

Notes on Macroeconomic Theory

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Introduction

These pages present static and dynamic analyses of some standard macroeconomic models. By static analysis we mean the analysis of events assumed to occur at a point in time. In effect, statics studies the alternative point-in-time or momentary equilibrium values for a set of endogenous variables associated with alternative possible settings for the exogenous variables at the particular point in time under consideration. Endogenous variables are those determined by the model at hand, while exogenous variables are those given from outside the model.

The task of dynamics is to study the time paths of the endogenous variables associated with alternative possible time paths of the exogenous variables. Thus, in a dynamic analysis, the behavior of a model is studied as time is permitted to pass. In contradistinction, in a static analysis, attention is confined to events assumed to occur instantaneously; i.e., at a given moment.

A third kind of analysis, that of stationary states, is a limiting form of dynamic analysis, and is directed toward establishing the ultimate tendencies of certain endogenous variables, such as the capital-output ratio, as time passes without limit and as certain critical exogenous variables remain constant through time. Stationary analysis ought not to be confused with statics.

The distinguishing feature of a static analysis is that it is capable of determining alternative values of the endogenous variables, taking as given only the values of the exogenous variables at that point in time, which may include values of endogenous and exogenous variables which were determined in the past and are thus given or predetermined at the present moment. As we shall see, some models for which a dynamic

analysis is possible simply cannot be subjected to static analysis. In order to perform static experiments, it is necessary partly to divorce current events from future events so that what happens in the future does not affect what happens now. This requires restricting the way in which people are assumed to form expectations about the future, and in particular requires that people not possess perfect foresight.

Generally, our models will consist of n structural equations in n endogenous variables $y_i(t)$, $i=1, \dots, n$ and m exogenous variables $x_i(t)$, $i=1, \dots, m$:

$$(1) \quad g_i(y_1(t), y_2(t), \dots, y_n(t), x_1(t), \dots, x_m(t)) = 0, \quad i=1, \dots, n.$$

A structural equation summarizes behavior, an equilibrium condition, or an accounting identity, and constitutes a building block of the model. In general, more than one, and possibly all n endogenous variables can appear in any given structural equation. The system of equations (1) will be thought of as holding at each moment in time t . Time itself will be regarded as passing continuously, so that t may be regarded as taking all values along the (extended) real line.

The exogenous variables $x_i(t)$, $i=1, \dots, m$ are assumed to be right-continuous functions of time, and furthermore are assumed to possess right-hand time derivatives of at least first, and sometimes higher order at all points in time. By right-continuity of the functions $x_i(t)$ we mean

$$\lim_{\substack{t \rightarrow \bar{t} \\ t > \bar{t}}} x_i(t) = x_i(\bar{t}),$$

so that $x_i(t)$ approaches $x_i(\bar{t})$ as t approaches \bar{t} from above; i.e., from

the future. However, the function $x_i(t)$ can jump at \bar{t} , so that we do not require

$$\lim_{\substack{t \rightarrow \bar{t} \\ t < \bar{t}}} x_i(t) = x_i(\bar{t}) .$$

For example, consider the function

$$\begin{aligned} x_i(t) &= 0 & , t < \bar{t} \\ x_i(t) &= 1 & t \geq \bar{t} , \end{aligned}$$

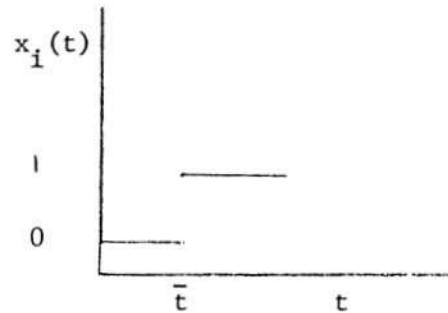


Figure 1

which is graphed in Figure 1. It is right-continuous everywhere even though it jumps; i.e., is discontinuous, at \bar{t} .

The right-hand time derivative of $x_i(t)$, which is assumed to exist everywhere, is defined as

$$\frac{d}{dt} x_i(\bar{t}) \equiv \lim_{\substack{t \rightarrow \bar{t} \\ t > \bar{t}}} \frac{x_i(t) - x_i(\bar{t})}{t - \bar{t}} .$$

For the function graphed in Figure 1, the right-hand derivative is zero everywhere, even though the function jumps and hence isn't differentiable at $t = \bar{t}$.

A model is said to be in static equilibrium at a particular moment if the endogenous variables assume values that assure that equations (1) are all satisfied. Notice that it is not an implication of this definition of equilibrium that the values of the endogenous variables are unchanging through time. On the contrary, since the values of the exogenous variables will in general be changing at some nonzero rates per unit time, the endogenous variables will also be changing over time.

Static analysis is directed toward answering questions of the following form. Suppose that one of the exogenous variables $x_i(t)$ takes

a (small) jump at time \bar{t} so that

$$\lim_{\substack{t \rightarrow \bar{t} \\ t < \bar{t}}} x_i(t) \neq x_i(\bar{t}) .$$

Then the question is to determine the responses of the endogenous variables at \bar{t} . The distinguishing characteristic of endogenous variables is that each of them is assumed to be able to jump discontinuously at any moment in time in order to guarantee that system (1) remains satisfied in the face of jumps in the $x_i(t)$'s. Thus, to be endogenous from the point of view of statics, a variable must be able to change instantaneously.

Notice that it is possible for the right-hand time derivative of a variable to be endogenous, i.e., to be capable of jumping discontinuously, even though the variable itself must change continuously through time

(Figure 2 gives an example). One way to view the difference between the classical and Keynesian models is that in the former the money wage is a variable in static experiments, while in the latter the right-hand time derivative of the money wage is a

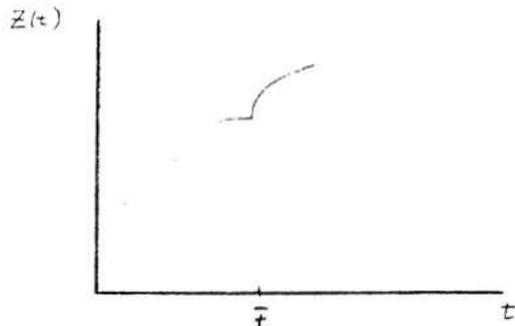


Figure 2: $z(t)$ is continuous, but the right-hand derivative jumps at \bar{t} .

variable but the level of the money wage is exogenous.

To answer the typical question addressed in statics, the reduced form equations corresponding to the system (1) must be found. The reduced form equations are a set of equations, each expressing one

$y_i(t)$ as a function only of the $x_i(t)$'s:

$$(2) \quad y_i(t) = h_i(x_1(t), x_2(t), \dots, x_m(t)) \quad i=1, \dots, n.$$

We will generally assume that the functions $g_i(\)$ in the structural equations (1) are continuously differentiable in all directions, that the n structural equations were satisfied at all moments immediately preceding the moment we are studying and that a certain function of the partial derivatives of (1), evaluated at the immediately preceding values of all variables, is not zero. To be more precise, we shall assume the hypotheses of the implicit function theorem.* Under these hypotheses, there exist continuously differentiable functions of the reduced form (2) which hold for $x_i(t)$'s sufficiently close to the initial (prejump) values of the $x_i(t)$'s. If these equations (2) are satisfied, we are guaranteed that the structural equations (1) are satisfied. For jumps in $x_i(t)$ sufficiently small, i.e., within the neighborhood identified in the implicit function theorem, the equations (2) hold and can be used to answer the characteristic question posed in static analysis. In particular, the reduced form partial derivative

$$(3) \quad \frac{\partial y_i(t)}{\partial x_j(t)} = \frac{\partial h_i}{\partial x_j} (x_1(t), \dots, x_n(t))$$

gives the response of $y_i(t)$ to a jump in $x_j(t)$ that occurs at t . We are generally interested in the sign of the partial derivative of the reduced form.

* See, e.g. Taylor and Mann, Advanced Calculus, Second Edition, p. 363.

Rather than using the implicit function theorem directly to calculate the reduced form partial derivatives (3), it will be convenient to use the following alternative technique that always gives the correct answer. First, take the differential of all equations in (1) to obtain

$$(4) \quad \frac{\partial g_i}{\partial y_1} dy_1 + \dots + \frac{\partial g_i}{\partial y_n} dy_n + \frac{\partial g_i}{\partial x_1} dx_1 + \dots + \frac{\partial g_i}{\partial x_m} dx_m = 0,$$
$$i=1, \dots, n.$$

all partial derivatives being evaluated at the initial values of the x_i 's and y_j 's. Then by successive substitution eliminated dy_2, \dots, dy_n from the above system (4) of linear equations to obtain an equation of the form

$$(5) \quad dy_1 = f_1^1 dx_1 + f_2^1 dx_2 + \dots + f_m^1 dx_m$$

where the f_j^1 's are functions of the partial derivatives appear in (4).

Now equation (5) is the total differential of the reduced form for y_1 , since dy_1 is a function only of dx_1, \dots, dx_m . Taking the differential of the first equation of (2) gives

$$(6) \quad dy_1 = \frac{\partial h_1}{\partial x_1} dx_1 + \dots + \frac{\partial h_1}{\partial x_m} dx_m.$$

From (6) and (5) it therefore follows that

$$f_j^1 = \frac{\partial h_1}{\partial x_j} \quad \text{for } j=1, \dots, n,$$

so that the f_j^1 's are the reduced form partial derivatives. Successive substitution in the system (4) will also, of course, yield the differentials of the reduced forms for the other endogenous variables, thereby enabling us to obtain the corresponding reduced form partial derivatives. The reduced form partial derivatives are often called "multipliers" in macroeconomics.