Coexistence of Money and Interest-Bearing Securities

S. R. Aiyagari, Neil Wallace, and Randall Wright

April 27, 1995

ABSTRACT. A pairwise random meeting model with money is used to study the nominal yield on pure-discount, default-free securities that are issued by the government. There is one steady state with matured securities at par and, for some parameters, another with them at a discount. In the former, exogenous rejection of unmatured securities by the government is necessary and sufficient for such a steady state to display a positive nominal yield on unmatured securities. In the latter, the post-maturity discount on securities induces a deeper pre-maturity discount even if there is no exogenous rejection of unmatured securities.

AFFILIATIONS. S.R. Aiyagari is Research Officer, Federal Reserve Bank of Minneapolis. Neil Wallace is Barnett Banks Professor of Money and Banking, University of Miami, and Senior Consultant to the Research Department, Federal Reserve Bank of Minneapolis. Randall Wright is Professor of Economics, University of Pennsylvania, and Senior Consultant to the Research Department, Federal Reserve Bank of Minneapolis.

Communication can be addressed to S.R. Aiyagari, Research Department, Federal Reserve Bank of Minneapolis, 250 Marquette Avenue, Minneapolis, MN 55480. FAX (612)-340-2366, E-mail: sra@res.mpls.frb.fed.us

Explaining coexistence of noninterest-bearing money and interest-bearing default-free securities is arguably *the* crucial and challenging problem for monetary economics. It is crucial because such coexistence is necessary for imperfect substitutability between money and default-free securities. It is challenging because, despite having been identified by Hicks more than 50 years ago, as the main challenge facing monetary theory, no one argues that a suitable model is in hand. The main difficulty is that models with complete markets are inconsistent with such coexistence-- unless, of course, there is resort to assumptions like money-in-utility or money-in-production functions. This paper explores the possibilities for such coexistence in a model with random matching and bilateral bargaining, one closely related to Shi (forthcoming) and Trejos-Wright (forthcoming), but in which, in addition to an asset that has the properties of a fiat money, there are payable-to-the-bearer, pure discount, default-free, small denomination claims to fiat money in the future.[1]

There are at least two reasons why it is interesting to explore the possibility of coexistence of noninterest-bearing money and interest-bearing default-free securities in a random-matching model. First, as already suggested, there is agreement that a suitable explanation requires a departure from complete markets. A random-matching model is a simple, although admittedly extreme, specification of market incompleteness. Second, random-matching models were not devised to explain such coexistence. They were devised to represent absence-of-double-coincidence, which, in almost all writings on money, has been considered an important ingredient for giving rise to a medium-of-exchange. Therefore, exploring whether random-matching models are consistent with coexistence is in the spirit of discovering additional implications of such models.

The basic model of the paper assumes a continuum of agents and a finite number of types of agents, where type determines both the kind of good consumed by the agent and the kind of good produced by the agent so that there is a potential for trade, but no double coincidence of wants. Goods are divisible, but not storable. In Shi and Trejos-Wright, there is only one storable object: indivisible units of fiat money. Moreover, in order to limit the state space, each agent is assumed able to store at most one unit of such money. The value of money or the price level is determined by bargaining in meetings between agents with money and those without money who can produce the good consumed by the money

[1]Instances in which governments isssued such securities include the U.S. during the Civil War (see Gherity 1993) and World War I (see the discussion of Liberty Bonds in Wallace (1988)), and France, 1915-27 (see Makinen and Woodward (1986)).

holder. We adopt all these assumptions, except that we add securities to the model. As in Aiyagari-Wallace (1994), we introduce a class of "government" agents whose behavior is exogenous. Their role is to issue and redeem pure discount securities. In addition, they sometimes reject trades involving securities that have not yet matured.[2]

Our results, in a sense, provide two explanations of coexistence of noninterest-bearing money and interest-bearing default-free securities. One explanation is a steady state in which matured securities and fiat money are perfect substitutes-- have the same value in all trades. We demonstrate that some exogenous rejection of unmatured securities (by government agents) is necessary and sufficient for such a steady state to display a discount on unmatured securities. The other explanation is a steady state in which matured securities and fiat money are imperfect substitutes in that matured securities trade at a discount in trades among private agents. In this steady state, which exists for some parameters, the post-maturity discount on securities induces a deeper pre-maturity discount even if there is no exogenous rejection of securities that have not matured.

Both explanations, of course, rest on the specific assumptions that we have made-- random pairwise meetings and the accompanying informational restrictions noted below, indivisible assets, and a unit upper bound on asset holdings. Although these assumptions constitute an extreme departure from complete markets, our formulation has one important virtue not found in most models of money: the same transaction frictions are applied to all the assets in the model, fiat money and securities. This contrasts with most existing monetary theory: the positing of a demand for "money" (a special case being the quantity theory equation), "money" in the utility function, "money" in the production function, transaction costs that are relatively low for trades involving "money" (if there are even trades), or "cash"-in-advance constraints. Those models provide trivial solutions to the problem of coexistence because the modeler must specify which assets yield utility, enhance production, have low transaction costs, or meet cash-in-advance constraints. Our explanations are less trivial because the common transaction frictions are permitted to interact with the intrinsic differences among the assets in our model. One consequence is that our model makes new predictions about episodes in which governments issued securities which resemble those in our model.

[2]Other models of credit in search models include Diamond (1990), Hendry (1992) and Shi (1994). Although credit in these models is a substitute for fiat money, the substitution is limited because credit instruments in those models are liabiliities of individuals and, therefore, cannot circulate.

The paper is set out as follows. In section 1, we describe the model. In section 2, we study a simple variant, one with two distinct fiat monies and no securities. This variant provides the ingredients for the results in the more complicated setting with money and securities, because matured securities can be regarded, for most purposes, as a distinct fiat money. In particular, the distinction between our two explanations, which rests on whether or not matured securities are treated as perfect substitutes for fiat money, is analogous to whether or not two distinct fiat monies are treated as perfect substitutes. In section 3, we present the two steady states which constitute our two explanations for coexistence in the money-securities model. In section 4, we discuss the robustness of our results to the assumptions that assets are indivisible and that each agent can hold at most one unit of one asset. We conclude in section 5 with the predictions of our model for historical episodes in which governments issued payable-to-the-bearer, default-free, small denomination securities. Proofs appear in the appendix.

1. The Model

We begin with the underlying discrete-time environment. We then introduce the assets and government agents. Then we describe how agents bargain in pairwise meetings.

1.1. The environment.

Time is discrete and the horizon is infinite. There are N distinct goods, actually services, at each date and there is a [0,1] continuum of each of N types of agents, where $N \geq 3$. Each type is specialized in consumption and production in the following way which rules out double coincidences: a type i agent, where i runs over integers from 1 to N, consumes service i and produces service i+1 (modulo N).

Each type i agent maximizes expected discounted utility with discount factor $\beta \in (0,1)$, where utility in a period is a function of the amount of service i consumed in the period and the amount of service i+1 produced in the period, $u(x) - y$, where x is the amount of service consumed and y is the amount of service produced.[3] The function u is defined on $[0,\infty)$,is increasing and twice differentiable and $u(0) = 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$.

[3]The assumption that the disutility of production is equal to the amount produced is without loss of generality. If we start with a different disutility, say, d(y) with d strictly increasing and convex, and a utility of consumption given by $u^*(x)$, then our specification results from letting $z = d(y)$, $u(z) = u^*[d^{-1}(z)]$ and having agents bargain over the disutility to the producer.

Agents meet pairwise at random and each agent's trading history is private information to the agent. Together, these assumptions rule out all but *quid pro quo* trade for optimizing agents. Services cannot be stored. The only storable objects are indivisible assets. Each agent has a storage capacity of one unit of some asset.

Notice that types are identical except as regards what they consume and produce; that is, β and u, the measure of each type, the matching process, and asset storage capacity are identical for all types. We have assumed such symmetry so that it makes sense to look for equilibria that are symmetric among types. In fact, those are the only kind of equilibria we consider in this paper.

1.2. Fiat money, securities, and government agents.

There are two types of assets in our model: indivisible units of fiat money and finite maturity claims to fiat money in the future, the securities. In order for there to be such securities in a steady state, they have to be created and retired. To accomplish that, we use a version of a device used by Aiyagari-Wallace. For each agent type, we let the subinterval [0,G], with $G \in (0,1)$ consist of a class of agents, called government agents. The government agents of type i are like the other type i agents, called private agents. Despite this, we impose the behavior of government agents, which, therefore, is not motivated by their preferences or those of private agents. Among other things, this exogenous behavior permits securities to be issued and redeemed, while the random matching feature of the model is retained.

In all but two respects, a type i government agent emulates exactly the actions of type i agents as described below. These two respects involve the issue and redemption of matured securities, and trades involving unmatured securities.

(i) Government agents sometimes "issue" securities and standy ready to "redeem" securities. At any date t, if a type i+1 government agent with a unit of money meets a type i private agent, then with probability q (for quantity of securities) the government agent offers a two-period pure discount bond, rather than the money held and, in effect, destroys the money held. This could be accomplished by having the government agent write on the money held "this matures at t+ 2", thereby converting it into a security. At maturity-- that is, at t+2 and thereafter-- a private agent holding this security who meets a government agent, can, if he or she wishes, exchange it for a unit of money. In effect, any government

agent stands ready to erase what was written on the money, thereby turning it back into a unit of money. This rule about how securities are issued and redeemed implies an unchanging total amount of assets. We assume that the total amount of assets per type is positive and less than one.

(ii) Government agents sometimes reject securities that have not matured. At any date t, government agents reject offers of securities that have not matured with probability r (for reject). That is, with probability r, they simply refuse to trade when offered a security that has not matured.

We assume that government agents issue securities which mature in two periods, because two periods is the shortest maturity that permits a discussion of the interesting possibilities and because longer-term securities complicate the presentation without affecting the conclusions. In particular, the conditions which produce a discount on newly issued two-period securities would produce a discount on longer-term securities whenever they have at least two periods until maturity, and the conditions under which newly issued two-period securities trade at par imply par values at all dates for longer-term securities. The rejection probability, r, is the crucial policy parameter that produces a discount on securities in the steady state in which matured securities trade at par. Instead of a rejection probability, it could have been assumed that government agents exogenously give less service when offered an unmatured security than when offered a unit of money. This too would produce a discount on newly issued securities.

1.3. The sequence of actions in a period and bargaining.

The sequence of actions within a period occurs as follows. Each agent, including a government agent, begins a period holding either one unit of one asset or nothing. Then agents meet pairwise at random. Agents in pairwise meetings bargain. If the outcome of bargaining involves the provision of a service, then consumption occurs. Then agents begin the next period.

In a symmetric equilibrium, any trade among private agents must involve the provision of a service. That is, it never pays to simply exchange one asset for another. Trade involving a service can occur only when a type i agent meets a type i+1 agent. In such a meeting, the type i agent, the potential producer, can supply what the type i+1 agent, the potential consumer, consumes but, since there are more than two types, the latter cannot supply

what the former consumes. Given the upper bound of unity on asset holdings, there are two potential trading situations: when the consumer has an asset and the producer does not, and when the consumer has a more valuable asset than the producer has. In the former situation, the asset may be traded for some amount of service. In the latter situation, the more valuable asset may be traded for some service and the less valuable asset.

We assume throughout the paper the following very simple bargaining rule: in a meeting between a consumer and a producer, the consumer makes a take-it-or-leave-it offer and the producer accepts if made no worse off by accepting. A consequence is that the producer of the service does not benefit from the exchange.

Except for the redemption of matured securities, which occurs at the behest of the holder of the matured security in a meeting with any government agent, all trades between government agents and private agents also involve the provision of a service. Each government agent either starts a period with a unit of money or nothing. (A consequence of our rules about security issue and redemption imply that a government agent never begins a period with a one-period security or with a matured security.) When the government agent can supply a service to a private agent (the government agent has nothing and has met a potential producer with an asset), we assume that the government agent has the same reservation value as a private agent of the same type, except that with probability r the government agent rejects an offer of a one-period security. When the government agent can consume a service provided by a private agent (the government agent has a unit of money), it makes the same take-it-or-leave-it offer that would have been made by a private agent with two exceptions: with probability q it offers a two-period security instead of the money and gets as much service for it as possible; if the private agent has a one-period security, then with probability r the government agent refuses to trade.

Before we define and present results for symmetric steady states for the above model, we discuss a simpler model with two fiat monies. This discussion serves two purposes. It provides the main ingredients for the results for the money-securities model and, because it is simpler, provides an introduction to the workings of the money-securities model.

2. Symmetric steady states with two fiat monies

As noted in the introduction, we study two monies because of the similarity between a matured security and a distinct kind of money. We begin by assuming that there are no

government agents and that there are given outstanding stocks of two kinds of money, money A and money B. These are distinguished by nothing but color. We let $p_A > 0$ be the per-type amount of money A and $p_B > 0$ the per-type amount of money B, where $p_A + p_B < 1$. We make all the assumptions made above. In particular, the monies are indivisible, each agent can hold at most one unit of one money, and bargaining is assumed to take the form of take-it-or-leave-it offers by potential consumers.

Since the labelling of monies is arbitrary, in this section we adopt a notation which presumes that money B at least as valuable as money A. We let c_j be the maximum amount of service that would be provided by someone who has no money in exchange for a unit of money j and we let c_{BA} be the maximum amount of service that would be provided by a holder of money A who is offered a unit of money B. We also let V_j be the discounted utility from beginning a period with a unit of money j. Notice that a consequence of the bargaining rule we use is that the discounted utility from beginning a period with no money, or, more generally, with no asset, is zero; a person with no asset must be indifferent between accepting a trade and not accepting a trade and not accepting implies a zero current period return (since $u(0) = 0$) and beginning the next period with no asset. To conserve on notation, we simply embed that zero value in all our definitions. We also let $\alpha \equiv (1 - p_A - p_B)/N$, the probability of meeting an agent of a particular type holding no money. Using this notation, we have the following definition of a symmetric steady state (in this case also a constant equilibrium).

Definition 1. A symmetric steady state in which money B is at least as valuable as money A is (c_A, c_B, c_{BA}) and (V_A, V_B) such that

(1) $V_A = \alpha \max[u(c_A), \beta V_A] + (1-\alpha)\beta V_A$

(2) $V_B = \alpha \max[u(c_B), \beta V_B] + (p_A/N)\max[u(c_{BA}) + \beta V_A, \beta V_B] + (1-\alpha-p_A/N)\beta V_B$

(3) $c_j = \beta V_j, j = A,B$

(4) $c_{BA} = \beta(V_B - V_A)$ •

In this definition, equation (1) includes the implication of the bargaining rule that suppliers of a service do not gain in a trade and the convention that money B is at least as valuable as money A. In equation (1), α is the probability of meeting a service supplier with no

money. Such a meeting gives the holder of money A the option between, on the one hand, a current period utility $u(c_A)$ and beginning next period with no money (which, as noted above, has discounted expected utility zero) and, on the other hand, choosing not to trade. With the remaining probability, a holder of money A gets the payoff from not trading. Although a holder of money A may also trade if he or she meets a holder of money B who consumes what the holder of money A produces, such a trade leaves the holder of money A indifferent. Therefore, in terms of expected utility, such a trade is equivalent to not trading. Equation (2) describes the probabilities and respective options for a holder of money B. Equations (3) and (4) are implications of take-it-or-leave-it offers by buyers; each expresses the condition that the disutility of performing the amount of service is equal to gain in discounted expected utility from changing asset positions.

2.1. Steady states in which the two monies are not distinguished.

There are exactly two steady states in which the two monies are not distinguished-- one with no trade and one with trade.

Lemma 1. There is a no-trade steady state with $(V_A,V_B) = (0,0)$ and exactly one steady state with $c_A = c_B = c^* > 0$, $c_{BA}= 0$, and $V_A=V_B = V^* > 0$.

The proof, which appears with all others in the appendix, consists of showing that equations (1)-(4) have solutions of the asserted kind.

2.2. Steady states in which money B is more valuable than money A.

There is always a steady state in which both monies are valuable and money B is more valuable.

Lemma 2. There exists at least one steady state in which both monies are valuable and money B is more valuable than money A.

In a lemma 2 steady state, the less valuable money, money A, necessarily exchanges for some amount of service because otherwise it could not have value. In fact, its value must be the same as in a lemma 1 steady state with valued money-- that is, $(V_A, c_A) = (V^*,c^*)$-- because, under the assumed bargaining rule, the option of giving up money A for money B does not enhance its value. It is also the case that in any lemma 2 steady state, money B

exchanges for money A plus some service, because otherwise money B would function just like money A, and, then, by lemma 1 would have to have the same value if both are valuable. It follows that there are only two possible trading patterns, which differ according to whether money B is given up solely for some amount of service. For some parameters, money B is not traded solely for services, while for others it is. For some parameters both kinds of steady states exist.[4]

Notice that a lemma 2 steady state is Pareto superior to a lemma 1 steady state. Those holding nothing or money A are equally well off, while those holding money B are better off in a lemma 2 steady state. An analogue of this result will carry over to the money-securities model.

Although we are primarily interested in steady states in which both monies are valuable and money B is more valuable than money A, for the sake of completeness, we note that there is a steady state in which money A is not valuable and money B is valuable. The argument is essentially the same as that for lemma 1.

2.3. Steady states consistent with government transactions.

Although we will not present a formal analysis of government agents and their trades in the version with two monies, it is instructive to describe government trades that are analogous to those in the money-securities model and their consequences for steady states. The analogue of the issue of new securities is having a government agent, with probability q, offer a unit of money A instead of a unit of money B. The analogue of redeeming matured securities is standing ready to turn a unit of money A into a unit of money B. For any $G > 0$, the only steady state consistent with both monies being held by private agents and having different positive values is one in which money B is traded directly for services.

In particular, there is no steady state in which money B is traded only for money A and some service, because such a trading pattern implies that government agents never acquire money B. That being so, the only steady state consistent with their trading rules is one in which private agents hold only money B. In the money-securities model, this would

[4]Shi (forthcoming) also displays steady states with two fiat monies with different values. However, such steady states are due to the existence in his model of multiple steady states with a single valued money. Ours are due to the possibility of trading the more valuable money for the less valuable money, a trade which Shi does not consider.

correspond to a steady state in which no securities are outstanding, which is not of interest to us.

Since we will be appealing, by way of a limiting argument in which G approaches 0, to existence of a steady state in which money B is more valuables than money A and money B is traded directly for services, we want a sufficient condition on parameters to assure that such a steady state exists. The following lemma, an easy consequence of lemma 2, provides such a sufficient condition.

Lemma 3. If $(1-\beta)/\beta \geq p_A/N$, then there exists exactly one symmetric steady state in which money B is more valuable than money A, both are traded solely for services, and money B is traded for money A and some service.

The hypothesis of the lemma 3 is that the discount factor is not too close to unity. It is not surprising that sufficient impatience gives rise to a steady state in which the chance to trade the valuable money directly for some service is not passed up. It can also be shown that for discount factors sufficiently close to one, any lemma 2 steady state is such that money B is not traded solely for services.

3. Steady states in the money-securities model

We begin by defining a symmetric steady state. Here, since the distribution of asset holdings can potentially be changing through time, we have to include in the definition the condition that the distribution is constant. We begin with our notation, which presumes that (i) all matured securities, independent of vintage, are treated identically, (ii) there is symmetry over types of agents, and (iii) a unit of money is at least as valuable as a security.

At the start of a period, each private agent is either holding a security with one-period until maturity or a matured security or a unit of money or nothing. Therefore, the state of the system at the start of a period can be described by 4 fractions. We let p_j be the fraction of each type who are private agents and hold a unit of asset j, where j = 1 means a one-period security, j = 0 means a matured security, and j = m means a unit of money. We also let p_{gm} be the fraction of each type who are government agents and who hold a unit of money.[5] We continue to let α denote the probability of meeting a particular type with no

[5]Since we assume that government agents never offer a matured security, p_{gm} is also the fraction of each type who are government agents and who hold an asset.

asset; that is $\alpha = [1- (p_1 + p_0+ p_m+ p_{gm})]/N$. As noted above, the total amount of assets per type is a parameter between 0 and 1. Since this total, in terms of α, is $1 - N\alpha$, it follows that α is a parameter and that $0 < \alpha < 1/N$.

We distinguish 5 amounts of services. There are 4 possibilities for trades of an asset for services alone. We let c_j be the amount that would be supplied for a unit of asset j, where j = 0,1,2, m and where j = 2 means a 2-period security and we let c_{m0} be the amount of service that would be supplied for a unit of money when the supplier of the service starts with a matured security. We also let V_j be the discounted expected utility of starting a period with a unit of asset j where j = 0,1,m.

We first describe the conditions that assure that the asset distribution is constant. These conditions are

(5) $p_1+ p_0+ p_m+ p_{gm} = 1 - N\alpha$

(6) $p_1 = (1 - G - p_1 - p_0 - p_m)(p_{gm}/N)q$

(7) $p_{gm}[(1-\delta_m)q + \delta_m](1 - G - p_1 - p_0 - p_m)/N =$

$$(G - p_{gm})[\delta_0(1-r)p_1 + \delta_m(p_0+p_m)]/N$$

(8) $p_0G = p_1\{1 - (1- r)[\delta_0(G - p_{gm}) + \delta_{m0}p_{gm}]/N\}$

where $\delta_m = 1$ if $u(c_m) > \beta V_m$ and 0 otherwise; $\delta_0 = 1$ if $u(c_0) > \beta V_0$ and 0 otherwise; and $\delta_{m0} = 1$ if $u(c_{m0})+\beta V_0 > \beta V_m$ and 0 otherwise.

Equations (5)-(8) are steady state versions of the law of motion. The first two are simple. Equation (5) expresses the constancy of per type holdings of assets in terms of α, the probability of meeting a particular type with no asset. Equation (6) equates the outflow and inflow into the measure of each type who are private agents and who hold a security with one-period until maturity. Since one-period securities mature in the next period, the outflow is equal to the measure at the last date, p_1 in a steady state. The inflow equals the measure of private agents of that type who were holding nothing at the previous date, $(1-G-p_1 - p_0 - p_m)$, and who met a potential government consumer who offered a security, the probability of such a meeting being $(p_{gm}/N)q$.

Equation (7) equates the outflow and inflow into the measure who are government agents of each type holding money. The outflow equals the measure of such agents in the last period who met and traded with a private potential supplier. We are assuming that government agents never offer matured securities; they either offer a two-period security or money. With probability q they offer a two-period security. With probability 1-q, they either offer a unit of money ($\delta_m = 1$) or nothing ($\delta_m = 0$). If $\delta_m = 1$, then they always give up their asset; if $\delta_m = 0$, then they give it up with probability q. Put differently, with probability (1-q), they emulate private agents with money. The inflow occurs from meetings in the last period between government agents holding nothing and private agents who are potential consumers and hold an asset. An inflow occurs from private agents holding one-period securities only if two conditions are met: the government agent accepts, which it does with probability 1-r, and the offer is made, which occurs if $\delta_0 = 1$; that is, if the private agent would rather consume what can be obtained for a matured security than not trade, matured because the one-period security is equivalent to a matured security to the recipient. If the private agent holds a matured security or money, then there is an inflow if that agent prefers giving up money for c_m to entering the next period with money.

Equation (8) equates the outflow and inflow from the measure of private agents of each type holding matured securties. The outflow is equal to that measure in the previous period who met a government agent, a meeting which occurs with probability G. That is, any meeting between a private agent holding a matured security and a government agent results in a decrease in private holdings of matured securities since, at worst, the government agent turns the matured security into a unit of money. The inflow, the right-hand side, is equal to the per type stock of one-period securities except for those traded to government agents. For a private agent of type i, there are two potential sources of trades with government agents: meetings with government agents who are type i-1 and who are not holding an asset and, hence, are potential suppliers, the probability of which is $(G-p_{gm})/N$; and meetings with government agents who are type i+1 and who are holding money and, hence, are potential consumers of i's service, the probability of which is p_{gm}/N. Trades with the former occur if the government agent accepts securities prior to maturity and if it is in the interest of the private agent to offer the security, $\delta_0 = 1$. Trades with the latter occur if the government agent accepts unmatured securities and if it is in that agent's interest to make the trade, $\delta_{m0} = 1$.

We now define a steady state. We list the conditions in the definition in an order that facilitates identifying matured securities and money in this version of the model with money A and money B of the two money version of section 2, respectively.

Definition 2. A symmetric steady state in which a unit of money is at least as valuable as a matured security is a list $(p_0, p_1, p_m, p_{gm}, c_0, c_1, c_2, c_m, c_{m0}, V_0, V_1, V_m)$ that satisfies (5)-(8) and:

(9) $$V_0 = [(1-G - p_1 - p_0 - p_m)/N]\max[u(c_0), \beta V_0] + [(G-p_{gm})/N]\max[u(c_m), \beta V_m] + [G - (G-p_{gm})/N]\beta V_m + [1 - G -(1-G - p_1 - p_0 - p_m)/N]\beta V_0$$

(10) $$V_m = \alpha\max[u(c_m), \beta V_m] + [(p_0 + p_1)/N]\max[u(c_{m0})+\beta V_0, \beta V_m] + [1- \alpha - (p_0 + p_1)/N]\beta V_m$$

(11) $$c_j = \beta V_j,\ j = 0,m$$

(12) $$c_{m0} = \beta(V_m - V_0)$$

(13) $$V_1 = [(1-G - p_1 - p_0 - p_m)/N + (1-r)(G - p_{gm})/N]\max[u(c_0), \beta V_0] + [1 - (1-G - p_1 - p_0 - p_m)/N - (1-r)(G - p_{gm})/N]\beta V_0$$

(14) $$c_1 = c_0$$

(15) $$c_2 = \beta V_1 \bullet$$

Although this definition contains a long list of conditions, they split up into what will be effectively three recursive blocks of equations: (5)-(8), (9)-(12), and (13)-(15). That is, when we apply our guess and verify procedure by guessing at a trading pattern and then verifying that there is a steady state with that trading pattern, we will first solve equations (5)-(8) for the steady state asset distribution. Then, given that asset distribution, we will

solve (9)- (12), which do not involve V_1, c_1, or c_2. Then finally, we will solve (13)-(15), which determine what happens in trades involving securities prior to maturity.

Equations (9)-(12) correspond closely to (1)-(4) provided we identify matured securities with money A and money with money B. That is, (10)-(12) correspond exactly to (2)-(4), respectively, provided we use the result, embedded in our definition, that the recipient of a one-period security is indifferent between it and a matured security because the one-period security becomes a matured security before it can be retraded. As for (9), it differs from (1) only because government agents treat matured securities as money. In particular, the first term on the right-hand side of (9) is the probability of meeting a potential private supplier with no asset times the payoff from such a meeting; the second term is the probability of meeting a potential government supplier with no asset times the payoff from such a meeting (the government supplier can be regarded as first redeeming the matured security and then possibly supplying a service for the money), the third term is the probability of meeting any other government agent times the payoff, which is beginning the next period with money; the last term is the probability of any other meeting times the payoff, which is equal to that of not trading.

Equations (13)-(15) not only form a recursive sub-block, but can be solved one equation at a time. In (13), we again use the fact that a one-period security becomes a matured security for the person who accepts it or who holds onto it. The first term is the probability of meeting a potential supplier who has no asset and, if a government agent, does not reject securities prior to maturity. With the remaining probability, the payoff is that of no trade. Notice that the right-hand side of (13) is determined by the solution to the other blocks of equations. Equation (14) says that the offer of a one-period security in a private trade is equivalent to an offer of a matured security. Equation (15) determines the amount of service that a government agent gets when it issues a two-period security.

If we compute the yield until maturity in the usual way and with current asset prices taken from trades of assets for services only, then the yield of one-period securities is $(c_m/c_1) - 1$, while the yield of two-period securities, our main concern, is $[(c_m/c_2) - 1]^{1/2}$. The former, by (14), is positive if and only if matured securities trade at a discount ($c_0 < c_m$); the latter, by (11) and (15), is positive if and only if $V_1 < V_m$.

We first describe a steady state in which matured securities and money are treated identically. Then we describe a steady state in which matured securities trade at a discount in private trades.

3.1. Matured securities exchange at par.

One would expect there to be a steady-state in which matured securities and money are treated identically. To show that there is, we begin by noting that there is a constant asset distribution under the trading pattern implied by identical treatment of money and matured securities. That trading pattern implies that both money and matured securities exchange for services alone; that is, $\delta_m = \delta_0 = 1$ in equations (5)-(8). Moreover, since with such identical treatment, it is not necessary to distinguish between holdings of matured securities and money, we need only find p_1, $p_0 + p_m$, and p_{gm}. The following lemma, which uses the fact that p_0 and p_m appear in equations (5)-(7) only as a sum, shows that those equations have a unique solution.

Lemma 4. If $\delta_m = \delta_0 = 1$, then there exists a unique solution to (5)-(7) with $(p_1, p_0+p_m, p_{gm}) \in R^3_{++}$. Moreover, the solution for p_{gm} is strictly decreasing in r, strictly decreasing in q if $r > 0$, and satisfies $p_{gm} < G$; while the solution for p_1 is strictly decreasing in r and q.

Our main result for steady states with matured securities trading at par is as follows.

Proposition 1. A necessary and sufficient condition for existence of a steady state in which matured securities exchange at par ($c_0 = c_m$) and newly issued securities exchange at a discount ($0 < c_2 < c_0$) is $r > 0$. Moreover, if $r = 0$, then there is a steady state in which securities always exchange at par ($0 < c_2 = c_0 = c_m$).

The necessity part of this proposition is established by contradiction. In particular, to prove it, we assume $r = 0$ and $c_0 = c_m > 0$ and show that a consequence is $c_2 = c_m$. The argument is straightforward backward induction. If a security will be equivalent to money at t+1, then a private potential supplier treats it as money at t, because t+1 is the earliest date at which it will be retraded; and, with $r = 0$, government agents, by assumption, have the same reservation value as private agents.

The sufficiency part, including existence for $r = 0$, involves showing that definition 2 can be satisfied by the kind of steady state described. This is done in three simple steps using the recursive structure of equations (9)-(15). The first step, already accomplished in lemma 4, is to show that there is a constant asset distribution under the hypothesized strategies, which imply $\delta_m = \delta_0 = 1$. Then, this solution is used in (9)-(12), which under the hypothesized strategies have a solution, the valued money solution in lemma 1. The third step is simply to read off the solutions for V_1, c_1, and c_2 from (13)-(15). In this solution, $r > 0$ implies $c_2 < c_m$, a discount on newly issued securities, because the potential supplier who accepts a newly issued security may meet a government agent at the next date who will reject a trade with probability r.

The necessity part of proposition 1 says that if matured securities are perfect substitutes for money, then so are unmatured securities unless there is exogenous nonacceptance by some agents. In other words, if matured securities are trading at par, then the belief that other agents will not accept unmatured securities at par is not an equilibrium belief unless supported by exogenous nonacceptance by some agents.

Although not germane to our main message, some features of the steady state displayed in the sufficiency part of proposition 1 deserve comment. In that steady state, a unit of money always exchanges for the same amount of service, the c^* of lemma 1, since (10) reduces to (1) when matured securities and money are perfect substitutes. It follows that in a proposition 1 steady state, the value of a unit of money depends only on α, the total amount of assets and not the composition of assets. This feature can be attributed partly to the absence of taxes in the model; if there were taxes in the model, then the value of money could depend on the composition of assets, as well as on the details of the tax system.

The policy parameters q and r affect the steady-state distribution of asset holdings as between government agents and private agents, the composition of private asset holdings as between one-period securities, on the one hand, and money and matured securities on the other hand, and, by way of those effects and directly via (13), the discount on newly issued securities. Since, by (13), and the argument in the proof, $V_1 = \alpha' u(c^*) + (1-\alpha')\beta V^*$, where, $\alpha' = \alpha - r(G - p_{gm})/N$, and since p_{gm} is decreasing in r and q (see lemma 4), it follows that V_1 is decreasing in r and q. This implies that the two-period nominal interest rate, $(c^*/c_2) - 1$, is increasing in r and, for any $r > 0$, is increasing in q.

3.2. Matured securities exchange at less than par.

Here we assume $r = 0$; that is, we now assume that government agents emulate private agents except that they sometimes issue two-period securities and stand ready to redeem matured securities. Proposition 1 implies that with $r = 0$ any steady-state in which securities trade at a discount has to be one in which matured securities do not exchange at par in trades between private agents. Therefore, we now explore that possibility. Notice that *if* there is a steady state in which matured securities trade at less than par in private transactions, then the value of securities when issued will be less than their value in private trades after maturity because it is only after maturity that government agents redeem securities.

The conjecture that there is a steady state in which matured securities exchange at less than par for some parameters is plausible because of the results for two monies in section 2. We now show that that conjecture is true, at least for some parameters. We begin by presenting some necessary conditions for a steady state in which matured securities trade at a discount.

Lemma 5. In any steady state with $V_m > 0$ and $V_0 \leq V_m$, (i) $V_m \geq V^*$ and (ii) $V_0 \geq V^*$, where V^* is the lemma 1 solution for valued money.

Conclusion (i) is obtained by noting that under the hypothesis, the worst that happens for money is that it plays the role of valued money in lemma 1. Conclusion (ii) is the same conclusion for matured securities. In particular, matured securities cannot be worthless if money has value, because a matured security can be redeemed with positive probability. One consequence of lemma 5 is that there is not a steady state in which money is valuable and matured securities play approximately the role of a distinct kind of money that has no value.

Lemma 6. In any steady state with $V_m > V_0 > 0$ and $p_1 + p_0 > 0$, (i) $u(c_{m0}) + \beta V_0 > \beta V_m$, (ii) $u(c_0) > \beta V_0$, and (iii) $u(c_m) > \beta V_m$.

Conclusion (i) is established by contradiction. The idea is that if it does not hold, then the only positive value for money is its positive value in lemma 1, V^*. But, then, the conclusion of lemma 5 contradicts the hypothesis. Conclusions (ii) and (iii) say that both matured securities and money trade for services alone. If not, then government agents

would never acquire money and there could not be a steady state in which securities are outstanding.

A consequence of lemma 6 is that there is only one possible trading pattern consistent with a steady state in which money has value, matured securities trade at a discount, and private agents hold securities. It is a steady state with $\delta_m = \delta_0 = \delta_{m0} = 1$. In particular, since $\delta_m = \delta_0 = 1$, the solution to (5)-(7) given in lemma 4 applies. Moreover, since we are now assuming $r = 0$, that solution takes a simple form which we now record along with some of its consequences.

Lemma 7. If $\delta_m = \delta_0 = \delta_{m0} = 1$ and $r = 0$, then the unique positive solution to (5)-(8) satisfies: (i) $p_{gm} = G(1-N\alpha)$; (ii) $p_1 + p_0 = q\alpha(1-N\alpha)(1-G)[1 + G(1-N^{-1})]$; (iii) $p_1 + p_0$, as given in (ii), is continuous and decreasing in G, with limit $q\alpha(1-N\alpha)$ as $G \rightarrow 0$ and limit 0 at $G \rightarrow 1$.

Conclusion (i) says that with $r = 0$ and both matured securities and money being traded for services only, the fraction of government agents with an asset is equal to the fraction of private agents with an asset. Conclusion (iii) describes the behavior of private holdings of one-period and matured securities. Notice that although $p_1 \rightarrow 0$ as $G \rightarrow 0$, p_0 does not because on average securities remain outstanding for a number of periods that approaches infinity as $G \rightarrow 0$. The limiting behavior as $G \rightarrow 0$ plays a role in proposition 2 which establishes existence of a steady state when G is sufficiently close to zero. The limiting behavior as $G \rightarrow 1$ plays a role in the subsequent discussion.

Proposition 2. Assume $r = 0$ (government agents treat one-period securities as do private agents). If G is sufficiently close to 0 and $(1-\beta)/\beta \geq q\alpha(1-N\alpha)/N$, then there exists a steady state with matured securities exchanging at less than par ($c_0 < c_m$ and $c_{m0} > 0$) and with newly issued securities valued at less than matured securities ($c_2 < c_0$) and with money traded for services only, and for securities and services.

The proof proceeds by verifying that there is a solution to the conditions in definition 2 that satisfy the inequalities hypothesized in lemma 6. It uses the recursive structure of the conditions in definition 2 that we noted above, and, therefore, has three steps. The first step, the solution for the asset distribution under the conditions in lemma 6, is accomplished in lemma 7. The second step is to show that equations (9)-(12) have a solution with $c_m > c_0 > 0$, a solution that is consistent with (i)-(iii) in lemma 6. That is

done via the implicit function theorem using the correspondance between (9)-(12) and (1) - (4). In particular, taking into account the dependence of p_{gm} and of $p_1 + p_0$ on G as given in lemma 7, the right-hand sides of (9) and (10) are easily shown to be continuous in G and to approach the right-hand sides of (1) and (2), respectively, as $G \rightarrow 0$. The third step amounts to noting obvious properties of the solutions to (13)-(15).

The other general result we have concerning steady states in which matured securities trade at a discount is that the discount, if it exists, approaches zero as either $G \rightarrow 1$ or $q \rightarrow 0$. It follows from lemma 7 that $p_1 + p_0 \rightarrow 0$ as either $G \rightarrow 1$ or $q \rightarrow 0$. But, then by, (10), it follows that $V_m \rightarrow V^*$. Therefore, by lemma 5, $V_0 \rightarrow V^*$, which implies that the discount on matured securities, if one exists, approaches zero as either $G \rightarrow 1$ or $q \rightarrow 0$.

We are not, of course, making any claim of necessity for the hypotheses of proposition 2. Indeed, as regards G, we have computed examples in which the conclusions of proposition 2 hold for all G's-- although, consistent with the claim just made, the discount on matured securities in these examples gets smaller as G gets larger. As regards the discount factor, some bound away from unity seems to be necessary to assure that money trades for services alone, which, as noted above, is necessary for a steady state with securities outstanding. Letting $\pi = N\alpha$, the fraction of each type without an asset, the sufficient condition on the discount factor in proposition 2 can be written as $(1-\beta)/\beta \geq q\pi(1-\pi)/N^2$ or $\beta \leq 1/[1 + q\pi(1-\pi)/N^2] \leq 1/[1 + 1/2N^2] = 2N^2/[1 + 2N^2]$. Hence, the stringency of that condition varies inversely with N^2. Since $N = 3$ is the smallest N we permit, $\beta \leq 18/19$ is sufficient.

There is a sense in which proposition 2 provides a welfare rationale for securities. Although the consumption and production of government agents varies across steady states, which in general implies that we should not ignore what happens to them in different steady states, what happens to them is not important if G is near zero. Ignoring government agents, the steady state of proposition 2 is Pareto superior to that of proposition 1. And, since with $r = 0$, the proposition 1 steady state is the same as a steady state with one money and no securities, we have the conclusion that there is a steady state with securities outstanding that is Pareto superior to any steady state with only one kind of money outstanding. Moreover, those steady states are comparable in the sense that the asset distribution of the proposition 2 steady state is the steady state asset distribution of proposition 1.

4. The role of asset indivisibility and the bound on asset holdings

Among the many extreme assumptions made in our model, perhaps the most troubling is the combination of asset indivisibility and the upper bound of unity on holdings of assets. Those assumptions seem to preclude direct choice among assets; for example, no one is ever in the position of choosing between offering a unit of money or a matured security for services. Here we discuss our suspicions about the consequences of weakening the upper bound on asset holdings and the indivisibility of assets.

It is straightforward to formulate versions of the model with indivisible assets and a general integer upper bound or no upper bound on individual asset holdings or a version with divisible assets and no bound on individual holdings. However, for such versions, we have not been able to describe and establish existence of steady states in which trade occurs. Nevertheless, we have some suspicions about the robustness of propositions 1 and 2 to such generalizations of the model.

As regards proposition 1, we suspect that the necessity part of that proposition will survive such generalizations of the model. That is, we suspect that the following is true: if there is a steady state in which matured securities and money are perfect substitutes, then absent exogenous nonacceptance, they are perfect substitutes prior to maturity. The sufficiency parts are more problematic, because they require existence arguments. That $r > 0$ is sufficient to generate a discount on matured securities could, in a sense, be explored without a general existence argument by considering the following conjecture: if there is a steady state in which matured securities and money are perfect substitutes and if $r > 0$, then newly issued 2-period securities necessarily sell at a discount. However, even such a limited claim does not seem easy to prove. This result in our version was easy to prove because the upper bound on asset holdings implies that someone with an unmatured security could meet a government agent and have only that asset to offer. With more general portfolios allowed, the conclusion about the discount will follow if portfolios and trades are such that $r > 0$ will be binding in some trades with government agents, but that will depend both on the portfolios and on the trades, both of which are endogenous. The result that $r = 0$ implies existence of a steady state in which securities and money are always perfect substitutes is simply the claim that the corresponding one money world has a steady state with valued money. Although that seems simple enough, it is not easy to prove. Nevertheless, despite these difficulties, we suspect that the essence of proposition 1 will survive weakening the upper bound on asset holdings and the indivisibility of assets.

We are less confident about proposition 2. Since proposition 2 rests on the results for two distinct kinds of money, let us consider a two money setting, but with divisible assets and no bound on each agent's holdings. If the initial condition is such that each agent holds the two monies in the same proportion, then we know that there is exchange rate indeterminacy in the following sense: if there is an equilibrium in which the two monies have value and are not distinguished, then for any arbitrary relative value between the two monies that is constant through time there is an equilibrium whose real aspects duplicate those of the equilibrium in which the two monies are not distinguished. Given this indeterminacy, we suspect that no analogue of proposition 2 is true if securities and money are divisible and there is no bound on individual portfolios. One basis for this suspicion is our finding that there is no money-securities approximation to a two money steady state in which one money is valuable and the other is not. A second basis is that indeterminacy of exchange rates is local nonuniqueness in the two money version, which precludes the kind of implicit function theorem argument we use to prove proposition 2.

We do not mean to suggest by these remarks that the limiting case of asset divisibility is the only case of interest. Most assets are to some extent indivisible. Although asset indivisibility can be ignored when markets are complete because the completeness permits assets to be shared, an appeal to sharing, using intermediation or some other credit device, is not available under the assumptions that imply that trade is *quid pro quo* in pairwise meetings. Therefore, results for the case of divisibility in such pairwise meeting models are of interest primarily as a limiting case of what happens as indivisibilities get less important.

Notice that the necessity part of proposition 1 provides, in a sense, two alternative necessary conditions for coexistence of noninterest bearing money and interest bearing default-free claims to money in the future. One is the exogenous nonacceptance of unmatured securities or some equivalent rule. The other is a discount on matured securities in private transactions. In accord with the discussion of proposition 1 above, these necessary conditions are almost certainly robust to weakening the upper bound on individual asset holdings and the indivisibility of assets.

5. Conclusion

The model described in this paper is one particular model in which there are transaction frictions that are not connected to particular assets. The frictions are random pairwise meetings, private information about trading histories, indivisible assets, and a unit storage capacity for assets. Although our model is extreme in many respects, it has some obvious virtues relative to most other models. We noted in the introduction that it gives a less trivial solution to explaining coexistence between noninterest-bearing money and interest-bearing default-free securities than models which simply posit that some assets overcome some explicit or implicit transaction frictions. Also, although our explanation that relies on exogenous nonacceptance by government agents is a form of legal restriction, it is a more appealing form of legal restriction than those that have been used in models of complete markets. In models with complete markets (for example, Sargent-Wallace (1982) or Bryant-Wallace (1984)), the legal restrictions have to directly limit the forms in which people can hold wealth. Here, because of the pairwise-meeting feature, we get coexistence from a much weaker and, therefore, more plausible form of legal restriction-- namely, non acceptance by the government of securities prior to maturity. Such non acceptance does not suffice in models with complete markets, because the price of securities adjusts to make the marginal holder, a private agent if both money and securities are outstanding, indifferent between the two.

Finally, our model makes some new predictions about historical episodes in which governments issued securities with features that made them suitable as forms of currency: securities which were in small denominations, were payable-to-the-bearer, and were default-free. Such securities were issued by the U.S. during the Civil War (see Gherity (1993)) and during World War I (see the discussion of Liberty Bonds in Wallace (1988)), and, most notably, in France during the years 1915-27 (see Makinen and Woodward (1986)). The French experience is notable because the government made widely available, at prices that implied roughly 5% yields until maturity, small denomination, pure discount, seemingly default-free, bearer securities with maturities of 3 months, 6 months, and 1 year.

In all these instances, the evidence seems to be that such securities did not trade at prices that implied zero nominal yields at all times. Therefore, our model makes the following predictions about these and other such episodes. The proposition 1 explanation of a discount predicts the existence of rules dictating that the government did not accept securities prior to maturity, or accepted them only at a pre-set positive price implying a

positive yield prior to maturity. The proposition 2 explanation predicts that matured securities traded at a discount in private transactions. So far as we know, these predictions have not been considered in existing discussions of episodes of coexistence between noninterest-bearing money and interest-bearing default-free, payable-to-the-bearer, small denomination securities.

Appendix

Lemma 1. There is a no-trade steady state with $(V_A,V_B) = (0,0)$ and exactly one steady state with $c_A = c_B = c^* > 0$, $c_{BA}= 0$, and $V_A=V_B = V^* > 0$.

Proof. We use a guess and verify style of argument. We solve equations (1)-(4) under the assumption that $c_j = c$, $V_j = V$ and $u(c) \geq \beta V$ and then verify that the solutions satisfy this inequality. Under the conjecture, equation (4) is satisfied and (2) reduces to (1). Therefore, we have only to find pairs (c, V) that satisfy $V = \alpha u(c)+ (1-\alpha)\beta V$ and $c = \beta V$. Upon eliminating V, we get $[\alpha + (1-\beta)/\beta]c = \alpha u(c)$. The assumptions about u imply that there are exactly two solutions: $c = 0$, the no-trade solution, and a positive solution, denoted c^*, which satisfies $c^* < u(c^*)$. The corresponding V's can be found from (3). That for $c = 0$ is zero, while that for c^*, which we denote V^*, is positive. Obviously, the no trade solution satisfies $u(c) \geq \beta V$. And, since $c^* < u(c^*)$, it follows from (3) that $u(c^*) > \beta V^*$. •

Lemma 2. There exists at least one steady state in which both monies are valuable and money B is more valuable than money A.

Proof. As in lemma 1, we use a guess and verify style of argument. We show that equations (1)-(4) have a solution satisfying $c_B > c_A > 0$.

Equation (1) and equation (3) for j = A involve only c_A and V_A. Therefore, as demonstrated in the proof of lemma 1, they have a unique positive solution, (c^*, V^*), which satisfies $u(c^*) > \beta V^*$. Our next task is to show that with $(c_A,V_A) = (c^*,V^*)$, equations (2), (4), and (3) for j = B have a solution for V_B, c_B, and c_{BA} that satisfies the conclusions in the lemma. After imposing that money B is traded for money A and some services and $(c_A,V_A) = (c^*,V^*)$, equation (2) can be written as

$$\text{(A1)} \quad (1-\beta)V_B = \alpha\max[u(c_B) - \beta V_B, 0]+ (p_A/N)[u(c_{BA}) - \beta(V_B - V^*)]$$

Then, letting $x \equiv \beta(V_B - V^*)$ and using (3) and (4), the indifference conditions, we can rewrite (A1), solely in terms of x, as $F(x) = H(x)$, where $F(x) = [(1-\beta)/\beta](x + \beta V^*)$ and $H(x) = \alpha\max[u(x + \beta V^*) - (x + \beta V^*), 0] + (p_A/N)[u(x) - x]$. Since $F(0) = H(0)$-- in effect, the lemma 1 valued money solution-- and $H'(0) = \infty$, it follows that $F(x) < H(x)$ for $x > 0$ and x sufficiently close to 0. If x'' is the unique positive solution for x to $u(x) - x =$

0, which exists given the assumptions about u, then $H(x'') = 0$. Therefore, $F(x'') > H(x'')$. Since F and H are continuous, it follows that there is at least one $x \in (0,x'')$ that satisfies $F(x) = G(x)$. Let x^* denote such an x. Since, by definition of x'', $u(x^*) > x^*$, the trade of money B for money A and the service amount x^* is, in fact, optimizing. •

Lemma 3. If $(1-\beta)/\beta \geq p_A/N$, then there exists exactly one symmetric steady state in which money B is more valuable than money A, both are traded solely for services, and money B is traded for money A and some service.

Proof. Let x' be the unique nonnegative solution for x to $u(x + \beta V^*) - (x + \beta V^*) = 0$. This solution exists and is positive because $u(\beta V^*) > \beta V^*$. Then $F(x') - H(x') = [(1-\beta)/\beta](x' + \beta V^*) - (p_A/N)[u(x') - x'] = [(1-\beta)/\beta]u(x'+\beta V^*) - (p_A/N)u(x') + (p_A/N)x' > 0$, where F and H are as defined in the proof of lemma 2 and where the inequality follows from the hypothesis. Therefore, using the facts about F(0) and H(0) noted in the proof of lemma 2 , there is a solution to $F(x) = H(x)$ with $x \in (0,x')$. By construction, this is a solution where the max term in (A1) is positive. Moreover, since H is strictly concave for $x \in (0,x')$, there is only one such solution. •

Lemma 4. If $\delta_m = \delta_0 = 1$, then there exists a unique solution to (5)-(7) with $(p_1, p_0+p_m, p_{gm}) \in R^3_{++}$. Moreover, the solution for p_{gm} is strictly decreasing in r, strictly decreasing in q if $r > 0$, and satisfies $p_{gm} < G$; while the solution for p_1 is strictly decreasing in r and q.

Proof. Letting $A \equiv 1- N\alpha$ and $p \equiv p_0 + p_m$ and using the hypotheses, (5)-(7) become, respectively,

(A2) $p_1 + p + p_{gm} = A$,

(A3) $p_1 = (1 - G - p - p_1)(p_{gm}/N)q$,

(A4) $p_{gm}(1 - G - p - p_1) = (G - p_{gm})[(1-r)p_1 + p]$

We begin by obtaining a single equation in p_{gm} that must be satisfied by any solution to (A2) - (A4). Solving (A2) for p_1+p and substituting the result into the right-hand side of (A3), we get

(A5) $p_1 = (1 - G - A + p_{gm})qp_{gm}/N$

Then substituting this expression for p_1 into (A2) and solving for p, we get

(A6) $p = A - p_{gm}[1 + (q/N)(1 - G - A + p_{gm})]$

Now substituting from (A5) and (A6) into (A4), we get a single equation in p_{gm} (x from now on) which we write as

(A7) $h(x) = f(x)$

where

(A8) $h(x) \equiv x(1- G - A +x)$

(A9) $f(x) \equiv (G-x)[A - x - r(q/N)x(1- G - A +x)]$

Let $x_0 = \max(0, G + A - 1)$ and let $x_1 = \min(A, G)$. It follows that $x_0 < x_1$. We will show that there is no solution to (A7) with $x \in [0, x_0]$, and that there is a unique solution with $x \in (x_0,x_1)$. Finally, we will show that that unique solution implies positive solutions for p_1 and p.

If $x \in [0, x_0]$, then $h(x) \leq 0$ and $f(x) \geq (G-x)(A-x) > 0$. Hence, there is no solution to (A7) with $x \in [0, x_0]$. As regards existence, we already have $h(x_0) < f(x_0)$. Obviously, $h(x_1) > 0$. As for $f(x_1)$, if $G \leq A$, then $f(x_1) = 0$; while if $G > A$, then $f(x_1) < 0$. Therefore, $h(x_1) > f(x_1)$. Therefore, by continuity of f and g, there exists at least one $x \in (x_0, x_1)$ that satisfies (A7). As regards uniqueness, straightforward calculation of first derivatives of h and f establishes that h is strictly increasing and f is strictly decreasing on (x_0, x_1). Let $x^*(q,r)$ denote the unique solution to (A7). Since $h(x_0) = 0$, it follows that $h[x^*(q,r)] > 0$.

The last step is to show that $p_{gm}= x^*(q,r)$ implies positive solutions for p_1 and p. Since, by (A5), $p_1 = (q/N)h[x^*(q,r)]$ and $h[x^*(q,r)] > 0$, the solution for p_1 is positive. We next show that $x^*(q,r) + (q/N)h[x^*(q,r)] < A$, which, by (A2), assures a positive solution for p. Since for any given $x \in (x_0, x_1)$, f is decreasing in r, it follows that $x^*(q,r)$ and $h[x^*(q,r)]$

are decreasing in r. Therefore, if $x^*(q,0) + h[x^*(q,0)] < A$, then $x^*(q,r) + (q/N)h[x^*(q,r)] < A$. It is immediate from (A7) that $x^*(q,0) = GA$ and $x^*(q,0) + h[x^*(q,0)] = GA(2 - G - A + GA) = A[2G - G^2 - AG(1 - G)]$. Straighforward calculation shows that $[2G - G^2 - AG(1 - G)]$ is strictly increasing in G for $G \in [0,1]$ and equals 1 at $G = 1$. Therefore, $x^*(q,0) + h[x^*(q,0)] < A$, as required.

The claims about the dependence of the solutions for p_{gm} and p_1 on r follow from the fact that f is decreasing in r. Since f is also decreasing in q for $x \in (x_0, x_1)$ and $r > 0$, $x^*(q,r)$ and $h[x^*(q,r)]$ are also decreasing in q for $r > 0$. This and the fact that $x^*(q,0) = GA$ imply the claims about the dependence on q. •

Proposition 1. A necessary and sufficient condition for existence of a steady state in which matured securities exchange at par ($c_0 = c_m$) and newly issued securities exchange at a discount ($0 < c_2 < c_0$) is $r > 0$. Moreover, if $r = 0$, then there is a steady state in which securities always exchange at par ($0 < c_2 = c_0 = c_m$).

Proof of necessity. (If there exists a symmetric steady state with $c_0 = c_m > 0$ and $c_2 < c_m$, then $r > 0$.) We do a proof by contradiction. In particular, we assume $r = 0$ and $c_0 = c_m > 0$ and show that a consequence is $c_2 = c_m$. By (15) and (11), we have only to show that $V_1 = V_m$. With $r = 0$, the right-hand side of (13) becomes $\alpha\max[u(c_0), \beta V_0] + (1-\alpha)\beta V_0$. However, by (11), $c_0 = c_m$ implies $V_m = V_0$, which, by (12), implies $c_{m0} = 0$. It follows that the right-hand side of (10) is also equal to $\alpha[u(c_0), \beta V_0] + (1-\alpha)\beta V_0$. •

Proof of sufficiency. (If $r > 0$, then there exists a (symmetric) steady state with $0 < c_2 < c_1 = c_0 = c_m$. Moreover, if $r = 0$, then there is a steady state in which securities always exchange at par ($0 < c_2 = c_0 = c_m$).) The conjectured trading pattern is very simple. Since one-period securities are equivalent to matured securities to potential recipients, which, in turn, are by hypothesis equivalent to money, there are no trades involving securities on both sides of the transaction. Moreover, we must have $\delta_m = \delta_0 = 1$. Given this trading pattern, Lemma 4 shows that equations (5)-(7) have a unique positive solution for (p_1, p_0+p_m, p_{gm}), a solution that satisfies $p_{gm} < G$.

We next show that (9)-(12) have a solution that satisfies the conclusions. If $c_0 = c_m$ and $V_0 = V_m$, then the right-hand side (9) can be written $\alpha\max[u(c_0), \beta V_0] + (1-\alpha)\beta V_0$. But, then, (9)-(12) correspond exactly to (1)-(4). Therefore, lemma 1 implies that there is a

unique positive solution with $c_0 = c_m = c^*$ and with $V_0 = V_m = V^*$. It remains only to examine the expression for V_1 in (13).

If $r = 0$, then the right-hand side of (13) is $\alpha[u(c_0), \beta V_0] + (1-\alpha)\beta V_0$, which implies that there is steady state with securities always at par. If $r > 0$, then the right-hand side of (13) can be written $\alpha'[u(c_0), \beta V_0] + (1-\alpha')\beta V_0$, where $\alpha' < \alpha$. Since, as shown in lemma 1, $u(c^*) > \beta V^*$, it follows that if $r > 0$, then $V_1 < V^*$. By (15), this implies that if $r > 0$, then $c_2 < c^*$. •

Lemma 5. In any steady state with $V_m > 0$ and $V_0 \leq V_m$, (i) $V_m \geq V^*$ and (ii) $V_0 \geq V^*$, where V^* is the lemma 1 solution for valued money.

Proof. By (10), $V_m \geq \alpha u(c_m) + \gamma\beta V_m + (1-\alpha-\gamma)\beta V_m = \alpha u(c_m) + (1-\alpha)\beta V_m$, where $\gamma = (p_1 + p_0)/N$. Therefore, by (11), $[\alpha + (1-\beta)/\beta]c_m \geq \alpha u(c_m)$. However, this implies that $c_m \geq c^*$, which by (11) gives conclusion (i). As for (ii), since, by hypothesis, $V_m \geq V_0$, it follows from (9) that $V_0 \geq \alpha u(c_0) + (1-\alpha)\beta V_0$. Then, by (11), we have $[\alpha + (1-\beta)/\beta]c_0 \geq \alpha u(c_0)$. Since $c_0 = 0$ is not consistent with $V_m > 0$ (according to (9)), it follows that $c_0 \geq c^*$. Then (11) implies (ii). •

Lemma 6. In any steady state with $V_m > V_0 > 0$ and $p_1 + p_0 > 0$, (i) $u(c_{m0}) + \beta V_0 > \beta V_m$, (ii) $u(c_0) > \beta V_0$, and (iii) $u(c_m) > \beta V_m$.

Proof. Suppose that (i) does not hold. Then, by (10) and lemma 5, $V_m = V^*$. It follows from lemma 5 that $V_0 \geq V_m$, a contradiction.

Conclusions (ii) and (iii) are established by showing that if either does not hold, then the steady state of the law of motion does not have securities outstanding. Suppose, first, that (ii) does not hold. Then, by (11), $c_0 \geq u(c_0)$. Since $c_m > c_0$, it follows that $\beta V_m = c_m > u(c_m)$. Thus, we have the conclusion that neither matured securities nor money are traded for services only. However, that implies that government agents never acquire assets; that is, the right-hand side of (7) is zero. It also implies, by (6), that the left-hand side of (7) is equal to p_1, which, therefore, is zero. Then, by (8), so is the stock of matured securities in private hands. It follows that if (ii) does not hold, then there is no steady state with securities in private hands. Suppose now that (iii) does not hold. Then $\delta_m = 0$. It follows that the left-hand side of (7) is p_1, while the right-hand side is a fraction of p_1. Again, there is no steady state with either p_1 or p_0 or both positive. •

Lemma 7. If $\delta_m = \delta_0 = \delta_{m0} = 1$ and $r = 0$, then the unique positive solution to (5)-(8) satisfies the following: (i) $p_{gm} = G(1\text{-}N\alpha)$; (ii) $p_1 + p_0 = q\alpha(1\text{-}N\alpha)(1\text{-}G)[1 + G(1\text{-}N^{-1})]$; (iii) $p_1 + p_0$, as given in (ii), is continuous and decreasing in G, with limit $q\alpha(1\text{-}N\alpha)$ as $G \rightarrow 0$ and limit 0 at $G \rightarrow 1$.

Proof. Conclusions (i) and (ii) follow from results in the proof of lemma 4 and (8). Conclusion (iii) is immediate from (ii). •

Proposition 2. Assume $r = 0$ (government agents treat one-period securities as do private agents). If G is sufficiently close to 0 and $(1\text{-}\beta)/\beta \geq q\alpha(1\text{-}N\alpha)/N$, then there exists a steady state with matured securities exchanging at less than par ($c_0 < c_m$ and $c_{m0} > 0$) and with newly issued securities valued at less than matured securities ($c_2 < c_0$) and with money traded for services only, and for securities and services.

Proof. Once again, we proceed by guessing and verifying. We conjecture that there is a solution satisfying the conclusions and verify that there is. The proof uses the recursive structure of the conditions in definition 2 that we noted above. We have already described, in lemma 7, the solution for the asset distribution under the conjectured trading pattern. The next step is to show that equations (9)-(12) have a solution with $c_m > c_0 > 0$, a solution that is consistent with the conjectured trading pattern. To do that we rely on the implicit function theorem, lemma 3, and the correspondance between (9)-(12) and (1)-(4) with $(c_0, c_m, c_{m0}, V_0, V_m)$ corresponding to $(c_A, c_B, c_{BA}, V_A, V_B)$.

We first show that the right-hand sides of (9) and (10) are continuous in G and approach the right-hand sides of (1) and (2), respectively, as $G \rightarrow 0$. Since $p_{gm} = G(1 - N\alpha)$, the right-hand side of (9) is continuous in G and approaches $\alpha u(c_0) + (1 - \alpha)\beta V_0$, which is the same as the right-hand side of (1). G appears in (10) implicitly by way of the sum $p_1 + p_0$, which is continuous in G and, as noted above, approaches $q\alpha(1\text{-}N\alpha)$ as $G \rightarrow 0$. Thus, the right-hand side of (10) is continuous in G and approaches the right-hand side of (2) as $G \rightarrow 0$, provided we replace p_A in (2) by $q\alpha(1\text{-}N\alpha)$.

To apply the implicit function theorem to (9)-(12), we need two other conditions. First, the solution at $G = 0$ must be in the interior of the domain. Since the solution in lemma 3 satisfies $0 < c^* = c_A < c_B$, that condition is met. We also need to satisfy the nonvanishing Jacobian condition (the local uniqueness condition) at $G = 0$. That condition is implied by

two features of the solution in lemma 3: first, the solution for c_A in lemma 3, which is c^*, satisfies $[\alpha + (1-\beta)/\beta] > \alpha u'(c^*)$; second, the positive solution to $F(x) = H(x)$ in lemma 3, denoted x", satisfies $F'(x") > H'(x")$.

It follows, then, from the implicit function theorem applied to (9)-(12) under the conjectured trading pattern that for G sufficiently close to zero those equations have a solution in the neighborhood of the lemma 3 solution. We denote that solution $(c_m^*(G), c_0^*(G), c_{m0}^*(G), V_0^*(G), V_m^*(G))$. Since, as established in lemma 3, the lemma 3 solution satisfies the optimization conditions with strict inequalities, it follows from continuity of the solution to (9)-(12) in G that those inequalities hold for the solution to (9)-(12) for G sufficiently close to zero.

The rest of the proof proceeds recursively to find V_1 and c_2. Existence is immediate. The properties of the solution follow from computing $V_0 - V_1$. By (9) and (13), $V_0 - V_1 = [(G - p_{gm})/N][u(c_m) - u(c_0)] + [G - (G - p_{gm})/N]\beta(V_m - V_0)$. Since this is positive when evaluated at the above solution to (9)-(12), it follows that the solution for V_1, denoted $V_1^*(G)$, satisfies $V_1^*(G) < V_0^*(G)$. It follows that the solution for c_2, denoted $c_2^*(G)$, satisfies $c_2^*(G) < c_0^*(G)$. •

References

Aiyagari, S. R., and N. Wallace, 1994. Government transaction policy and the medium of exchange. *Manuscript*, Federal Reserve Bank of Minneapolis.

Bryant, J. B. , and N. Wallace, 1984. A price discrimination analysis of monetary policy. *Review of Economic Studies*, April, 279-88.

Diamond, P., 1990. Pairwise credit in search equilibrium. *Quarterly Journal of Economics*, May, 285-319.

Gherity, J.A., 1993. Interest-bearing currency: evidence from the Civil War experience. *Journal of Money, Credit and Banking* , February, 125-131.

Hendry, S., 1992. Credit in a search model with money as a medium of exchange. *Manuscript*, University of Western Ontario.

Makinen, G. E., and G. T. Woodward, 1986. Some anecdotal evidence relating to the legal restrictions theory of the demand for money. *Journal of Political Economy*, April, 260-65.

Sargent, T. J., and N. Wallace, 1982. The real-bills doctrine versus the quantity theory: a reconsideration. *Journal of Political Economy*, December, 1212-36.

Shi, S. , 1994. Credit and money in a search model with divisible commodities. *Manuscript*, Queen's University.

Shi, S., forthcoming. Money and prices: a model of search and bargaining. *Journal of Economic Theory*.

Trejos, A. and R. Wright, forthcoming. Search, bargaining, money and prices. *Journal of Political Economy* , February, 1995.

Wallace, N., 1988. A suggestion for oversimplifying the theory of money. *Economic Journal* , February, 25-36.