

where g is government expenditures per unit of capital.

There are three paper assets that individuals own: money, variable-interest-rate bonds, and equities. Money has a nominal yield of zero while bonds, issued by the government and individuals, have a nominal yield of r and, like savings deposits, have a fixed nominal value. Equities are issued by firms and constitute a claim on firms' net cash flow, all of which the firms pay out as dividends. We assume that individuals regard equities and bonds as perfect substitutes when their real yields are equal.

Portfolio balance is described by the equation

$$\frac{M}{pK} = m(r, y) \qquad m_r \leq 0,\ m_y > 0 \tag{4.6}$$

where M is the supply of money. Equation (6) posits that the demand for real balances is linearly homogeneous in the capital stock, which in the long run equals the real value of the public's equities. Once equation (6) is satisfied, it follows that individuals are happy with the division of their portfolios between money, on the one hand, and bonds and equities, on the other hand.

The value of equities, S, is the discounted present value of firms' dividends. Since wealth-holders regard bonds and equities as perfect substitutes the appropriate discount rate is the bond rate r:

$$S = \int_0^\infty e^{-rt}[p(t)Y(K(t),N(t))-w(t)N(t)-p(t)\delta K(t)]dt$$

We assume that p(t) and w(t) are expected to follow the paths

$$p(t) = pe^{\pi t},$$

$$w(t) = we^{\pi t}$$

where π is the anticipated rate of inflation. Then we have

$$S = \int_0^\infty e^{-(r-\pi)t}dt\cdot[pY-wN-\delta pK]$$

$$= \frac{pY-wN-\delta pK}{r-\pi}$$

$$= \frac{pY-wN-p(y(\lambda)-\lambda y'(\lambda))K}{r-\pi} + \frac{p(y(\lambda)-\lambda y'(\lambda)-(r+\delta-\pi))}{r-\pi} + pK,$$

which by the linear homogeneity of Y(K,N) yields

$$S = \frac{p(y(\lambda)-\lambda y'(\lambda)-(r+\delta-\pi))}{r-\pi} + pK \tag{4.7}$$

When the marginal product of capital exceeds (falls short of) the cost of capital, the value of equity exceeds (falls short of) the replacement value of the capital stock, pK.[25/] Some writers, most notably Tobin [36], view the Keynesian investment demand schedule as postulating a direct dependence of investment demand on the difference between the value of equity (i.e., individuals' claims on existing capital) and the capital stock evaluated at the price of newly produced capital. The investment function (3') can obviously be given such an interpretation.

The monetary authorities can conduct open market operations at a point in time subject to the constraint

$$dM = -dB$$

where B is the stock of bonds. Government expenditures are made subject to the flow budget constraint

$$g = t + \frac{\dot{M}}{p\cdot K} + \frac{\dot{B}}{pK}$$

where $\dot{M}$ and $\dot{B}$ are the rates at which the government is adding to the stocks of money and bonds, respectively.

The rate of wage inflation is determined by the Phillips curve

$$\frac{\dot{w}}{w} = h\left(\frac{N-N_s}{N_s}\right) + \gamma\pi \qquad h' > 0,\ \gamma > 0 \tag{4.8}$$

where N_s is the full-employment labor supply, which we have assumed follows the path

$$N_s(t) = N_s(0)e^{nt}.$$

The full-employment labor supply is a construct that is assumed to make allowance for normal hours worked, normal turnover rates, etc. As a consequence, employment in man-years can exceed the full-employment labor supply if aggregate demand is high enough and if there is sufficient rigidity in the money wage rate. We will assume that γ equals unity, which implies that there is no long-run tradeoff between wage inflation and employment. However, the implications of our model for all variables except the employment level would be unchanged if we dropped the assumption that γ is unity. We will usually assume that the anticipated rate of inflation, π, that appears in (8) and (3') is determined via some version of the adaptive expectations scheme, e.g.,

$$(4.9) \qquad \dot{\pi} = \beta\left(\frac{\dot{p}}{p} - \pi\right) \qquad \beta > 0$$

where $\dot{p}$ is interpreted as a left-hand derivative. Given (9), if γ equals unity in equation (8), real wages can be constant over time only if employment N equals the full-employment labor supply N_s.

The model can now be summarized by the following equations:

$$(4.1) \quad y = y(\lambda)$$

$$(4.2) \quad \frac{w}{p} = y'(\lambda)$$

$$(4.3') \quad i = n + I(y(\lambda) - \lambda y'(\lambda) - (r+\delta-\pi))$$

$$(4.4) \quad c = z(y-t-\delta)$$

$$(4.5) \quad y = c + i + g + \delta$$

$$(4.6) \quad \frac{M}{pK} = m(r,y)$$

$$(4.8) \quad \frac{\dot{w}}{w} = h\left(\frac{N-N_s}{N_s}\right) + \gamma\pi$$

$$(4.9) \quad \dot{\pi} = \beta\left(\frac{\dot{p}}{p} - \pi\right)$$

(4.10) $\dot{K} = I$

(4.11) $\frac{\dot{N}_s}{N_s} = n$

These equations are assumed to hold at each moment.

At a point in time, K, π, and w are fixed. At any moment the positions of y, λ, w/p, i, c, and r are determined by equations (1), (2), (3'), (4), (5), and (6), while equations (8), (9), (10), and (11) govern the evolution of the system over time. The fact that $\frac{\dot{w}}{w}$ is finite permits disturbances to affect the level of output and employment at a point in time.

The momentary equilibrium of the system can be determined by solving equations (1) - (6) for versions of "IS" and "LM" curves. The IS curve gives the combinations of r and y that make equal the demand for and supply of output. It is derived by substituting (3') and (4) into (5):

$$y = z(y-t-\delta)+n+I(y(\lambda)-\lambda y'(\lambda)-(r+\delta-\pi))+g+\delta$$

Since $y'(\lambda) > 0$, we can invert (1) and obtain

$$\lambda = \lambda(y) \qquad \lambda'(y) = \frac{1}{y'(\lambda)} > 0, \; \lambda''(y) = \frac{-y''(\lambda)\lambda'(y)}{y'(\lambda)^2} > 0.$$

Substituting this into the above expression yields the IS curve:

(4.12) $$y = z(y-t-\delta)+n+I(y - \frac{\lambda(y)}{\lambda'(y)} - (r+\delta-\pi))+g+\delta.$$

The slope of the IS curve in the y - r plane is given by

$$\left.\frac{dy}{dr}\right)_{IS} = \frac{-I'}{1-z-I'\frac{\lambda(y)\lambda''(y)}{\lambda'(y)^2}} \quad ,$$

which is of ambiguous sign since $\lambda''(y) > 0$. The denominator of the above expression is the reciprocal of Hicks's "supermultiplier," the term $I'\frac{\lambda\lambda''}{\lambda'^2}$ being the marginal propensity to invest out of income. We will assume that this term is less than the marginal propensity to save, so that the IS curve is downward sloping. The position of the IS curve depends on the parameters t, g, and π in the usual way. An increase in π shifts the IS curve upward by the amount of that increase.

We can write the marginal productivity condition for labor as

$$p = w\lambda'(y).$$

Substituting this expression for p into (6) yields the LM curve:

$$M = w\lambda'(y)\cdot Km(r,y), \tag{4.13}$$

the slope of which is easily verified to be positive in the r - y plane. The LM curve shows the combinations of r and y that guarantee portfolio balance. Its position depends on M, w, and K, all of which are parameters at a point in time.

The momentary equilibrium of the system is determined at the intersection of the IS and LM curves. That equilibrium will in general be a nonstationary one, the interest rate, the real wage, and the capital-labor ratio possibly changing over time. However, given fixed values of g, t, and $\dot{M}/M$, the system may over time approach a "steady state" in which the interest rate, real wage, and employment-capital ratio are fixed, while prices and wages change at a rate equal to $\dot{M}/M$ minus n. We will use two curves to characterize the steady-state growth path in the r - y plane. The first is simply a vertical line at the steady state output-capital ratio, which we denote by y^*. From (5), the rate of growth of capital is

$$i = y-z(y-t-\delta)-g-\delta$$

Subtracting n from i yields the proportionate rate of growth of the capital-labor ratio. Setting it equal to zero and solving for y yields the value of y^*:

$$y^* = \frac{n+g+\delta(1-z)-zt}{(1-z)}. \tag{4.14}$$

We show y^* graphically as a vertical line in figure 1. On our assumptions, the steady-state value of y is independent of the interest rate.

If investors are to be content to increase the capital stock precisely at the rate n, so that i - n equals zero, the real cost of capital must equal the marginal product of capital:

$$(4.15) \qquad \begin{aligned} (r+\delta-\pi) &= y(\lambda)-\lambda y'(\lambda) \\ &= \frac{\lambda(y)}{\lambda'(y)} \quad , \end{aligned}$$

which is an equation that tells us what $r + \delta - \pi$ must be if the system is to be in a steady-state equilibrium at a given y. The slope of (15) is

$$\frac{dr}{dy} = \frac{\lambda(y)\lambda''(y)}{\lambda'(y)^2} > 0,$$

which is positive, reflecting the direct dependence of the marginal product of capital on the output-capital ratio. We call (15) the capital-market equilibrium curve, and label it KE. An increase in π causes the KE curve to shift upward by the full amount of the increase.

The determination of equilibrium is depicted in figure 1.

[figure 1 goes here]

Notice that the IS curve has been drawn so that it intersects the KE curve at y^*, the steady-state output-capital ratio. That our IS curve has this property can be verified by setting $r + \delta - \pi$ equal to $y(\lambda)-\lambda y'(\lambda)$ in (12) and solving for y, which turns out to equal y^*.

To illustrate how the model works, suppose that the system is initially in a full steady-state equilibrium, the IS, LM and KE curves all intersecting

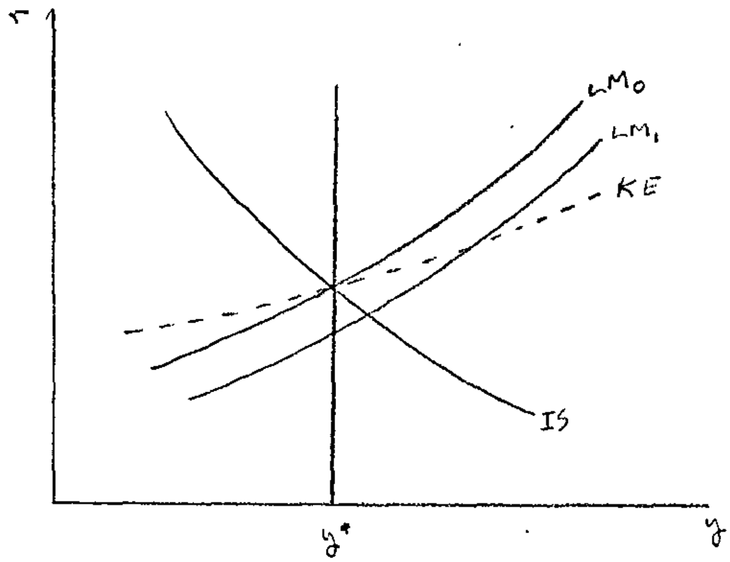


Figure 4.1

at y^*, as in figure 1. Suppose that $\frac{\dot{M}}{M} = n$, so that the equilibrium rate of inflation is zero. Now suppose that at some point in time there occurs a once-and-for-all jump in M, engineered via an open-market operation, that leaves $\dot{M}/M$ unaltered. To simplify matters, we will suppose that π remains fixed at zero, its steady-state value, during the movement to a new steady-state equilibrium. We also assume that g and t are constant over time. The effect of the jump in the money supply is to shift the LM curve to the right, say to LM_1 in figure 1. The result is an instantaneous jump in employment and in the output-capital ratio. Employment now exceeds the labor supply, causing wages to adjust upward over time. In addition, the nominal interest rate has fallen, creating a positive discrepancy between the marginal product of capital, which has risen, and the real cost of capital. Firms respond by adding to the capital stock at a rate exceeding n. Since capital is growing faster than the money supply, and since money wages are rising over time, the LM curve shifts upward over time, from LM_1 toward LM_0 in figure 1. The upward movement of the LM curve depresses y and causes employment N to approach the labor supply N_s from above as time passes. The real wage rises as LM shifts from LM_1 back toward LM_0, so we know that $\dot{p}/p$ falls short of $\dot{w}/w$ during the movement back to the steady-state equilibrium. The LM curve will stop shifting when it finally reaches LM_0, since then $I/K = n = \dot{M}/M$, and $\dot{w}/w = 0$. The final result of the once-and-for-all increase in the money supply is thus to drive the level of money wages and prices upward proportionately, and to leave the interest rate, the real wage, and the output-capital ratio unchanged. Employment will equal the labor supply when the steady-state is achieved.

The adjustment process would be considerably more complex if we were to permit π to respond to the occurrence of actual inflation, as in equation (9). For then the IS curve and the KE curve would shift upward as π increased in response to the emergence of inflation. It is easily verified that when π changes, the new IS curve continues to intersect the new KE curve at y^*. The result of allowing π to depend on past values of $\dot{p}/p$ is likely to be the emergence of "overshooting." Following the original jump in M, the LM curve will be shifting toward an intersection at y^* with an IS curve associated with a positive anticipated rate of inflation. But at that intersection at y^*, $\dot{w}/w$ will equal π, which is likely to be positive since it depends heavily on recently past values of $\dot{p}/p$, which were positive. Since the relative rate of change of the money wage exceeds the difference between $\dot{M}/M$ and $\dot{K}/K$, which equals zero at y^*, the LM curve must be shifting upward over time.[26/] This means that the output-capital ratio will fall below y^*, employment will fall below the labor supply and $\dot{w}/w$ will fall below π, as will $\dot{p}/p$, causing π to fall. That will cause the IS curve to shift downward. Assuming that the system is dynamically stable, the final resting place for all variables will be the same as if π had remained at its steady-state value throughout the adjustment process; but the path to steady-state equilibrium may be much different.

In steady-state equilibrium the nominal interest rate is related to the anticipated rate of inflation by (15), which is a version of Fisher's equation. When the system is not at steady-state equilibrium, however, (15) need not hold. Moreover, it is clear that if π were to increase at a point in time, the immediate effect would in general be to drive r up by an amount less than the increase in π. That would be associated with an increase in the value of equity as compared with the replacement value of the capital stock and increases in the demand for investment goods and in output and employment.

Using the same techniques employed in section III, we have analyzed the relationship between the nominal interest rate and the inflation rate in a discrete-time, stochastic version of our model. The equations of the discrete-time model are:

(i) $y = Ax^{\alpha}$

(ii) $\frac{w}{p} = y'(\lambda)$

(iii) $i = n+g\cdot(y(\lambda)-\lambda y'(\lambda)-(r+\delta-\pi)) + \epsilon_i$

(iv) $c = z(y-t-\delta)+\epsilon_c$

(v) $y = c+i+g+\delta$

(vi) $\frac{M}{pK} = By^{m_1}r^{m_2}\epsilon_m$

(vii) $\frac{w-w_{-1}}{w_{-1}} = h\cdot(\frac{N-N_s}{N_s}) + \gamma\pi + \epsilon_w$

(viii) $\pi = \xi_0\pi_{-1} + \xi_1(\frac{p_{-1}-p_{-2}}{p_{-2}})$

(ix) $K_{+1}-K = iK$

(x) $N_s = N_{s0}(1+n)^t$

where variables with no time subscripts correspond to the current period, while those with numerical subscripts denote the corresponding variables shifted foward or backward in time the appropriate number of periods. The variables ϵ_i, ϵ_c, ϵ_w, and log ϵ_m are stochastic terms that are assumed to be mutually and serially uncorrelated, and normally distributed. Notice that our specification of the discrete-time Phillips curve makes the current money wage rate a function of the current level of employment.

Tables 4.1, 4.2, 4.3, and 4.4 and figure 4.2 report the results of simulating the model where the parameter values assumed the following values:

$A = 759.836$	$\gamma = 1.0$	$M_o = 100$ (billions)
$\alpha = .75$	$\xi_0 = 0.$	$K_o = 1500$ (billions)
$n = .03$	$\xi_1 = 0.$	$N_{so} = .05$ (billion)
$z = .8$	$\sigma_{\epsilon i} = .003$	$\delta = .02$
$B = .025$	$\sigma_{\epsilon c} = .01$	
$m_1 = 1.0$	$\sigma_{\epsilon m} = .00067$	
$m_2 = -.75$	$\sigma_{\epsilon w} = .01$	
$h = 1.5$	$t = .0217$	
$M_t = (1+n)^t M_0$	$g = .05$,	

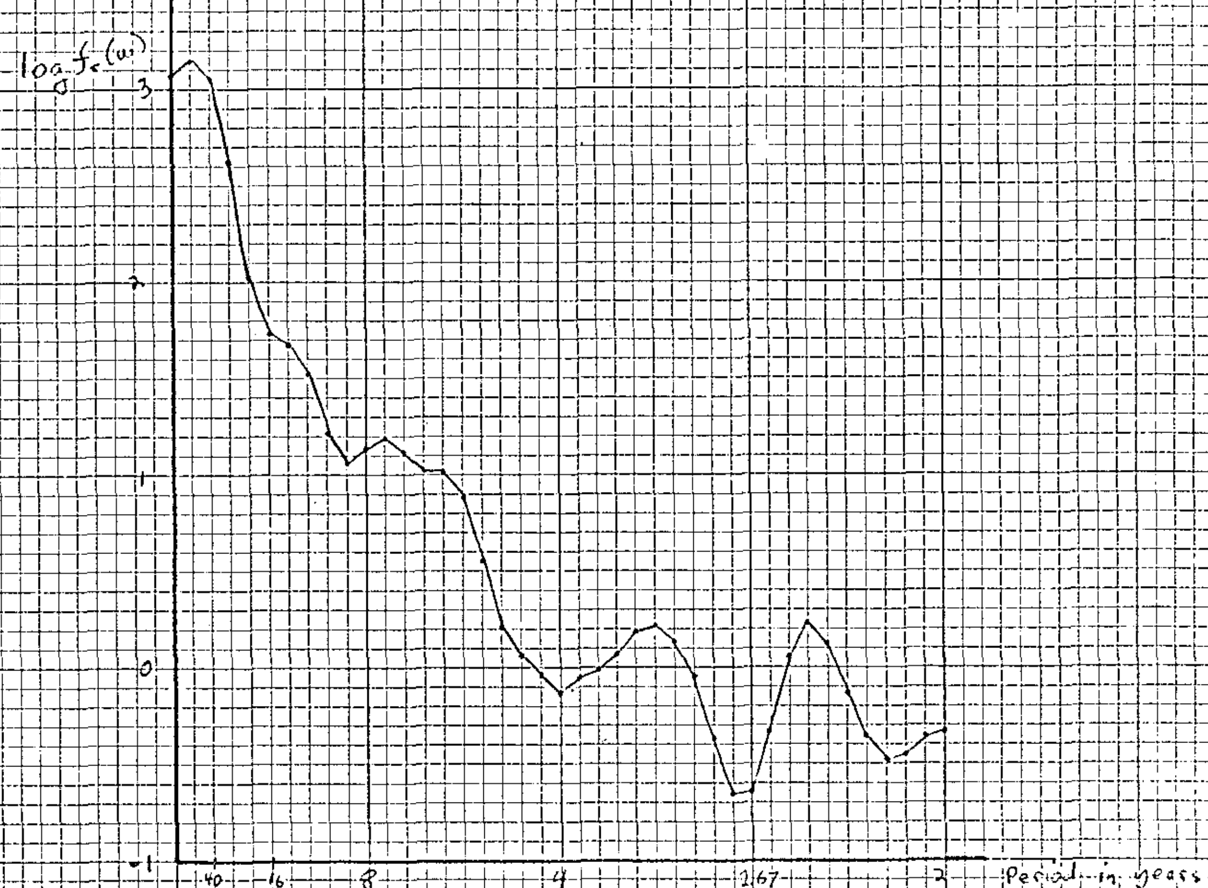
where $\sigma_{\epsilon i}$ denotes the standard error of ϵ_i, etc. The initial conditions were chosen so that the system was initially on a steady-state growth path. Notice that the money supply grew at the same rate as the labor force, so that there was no trend in the price level.[27/] In addition, note that anticipated inflation is held at zero throughout this simulation, so that it exerts no influence on the interest rate. The parameter h takes the value 1.5, which implies that, ceteris paribus, a one-percent unemployment rate would cause money wages to fall by 1.5 percentage points. We believe that this much response in the wage level would occur only over a period of at least a year. As a consequence, we think of the model as generating annual data.

The logs of the estimated spectral densities of prices, inflation, and interest are recorded in figures 4.2(a-c), while table 1 reports the estimated cross-spectral statistics. Tables 2, 3, and 4 record our estimates of two-sided distributed lag functions between r_t and x_t, and between r_t and log p_t.

Reference to these figures and tables establishes that the relationship between the interest rate and inflation rate in these artificial data in important ways resembles that found for the historical data analyzed in section III. Reference to figures 4.2a and 4.2c and the gain of r over x

log of spectral density of
inflation rate
Figure 4.2a

log of spectral density of one
hundred times log of price level
Figure 4.2b



Log of spectral density of interest rate

Figure 4.2 c

Table 4.1

Cross Spectrum Between Interest and Inflation

Period In Years	Coherence	Phase* (radians)	Gain
80	.311	-1.51 (.25)	.264
40	.514	-1.56	.261
26.67	.607	1.55	.226
20.	.657	1.47	.198
16	.621	1.33	.175
13.3	.550	1.25	.149
11.43	.478	1.17	.116
10.	.492	1.09	.095
8.89	.600	1.04	.095
8.0	.730	1.02	.103
7.27	.773	.98	.101
6.67	.687	.87	.087
6.15	.583	.77	.074
5.71	.550	.74	.074
5.33	.602	.69	.078
5.00	.675	.64	.075
4.71	.650	.62	.066
4.44	.521	.58	.059
4.21	.441	.63	.055
4.0	.466	.69	.053
3.81	.514	.52	.055
3.64	.598	.33	.062
3.48	.700	.27	.068
3.33	.726	.29	.069

Table 4.1 (Continued)

Cross Spectrum Between Interest and Inflation

Period In Years	Coherence	Phase (radians)	Gain
3.20	.696	.38	.063
3.08	.631	.49	.053
2.96	.473	.47	.042
2.86	.338	.35	.031
2.76	.373	.35	.027
2.67	.462	.32	.031
2.58	.580	.28	.045
2.50	.728	.31	.059
2.42	.770	.33	.059
2.35	.731	.29	.054
2.29	.645	.15	.049
2.22	.600	.05	.047
2.16	.639	.25	.047
2.11	.748	.37	.054
2.05	.783	.23	.061
2.0	.796	.00	.065

$$H(0) = \frac{\mathrm{Cosp}(0)}{f_x(0)} = \frac{-2.0393}{69.7926}$$

*negative phase indicates that interest rate "leads" inflation rate at frequency in question.

Table 4.2

Regression of interest rate on future and past values of inflation

$$r_t = \sum_{j=-20}^{20} h_j x_{t-j}$$

$\lvert j \rvert$	Coefficients on future rates of inflation	Coefficients on past rates of inflation
0		.044
1	-.033	.026
2	-.033	.020
3	-.031	.020
4	-.023	.018
5	-.023	.016
6	-.019	.014
7	-.014	.018
8	-.017	.010
9	-.013	.011
10	-.008	.013
11	-.009	.008
12	-.005	.005
13	-.006	.009
14	-.003	.005
15	-.003	.008
16	-.006	.005
17	-.005	.008
18	-.005	.004
19	-.003	.005
20	-.003	.003

Estimated standard error of estimated coefficients equal .006

Table 4.3

Regression of inflation on future and past values of interest rate

$$x_t = \sum_{j=-20}^{20} k_j r_{t-j}$$

$\lvert j \rvert$	Coefficients on future interest rates	Coefficients on past interest rates
0		7.508
1	.028	-5.679
2	-.159	-.745
3	.762	-.537
4	-.392	-.072
5	.412	.383
6	.069	.060
7	.469	.226
8	-.202	.143
9	.580	-.037
10	.149	.022
11	.504	.298
12	-.489	-.007
13	.637	.121
14	-.150	.395
15	.331	.157
16	-.030	.089
17	.385	.034
18	-.166	.194
19	.518	.446
20	-.040	-.202

Estimated standard error of estimated coefficients equals .420

Table 4.4

Regression of interest rate on past and future values of log of price level

$$r_t = \sum_{j=-10}^{10} h_j p_{t-j}$$

$\|j\|$	Coefficients on future values of log of price level	Coefficients on past values of log of price level
0		.080
1	-.017	.003
2	-.002	.003
3	.003	-.005
4	.001	.002
5	.001	.001
6	-.001	-.003
7	.005	.008
8	-.006	-.001
9	.003	-.001
10	.003	.003

Estimated standard error of estimated coefficients equals .003

recorded in table 1 shows that the spectrum of the inflation rate is much flatter than that of the interest rate, which is fairly "typical" in shape, with power generally decreasing with increases in angular frequency. The gain statistics and the phase statistics both indicate that the impulse-response function is characterized by a very long mean lag. At low frequencies, the interest rate and inflation rate are approximately $\pi/2$ radians out of phase, which is consistent with the interest rate being related to the (log of the) price level, rather than with the rate of inflation. Thus, the Gibson paradox characterizes the artificial data that we have generated.

The regression of r on future and past values of x, which is reported in table 4.2, appears generally similar in configuration with those estimated using historical data. Lagged values of x pick up mostly positive coefficients while negative coefficients are associated with future values of x. Using the cospectrum divided by the spectral density of inflation at zero frequency to estimate the sum of the weights produce a value of -.028. One notable difference between these estimates and those reported in section III is the sizable coefficient that appears on the current value of x. In section III that coefficient was estimated to be relatively much smaller.

The regression of r on past and future values of log p, which is shown in table 4.4, is characterized by a dominant coefficient on the current value of log p. While there is evidence of feedback from log p to r, the relationship is well approximated simply as a contemporaneous one between r and log p.

While the relationship between inflation and interest in the artificial data is broadly similar to that characterizing the historical data, it is clear that anticipations of inflation play no role in shaping the relationship for the artificial data, since they are assumed to be zero throughout the simulation. In this case, the long mean lag that characterizes the relationship between interest and inflation has nothing to do with long lags in adjusting expectations of inflation in response to the occurrence of actual inflation.

The simulation results give content to our earlier assertion that what appears to be feedback within the context of the class of bivariate models utilized in section III can be interpreted as indicating that r and x are both being influenced by some other variables. In our model, r and x are mutually determined, both being influenced by the stochastic terms ϵ_i, ϵ_c, ϵ_m and ϵ_w. The model generates realizations of r and x that, within the context of a bivariate model, are inconsistent with the notion that x is influencing r, with no reverse feedback occurring.

In summary, our model is capable of generating realizations in which the interest-inflation relationship broadly resembles that characterizing the actual historical data. It follows that the model is capable of providing an explanation of the Gibson paradox which is an alternative to Irving Fisher's. The key reason that the Gibson paradox may infest the data generated by the model is the failure of wages and prices to adjust sufficiently quickly to keep output always at its full-employment level. Thus, consider again the effects of a once-and-for-all increase in the money supply, which occurs at time $\bar{t}$ and is assumed to leave $\dot{M}/M$ unchanged. Assume that the system is in steady-state equilibrium up to $\bar{t}$, with $\dot{M}/M = n$. In addition, assume that events are described by the nonstochastic, continuous-time version of our model. In a "classical" version of our model, in which the Phillips curve is dropped and replaced by the assumption that output is fixed at its full-employment level at any moment, wages and prices being perfectly flexible, the effect of the increase in M is to leave r unaltered, and to cause w and p to jump once and for all at $\bar{t}$, as depicted in figure 4.3.

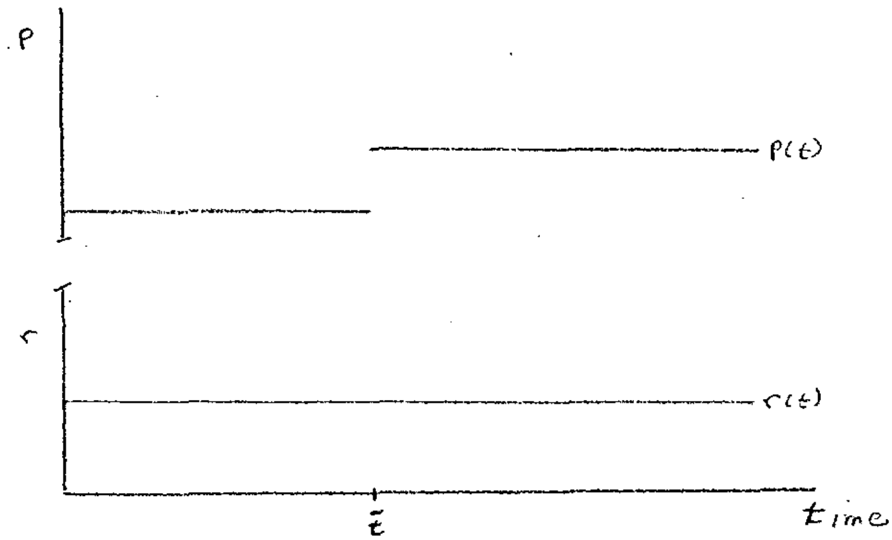


Figure 4.3

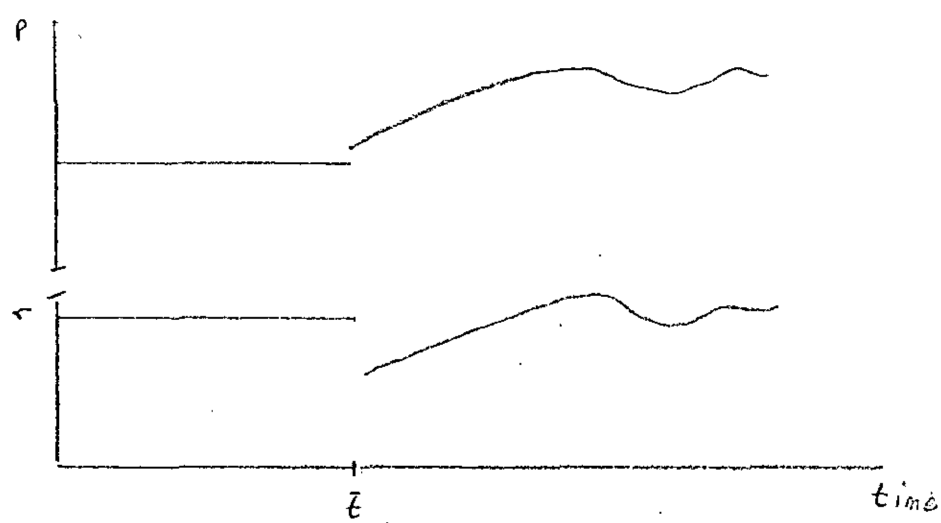


Figure 4.4

[figure 4.3 goes here]

The response of our "Keynesian" model to a jump in M is depicted in figure 4.4. The jump in M causes r to fall at $\bar{t}$, but then to rise over time back toward its steady-state value. The price level jumps at $\bar{t}$ and then continues to rise over time as r rises toward its steady-state level. During the transition back to the steady state, both r and p are too low (or else too high in the case in which overshooting emerges) to be steady-state values, and they tend to move together over time. The result is a tendency for r and p to be positively correlated, even though their steady-state values are not related.

[figure 4.4 goes here]

The results of this section demonstrate that it is possible to construct an explanation of the Gibson paradox that does not, like Fisher's, rely importantly on hypothesized movements in the anticipated rate of inflation. Thus it does not seem necessary to stress differences between nominal and real rates of return in order to explain the Gibson paradox. This finding carries the implication that the Gibson paradox ought to characterize measures of the real rate of return on equities as well as the nominal yields that we examined in sections I and III. To determine whether this implication is borne out, we have analyzed the relationship between wholesale commodity price inflation, and the real return on equities, as measured by Cowles' [42] dividend-price index or earnings-price index for the United States. A sample of the results is reported in Tables 4.5, 4.6, 4.7, and 4.8. Table 4.5 records the cross spectrum between inflation in wholesale commodity prices and Cowles' dividend-price ratio for railroads over the period 1871-1929. Thirty was the maximal lag in the covariograms and cross covariograms used in estimating the cross spectrum. The cross spectrum resembles those reported for nominal yields in section III. The gain of the dividend-price ratio over inflation falls off sharply as frequency increases, while the two series are approximately $\pi/2$ radians out of phase at low frequencies. These findings imply that the relationship between the dividend-price ratio and the rate of inflation was characterized by the Gibson paradox.

Table 4.6, 4.7, and 4.8 record various estimates of the two-sided distributed lag

$$r_t' = \sum_{j=-m}^{m} h_j x_{t-j} + w_t$$

where r_t' is the "real" rate of return on equities and w_t is a statistical

Table 4.5

Cross Spectrum Between Railroad Dividend-price Ratio and

Inflation in Wholesale Commodity Prices (1871-1929)

Period in Years	Coherence	Phase (radians)	Gain
60	.396	1.52 (.46)	.195
30	.534	1.51	.212
20	.324	-1.44	.111
15	.432	-.97	.082
12	.505	-.83	.076
10	.343	-.93	.052
8.57	.208	-1.20	.033
7.5	.143	-1.49	.021
6.67	.240	-1.30	.019
6.0	.587	-1.13	.030
5.45	.511	-1.24	.037
5.0	.163	-1.55	.029
4.62	.069	.06	.026
4.29	.311	-.05	.064
4.0	.229	.42	.045
3.75	.189	1.33	.030
3.53	.450	-1.43	.041
3.33	.575	-1.44	.045
3.16	.411	-1.39	.034
3.0	.192	.84	0.19
2.86	.185	-.58	.016
2.73	.207	-.94	.014
2.61	.302	-1.17	.014
2.50	.366	-1.31	.015
2.40	.292	-1.38	.015

Table 4.5 (Continue)

Period in Years	Coherence	Phase (radians)	Gain
2.31	.148	-1.35	.014
2.22	.030	-.82	.008
2.14	.105	.73	.018
2.07	.292	.77	.029
2.00	.195	.00	.021

Table 4.6

Regression of Total Dividend-price Index on Future and

Lagged Rates of Inflation (1871-1929)

$$r_t' = \sum_{j=-20}^{20} h_j x_{t-j}$$

$\|j\|$	Coefficients on Future Rates of Inflation	Coefficients on Lagged Rates of Inflation
0		.0069
1	-.0113	.0228
2	-.0044	.0045
3	-.0135	.0094
4	-.0178	.0220
5	-.0044	.0181
6	.0011	.0026
7	-.0099	.0105
8	-.0160	.0176
9	-.0060	.0037
10	-.0070	.0024
11	-.0080	-.0025
12	-.0095	.0028
13	-.0072	.0047
14	-.0049	.0043
15	-.0010	-.0019
16	-.0023	.0011
17	-.0031	.0026
18	-.0027	.0019
19	.0005	-.0004
20	.0003	-.0004

Estimated standard error of estimated coefficients = .0119

Estimated mean lag of real interest behind inflation = 14 years

Table 4.7

Regression of Railroad Dividend-price Ratio on Future and

Lagged Rates of Inflation (1871-1929)

$$r_t' = \sum_{j=-20}^{20} h_j \; x_{t-j}$$

$\lvert j \rvert$	Coefficient on Future Rates of Inflation	Coefficients on Lagged Rates of Inflation
0		-.0048
1	-.0191	.0158
2	-.0172	.0038
3	-.0133	.0125
4	-.0097	.0235
5	-.0072	.0192
6	-.0134	.0101
7	-.0115	.0150
8	-.0101	.0204
9	-.0027	.0089
10	-.0112	.0053
11	-.0138	.0034
12	-.0110	.0083
13	-.0057	.0053
14	-.0074	.0035
15	-.0055	-.0010
16	-.0036	.0035
17	-.0030	.0026
18	-.0019	.0011
19	-.0015	-.0018
20	-.0005	.0004

Estimated standard error of estimated coefficients = .0118

Estimated mean lag of real interest behind inflation = 15 Years

Table 4.8

Regression of Railroad Earnings-price Index on Future and

Lagged Rates of Wholesale Price Inflation (1872-1929)

$$r_t' = \sum_{j=-20}^{20} h_j \, x_{t-j}$$

$\lvert j \rvert$	Coefficients on Future Rate of Inflation	Coefficients on Lagged Rates of Inflation
0		.0070
1	.0373	.0309
2	-.0071	-.0082
3	-.0275	.0059
4	-.0314	-.0006
5	-.0407	.0227
6	-.0338	.0490
7	-.0041	.0373
8	-.0041	.0232
9	-.0259	.0384
10	-.0147	.0341
11	-.0177	.0196
12	-.0062	.0117
13	-.0058	.0048
14	-.0103	.0064
15	-.0108	.0040
16	.0002	.0034
17	-.0044	-.0026
18	-.0043	-.0018
19	-.0054	-.0013
20	.0006	.0004

Estimated standard error of estimated coefficients = .0226

Estimated mean lag of real interest behind inflation = 12 Years

residual. The estimates were obtained using Hannan's inefficient estimator, while the estimated mean lags were calculated using the method of equation (3.12). As measures of the real rate of return we took Cowles' dividend-price ratio (total index), Cowles' dividend-price ratio for railroads, and Cowles' earnings-price ratio for railroads. In each case, the estimated distributed lags resemble those reported for nominal yields in section III. The distributed lags are long and apparently two-sided, with lagged rates of inflation obtaining predominately positive coefficients while subsequent rates of inflation pick up negative coefficients.[28]

These results provide support for the proposition that the relationship between interest rates and commodity price inflation cannot in large measure be explained by appealing to hypothesized movements in the anticipated rate of inflation. The Gibson paradox appears to have characterized nominal and real interest rates alike. It follows that it is desirable to have an explanation of the Gibson paradox that focuses on the relationship between movements in real rates of return and the price level.

V. Conclusions

Our empirical results imply that to explain the Gibson paradox it is not adequate to hypothesize a one-way influence directed from inflation to the interest rate (or, for that matter, from interest to inflation). Instead, within the context of bivariate models, interest and inflation appear mutually to influence one another. One implication of this finding is that Irving Fisher's explanation of the Gibson paradox, which posits a unidirectional influence flowing from inflation to interest, is inadequate. Instead, to explain the paradox, it is necessary to view interest and inflation as being mutually determined. The model set forth in section IV illustrates a way apparent feedback emerges in a system in which interest and inflation are both endogenous variables. Moreover, for "plausible" values of its parameters, the model generates data characterized by the Gibson paradox. Long lags in forming expectations of inflation play no role in producing the paradox in these artificial data. This suggests that in general there is no reason to expect a regression of interest against current and lagged rates of inflation to reveal very much about the expectations of inflation held by the public. In addition, the data on interest and inflation indicate that the interest rate contains information, over and above that contained in lagged rates of inflation, that is useful in predicting the rate of inflation. This implies that it is probably inadequate to hypothesize that expectations of inflation are simply naive extrapolations of past rates of inflation, since that involves supposing that readily available information about the subsequent course of inflation goes unused.

FOOTNOTES

1. For example, see Milton Friedman [9], David Meiselman [25], Phillip Cagan and A. Gandolphie [4], William Gibson [11], W. Yohe and D. Karnosky [39].

2. Keynes named the paradox after A. H. Gibson, a businessman who had written several articles about the correlation between interest rates and prices. See Keynes [20, pp. 198-210].

3. See Irving Fisher [6].

4. In particular, see Yohe and Karnosky [39].

5. Our procedure here was first to search over λ's ranging from .1 to .9 at steps of .1. Having found the value of λ, say λ_0, that, among these nine values of λ, delivered the smallest residual variance, we then searched again over $[\lambda_0 - .09, \lambda_0 + .09]$ at steps of .01 for the λ associated with the minimum residual variance. This value was taken as our estimate of λ. Notice that our search never extended beyond the limits of λ's of .01 and .99.

6. See Friedman [9] and Sargent [30].

7. See Cagan [2]. For a very unsympathetic view of Fisher's work, see Macaulay [24]. Also, see Keynes [20].

8. Throughout the rest of this section and also section III, we are measuring the data in deviations from their means.

9. Nerlove [29] has extensively discussed using this model to implement Muth's argument.

10. The Fourier transform of the v's of (1) is related to the Fourier transform of the c's of (2) by $1 - \sum_{j=1}^{\infty} v_j e^{-iwj} = (\sum_{j=0}^{\infty} c_j e^{-iwj})^{-1}$, provided that both exist.

11. Whittle [37] is the fundamental reference on the subject of forecasting processes like (1).

12. For these mixed autoregressive, moving average error processes, a simpler version of equation (8) can be easily obtained. The process can be written in the alternative forms

$$x_t = \frac{B(L)}{A(L)} u_t$$

or

$$\frac{A(L)}{B(L)} x_t = u_t$$

where $A(L) = 1 - a_1L - a_2L^2 \cdots - a_mL^m$, $B(L) = 1 + b_1L + b_2L^2 + \cdots + b_nL^n$. Let $I + G(L) = A(L)/B(L)$. Solving for $G(L)$ yields

$$G(L) = \frac{A(L) - B(L)}{B(L)} .$$

Then the autoregressive form can be written

$$x_t = \left[\frac{B(L) - A(L)}{B(L)}\right] x_t + u_t = \left[\frac{(b_1 + a_1)L + (b_2 + a_2)L^2 + \cdots}{1 + b_1L + b_2L^2 + \cdots}\right] x_t + u_t$$

The polynomial on the right involves only powers of L of degree greater than or equal to one, and thus equals $LV(L)$.

13. Points inside the $(1-\alpha)$ confidence region satisfy

$$SSR(\theta) \leq SSR(\hat{\theta}) \ (1 + \frac{k}{N-k} F_{k,n-k} \ (\alpha))$$

where $SSR(\theta)$ is the sum of squared residuals associated with the vector θ of parameter values, $\hat{\theta}$ is the least-squares vector of parameter values, k is the number of parameters estimated, N is the number of observations, and $F_{k,n-k}(\alpha)$ is the value of the F distribution with k, N-k degrees of freedom. See Jenkins and Watts [18].

14. The long confidence region around $(\hat{a}_1, \hat{b}_1)$, which includes distributed lags with widely different shapes, is reminiscent of the findings of Griliches and Wallace [14] and Griliches [15], who point out that different points within the confidence regions surrounding estimates of the parameters of rational distributed lag functions are often associated with distributed lags with very different shapes. Griliches and Wallace were concerned with the case in which one variable is regressed on current and past values of another variable, while the point made in the text pertains to autoregressions.

15. The results of this section raise doubts about the promise of Nerlove's [29] recommendation that the optimal autoregressive forecasting scheme be imposed in structural equations in which distributed lags arise from the presence of expectations.

16. The Fourier transforms of the matrices appearing in (3) and (4) are related by

$$\text{(a)} \quad \begin{bmatrix} 1 - a(e^{-iw}) & -b(e^{-iw}) \\ -c(e^{-iw}) & 1 - d(e^{-iw}) \end{bmatrix} = \begin{bmatrix} \alpha(e^{-iw}) & \beta(e^{-iw}) \\ \gamma(e^{-iw}) & \delta(e^{-iw}) \end{bmatrix}^{-1}$$

where $a(e^{-iw}) = \sum_j a_j e^{-iwj}$, etc. Taking the inverse Fourier transforms of these matrices permits one to recover the matrices appearing in (3) and (4). Notice that if $c(e^{-iw}) = 0$, then $\gamma(e^{-iw}) = 0$, since the inverse of a triangular matrix is also triangular.

17. Here is Sims's proof. Suppose that r_t can be written as

$$r_t = g(L)\ x_t + w_t$$

where w_t is a disturbance process independent of x_t for all t and $g(L) = \sum_{i=0}^{\infty} g_i\ L^i$. Let u_t be the fundamental white noise process in the moving-average (Wold) representation of x_t alone, so that $x_t = k(L)\ u_t$; and let ϵ_t be the fundamental white noise process in the moving average representation of w_t, so that $w_t = h(L)\ \epsilon_t$. Then we have

$$r_t = g(L)\ k(L)\ u_t + h(L)\ \epsilon_t$$

$$x_t = k(L)\ u_t$$

where u_t and ϵ_t are mutually uncorrelated processes. The above equations are in the form of (4) with $\gamma(L) = 0$; but $\gamma(L)$ will be 0 if and only if $c(L) = 0$.

Now suppose $c(L)$ is zero. From (4) we have

$$r_t = \alpha(L)\ \epsilon_t + \beta(L)\ u_t.$$

From the second equation of (3) we have

$$(1 - d(L))\ x_t = u_t$$

which implies

$$r_t = \alpha(L)\ \epsilon_t + \beta(L)\ (1- d(L))\ x_t$$

in which r_t is expressed as a one-sided distributed lag of x_t with x_s independent of ϵ_t for all s.

18. The spectrum of the white noise u_t is

$$f_u(w) = E \left| \sum_t u_t e^{-iwt} \right|^2$$

$$= \frac{1}{2\pi}\ \sigma_u^2$$

where σ_u^2 is the variance of u_t.

19. Angular frequency w is related to the period of oscillation N by

$$w = \frac{2\pi}{N} .$$

20. The asymptotic standard errors are estimated by substituting the estimated coherence in the following formula for the asymptotic standard error:

$$\text{s.e.} \quad (\hat{\phi}(w)) = \left[\frac{1}{b\ T} \left[\frac{1}{\text{coh}(w)} - 1 \right] \right]^{1/2}$$

where b is the bandwidth of the spectral window and T is the sample size. See Jenkins and Watts [18] for a discussion of this formula. It should be noted, however, that coherence will generally be underestimated for processes that, like interest and inflation, are badly out of phase. Hence the standard errors reported in the text may be too large.

21. An estimate of the asymptotic covariance between $\hat{h}_\ell$ and $\hat{h}_s$ is

$$(2nT)^{-1} \sum_{j=-n+1}^{n} \frac{\hat{f}_r(w_j)\ [1-\text{coh}\ (w_j)]}{f_x(w)}\ e^{iw_j(\ell-s)}, \quad w_j = \frac{\pi j}{n} ,$$

where n is the maximal lag and T is the number of observations. See Fishman [7, p. 161].

22. The source of the data is B. Mitchell [26].

23. *On page 422, Fisher reports: "Experiment proved that when price changes* were lagged behind the distributed influence of changing interest rates, the correlation coefficients were too small to have any significance." Again, on page 425 he reports: "Experiments, made with United States *short-term interest rates, to test the alternative hypothesis of distributed* influence of interest rate changes instead of price changes, gave results of negligible significance."

24. Wicksell [38] and Keynes [20] are most closely associated with the view that to explain the Gibson paradox it is necessary to posit that interest and inflation are mutually determined. The quotation of Fisher that we cited in section III reveals that he saw the merits in such an approach.

25. Notice that the earnings-price ratio for equities equals $r - \pi$.

26. For fixed y, differentiate the LM curve (13) logarithmically to obtain

$$\frac{\dot{M}}{M} = \frac{\dot{w}}{w} + \frac{\dot{K}}{K} + \frac{m_r \dot{r}}{\frac{M}{pK}}$$

or

$$\dot{r}\Big|_{LM} = \frac{pK}{m_r M} \left[\frac{\dot{M}}{M} - \frac{\dot{K}}{K} - \frac{\dot{w}}{w} \right] .$$

If the expression in brackets is negative, then at each value of y, the r that maintains portfolio balance increases over time.

27. The simulation was carried out for 400 periods, the last 380 periods being analyzed by cross-spectral methods. A pseudo-random-number generator was used to produce the random terms. Forty was the maximal lag in calculating the spectral and cross-spectral statistics from the estimated covariograms and cross covariograms.

28. Results similar to those reported in the text were obtained for Cowles' dividend-price ratio for utilities, total earnings-price ratio index, and earnings-price ratio for utilities. Cowles' earnings-price ratio and dividend-price ratio for industrials, however, were not so clearly characterized by the Gibson paradox.

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