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CLASSICAL COMPETITIVE ANALYSIS IN A GROWTH ECONOMY WITH SEARCH

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ABSTRACT

Arrow-Debreu competitive equilibrium analysis is extended to environments with information sets differing in space as well as in time and with people moving between locations. Equilibrium is shown to exist and to be optimal and the equilibrium price system is characterized. Such environments include many of those studied in the equilibrium search literature.

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1. Introduction

The role of competitive general equilibrium theory in economics has been continually expanding since its first rigorous formulation in the forties and fifties. Initially it was seen as a static, highly stylized concept. Although the idea of indexing goods by date and location, making them different goods, can be traced back to Lindahl in the twenties and Hicks and Tintner in the thirties, only recently has this theory been taken seriously as a tool to study dynamic problems. The introduction of uncertainty by Arrow and Debreu in the fifties by considering trades of goods contingent on states of nature was a major expansion in the scope of the problems that this theory could address. This approach has become a standard tool of economic analysis in macroeconomics, financial economics and public finance.

Subsequently, beginning in the late sixties there have been a number of important advances in general equilibrium analysis permitting the analysis of economies with large numbers of products and households. These advances (in particular, those of Bewley (1972) and Mas-Colell (1975)) were exploited in an extension of the classical approach to a class of economies with private information (Prescott and Townsend (1984a) and (1984b)). With their approach, traded and priced are incentive compatible contracts in the sense of Hurwicz (1971). These contracts are lotteries on the underlying commodity space.

Other economies in which contracts with lotteries are the appropriate commodity point for competitive analysis include ones with indivisibilities in individual labor supply possibili-

ties. (See R. Rogerson (1988) for this development and G. Hansen (1985) for an application to business cycle modeling.)

The purpose of this paper is to show that an even wider class of economies can be addressed using competitive general equilibrium analysis. In particular we are interested in economies in which people, production, and information are geographically dispersed and in which people and goods can move between locations.[1] This work is in part motivated by the desire to apply classical competitive analysis to the study of search environments previously studied using other techniques. An example of such an environment is the one studied by Lucas and Prescott in their equilibrium search and unemployment paper (1974).[2] It also is motivated by the absence of applied general equilibrium tools to assess the consequences of more timely business and employment statistics. Needed to carry out such analyses are general equilibrium tools that are capable of dealing with environments with informationally decentralized production and consumption.

Key to the extension is the choice of a commodity space which permits the representation of the environment as a Debreu (1954) economy. A point in our commodity space is an infinite sequence of signed measures with elements being indexed by type, date, location, and date-location event. A crucial feature of the environments considered is that the aggregate production possibility sets depends only upon the first moments of these signed measures. This feature preserves standard production theory, with its empirically determined production functions, while permitting rich contractual arrangements between firms and households.

Standard consumer behavior theory, which requires convexity and no private information, however, is not preserved. Section 2 formally describes the environment and represents it as an economy. With this commodity space, production and consumption possibility sets are convex, as are preferences.

In Section 3 a topology for the commodity space is introduced for which the utility functions are continuous and the consumption and production possibility sets closed. In Section 4, by establishing that the set of feasible allocation is compact and nonempty, the set of Pareto optima is shown to be nonempty. In Section 5, the first welfare theorem is proved for our economies. It is a straightforward application of Theorem 1 in Debreu (1954). In Section 6, the second welfare theorem is proved, namely, for our economies any Pareto optimum can be supported as a quasi competitive equilibrium with transfers. A stronger topology than the one previously used is required in order that the aggregate production possibility set have nonempty interior. The fact that the constraints defining this set involve only the first moments of the signed measures is the reason why we can show the aggregate production possibility set has nonempty interior. With this result the proof is an application of Theorem 2 of Debreu (1954).

In Section 6 existence of a quasi competitive equilibrium without transfers is established. The proof, unlike the welfare theorems, is not an application of an existing theorem. It adopts the proof strategy developed by Bewley (1969), McGill (1981), and Mas-Colell (1986) for existence of equilibrium with an

infinite dimensional commodity space. Under additional conditions, a cheaper point for the households is shown to exist. This insures a quasi competitive equilibrium is a competitive equilibrium.

Finally in Section 7 we address the problem of representing the price system. Theorem 1 of Prescott and Lucas (1971) is extended and used to guarantee existence of a quasi competitive equilibrium with prices being the sum of the values of the signed measures that constitute a commodity point. Theorem 7 is another representation result. It shows that equilibrium prices for our economy can be written as linear functions of the first moments of the measures. Furthermore the coefficients of these linear function are derived from marginal rates of substitution and transformation.

2. The Economy

The economy is an infinite-period one, with a continuum of identical agents that is taken to be of measure one. There is a finite number I of agent types. Each type $i \in \{1,2,\dots,I\}$ of agents has Lebesgue measure λ^i. There are L islands and agents can go from island ℓ at date t to island ℓ' at date t + 1. Agents care about leisure and consumption of the produced good. They have standard preferences over such pairs; their endowment is one unit of time per period. There is a neoclassical production function with inputs capital and labor. This function is subject to date-location specific technology shocks. These shocks are observed only on the island they affect; however, next period their values will be known everywhere. The capital once installed

cannot be moved. Investment can flow from one island to another. The consumption good, however, has to be consumed at the same date-location as it is produced.

Consumers choose probabilities over pairs of labor-consumption contingent on available information on each island.

There is only one firm or technology but, with a constant returns of scale technology, one and many firms are essentially the same. This firm unlike the households, chooses quantities of the labor and consumption goods contingent on the history of shocks at each date-location. The way to reconcile the objects that interest the firm and the consumers is by letting the firm choose signed measures over consumption-labor pairs. In particular, it can choose an atomic one.

A further argument is needed to be able to say that an allocation is feasible when consumers choose probabilities and the firm chooses measures--in particular, something that guarantees the realization of the lotteries leads to an ex post distribution of the agents that is the same as the ex ante probabilities. In short we need a law of large numbers. It is well-known (see for example Judd (1985)) that with a continuum of random variables (our case since we have a continuum of consumers) severe measurability problems appear. However, in a very recent paper Uhlig (1987) has proved a version of the L_2 law of large numbers for a continuum of random variables that are identically and independently distributed. This fits our problem permitting us to equate ex ante probabilities to ex post realizations and hence measures of agents. This allows us to talk about feasible and equilibrium

allocations in terms of signed measures on both the consumption and the production sides of the economy.[3]

As of time 0 there is a probability assigned to shocks in each date location. Let's call these shocks $z_{\ell t}$. They have support on a finite set Z. Let $z_t = \{z_{1t},\ldots,z_{L,t}\}$, $z^t = \{z_1,\ldots,z_t\}$, and $h_{\ell t} = \{z^{t-1}, z_{\ell t}\}$. The last element is the information available at (ℓ,t). By Kolmogorov's Extension Theorem there exists a probability space $(\Omega,\sigma(\Omega),\pi)$ with the property that for each (ℓ,t) and possible history $h_{\ell t} \in H_{\ell t}$, $\pi(h_{\ell t})$ is the probability of history $h_{\ell t}$ happening. (We write $\pi(h_{\ell t})$ instead of the cumbersome notation $\pi(\{\omega\colon \mathrm{proj}_{\ell t}(\omega) = h_{\ell t}\})$).

We can identify the set Ω as the set of all possible elements of $\prod_{t=0}^{\infty} Z^L$. Let $\sigma(\Omega)$ be the Borel σ-field generated by Ω.

In the same fashion, let $\sigma(z^t)$ and $\sigma(h_{\ell t})$ be the smallest σ-fields on Ω that make z^t and $h_{\ell t}$ measurable. Clearly $\{\sigma(z^t)\}_{t=0}^{\infty}$ is an increasing family of σ-fields, a filtration on the probability space $(\Omega,\sigma(\Omega),\pi)$.

Properties of these σ-fields are:

$$\sigma(z^{t-1}) \subset \sigma(h_{\ell t}) \subset \sigma(z^t) \text{ and } \sigma(h_{\ell t}) \subset \sigma(h_{\ell',t+1})$$

for all ℓ, ℓ', t. We can think of $\sigma(h_{\ell t})$ as the information available at ℓ,t.

Now for each (ℓ,t) the set $H_{\ell t}$ of possible histories $h_{\ell t}$ is a finite set, and $H = \bigcup_{\ell,t} H_{\ell t}$ is a countable set.

There is an underlying consumption set of the agent. It is C, a closed subset of R^2. Moreover, $C \subset \{[-1,0]\times[0,\bar{c}]\}$ where the first component is the negative of the length of time within

each period devoted to work and hence $(1+c_1)$ is an agent's leisure.[4] The second component is the consumption good where $\bar{c}$ is an upper bound for consumption.

Let M be the set of finite signed measures defined on the Borel sets $\mathbf{B}(C)$ of C, the underlying consumption possibly set. Let $S_{\ell t}(\Omega, \sigma(h_{\ell t}), M)$ be the space of functions $\Omega \to M$ such that they are measurable with respect to $\sigma(h_{\ell t})$.

The <u>commodity space</u> S is:

$$S \equiv \{S_{\ell t}(\Omega, \sigma(h_{\ell t}), M)\}_{\ell=1}^{L} {}_{t=0}^{\infty}.$$

Note that S can be characterized as sequences of signed measures indexed by $h \in H$.

The <u>consumption possibility set</u> X is:

$$X = \{x \in S_+ : x \text{ satisfies } (1)\text{-}(4)\},$$

where the $x_{\ell t}(h_{\ell t}, A)$ are the probabilities of being at island ℓ at date t and consuming some element c belonging to Borel set $A \subset C$ given $h_{\ell t}$. The four conditions are as follows:

(1) The $x_{\ell t}(h_{\ell t}, C)$ are $\sigma(z^{t-1})$ measurable.

This condition requires that the probability of being at (ℓ, t) is not contingent on the actual shock $z_{\ell t}$, as the decision to be at (ℓ, t) is previous to its realization.

(2) $\sum_{\ell} x_{\ell t}(h_{\ell t}, C) = 1$ for all t, all z^t.

(Note z^t defines $h_{\ell t}$ for all ℓ.) This insures that with probability one a person is always in one and only one of the locations.

(3) There exist functions $b_{\ell,t+1,\ell'}$: $\Omega \to \mathbf{R}_+$, measurable with respect to $\sigma(h_{\ell' t})$ for all (ℓ,ℓ',t) such that:

$$\sum_{\ell} b_{\ell,t+1,\ell'}(h_{\ell' t}) = x_{\ell' t}(h_{\ell' t}, C) \text{ for all } (\ell', t, h_{\ell' t}).$$

The $b_{\ell,t+1,\ell'}$ functions are the joint probabilities of being at (ℓ',t) and $(\ell,t+1)$ given $(h_{\ell' t})$. Thus, condition (3) is a consistency property for the probabilities of being in each location every period.

(4) $$\sum_{\ell'} b_{\ell,t+1,\ell'}(h_{\ell' t})$$

$$= x_{\ell,t+1}(h_{\ell,t+1}, C) \text{ for all } (\ell', t, h_{\ell,t+1}).$$

This is another consistency property for the probabilities of being at the different locations. Note that the arguments of the functions on the two sides of (4) are different. This reflects the fact that the decisions to move and to choose labor consumption pairs are based on different information sets. Finally, from conditions (2) and (3) or (2) and (4)

$$\sum_{\ell'} \sum_{\ell} b_{\ell,t+1,\ell'}(h_{\ell' t}) = 1 \text{ for all } t,\ z^t.$$

Preferences of a type i agent are discounted expected utility:

$$E_0\{\sum_{t=0}^{\infty} \beta^t\ U^i(c_t)\},$$

where $0 < \beta < 1$ and U^i: $C \to R_+$ is bounded and strictly increasing.

The <u>aggregate production possibility set</u> Y is:

$Y = \{y \in S$: there exist measurable $k_{\ell t}: Z^{t-1} \to R_+$ and measurable $a_{\ell,t+1,\ell'}: H_{\ell t} \to R_+$ for all (ℓ,ℓ',t) satisfying (5), (6), and (7)$\}$.

In the above $a_{\ell,t+1,\ell'}(h_{\ell t})$ is the amount of the investment good produced at (ℓ,t) contingent on event $h_{\ell t}$ and shipped to location ℓ' for use at time $t+1$ while $k_{\ell t}(z^{t-1})$ is beginning of period capital stock at (ℓ,t) given history z^{t-1}. Condition (5) is

$$k_{\ell t}(z^{t-1}) = k_{\ell,t-1}(z^{t-2}) + \sum_{\ell'} a_{\ell' t\ell}(h_{\ell',t-1}) \tag{5}$$

for all $t \geq 1$, ℓ, and all z^{t-1}.

This is just the law of motion for the capital stock given the initial capital stocks $k_{\ell 0}$. Condition (6) is

$$\int_C c_2\, y_{\ell t}(h_{\ell t},dc) + \sum_{\ell'} a_{\ell',t+1,\ell}(h_{\ell t}) \leq z_{\ell t}\, f\left[k_{\ell t}(z^{t-1}), -\int_C c_1 y_{\ell t}(h_{\ell t},dc)\right] \tag{6}$$

for all $(\ell,t,h_{\ell t})$.

The function f is a constant returns to scale neoclassical production function whose first argument is the capital input and the second is the labor input. Note that c_1 is the first component of C and c_2 the second. Consequently $-\int c_1 y_{\ell t}(h_{\ell t},dc)$ is the event contingent labor input at (ℓ,t) while the left side of (6) is the event contingent production of the consumption good also at (ℓ,t). We assume $f(k,1)$ is bounded. This assumption insures a uniform bound for output, and therefore for consumption. Condition (7) is

(7) $$-\int_C c_1 y_{\ell t}(h_{\ell t}, dc) \geq 0 \text{ for all } (\ell, t, h_{\ell t}).$$

This condition guarantees that labor cannot be an output of the production process.

All agents of the same type choose the same commodity point. This is not a restriction given that there are lotteries as the law of large numbers guarantees that the ex post distribution is the same as the ex ante distribution for each of the I classes of consumers.

An allocation $[(x^i), y]$ is <u>feasible</u> if $x^i \in X$ for all i, if $y \in Y$, and if for all $(\ell, t, h_{\ell t})$,

$$\sum_i \lambda^i x^i_{\ell t}(h_{\ell t}, A) = y_{\ell t}(h_{\ell t}, A) \text{ for all } A \in \mathbf{B}(C).$$

This is the standard requirement that objects in the production side have to equal those in the consumption side.

3. Some Preliminary Mathematical Results

Our sets X and Y are the projections of sets onto S. Formally, let R be

$$R \equiv \{r = \{r_h\}_{h=1}^{\infty} : r_h \in \mathbf{R}^{L+1},\ \sup_{\ell} |r_{\ell h}| < \bar{r}_h \text{ for all } h\}.$$

Define T_1, T_2 as

$$T_1 \equiv \{(s,b,0) \in S_+ \times R : (s,b) \text{ satisfies (1) to (4)}\}$$

$$T_2 \equiv \{(s,a,k) \in S \times R : (s,a,k) \text{ satisfies (5) to (7)}\}.$$

Then

$$X = \mathrm{Proj}(T_1 \text{ onto } S)$$

$$Y = \mathrm{Proj}(T_2 \text{ onto } S).$$

The sets T_i are convex and closed in the product topology over sequences whose components have the weak* topology. The reason is that they are defined as the intersection of a countable number of closed and convex constraints.

In this topology the set R is compact. This insures the projections of closed subsets of $S \times R$ onto S are closed. Projections of convex sets are convex. All this can be summarized in the following Lemma.

Lemma 1: The sets X and Y are convex and are closed in the product topology with the weak* topology for components.

Regarding preferences, the following result holds:

Lemma 2: Preferences can be represented by u^i: $X \to R_+$,

$$u^i(x) = E_0\{\sum_{t=0}^{\infty} \beta^t \int_C U^i(c)[\sum_{\ell} x_{\ell t}(h_{\ell t}, dc)]\}.$$

Furthermore, in the product topology generated by the weak* topology on M(C), u^i is a continuous and linear function.

Proof: See Appendix. □

4. Existence of Pareto Optimal Allocations

Theorem 1: For any $\mu \in \mathbf{R}_+^I$ with $\sum_i \mu^i = 1$ the problem

$$\max_{[(x^i),y]} \sum_i \lambda^i \mu^i u^i(x^i)$$

subject to feasibility, has a solution. The solutions are necessarily Pareto optimum allocations if $\mu^i > 0$ for all i.

Proof: By Theorem 6-4 of Parthasarathy (1967), a version of the Banach-Alaoglu Theorem, the set

$$T_h \equiv \{s_h \in M_+(C): \int_C ds_h \le 1\}$$

is weak* compact. By Tychonov's Theorem,

$$T \equiv \{(s,b) \in S \times R: s_h \in T_h \text{ for all } h\}$$

is compact in the product topology over components. T_1 is a closed subset of T, a compact set, hence compact itself. The projection operator is continuous, hence X is a compact set.

From our definition of feasibility, allocation $[(x^i),y]$ being feasible implies $x^i \in X$ all i, $y \in Y$, and $y = \sum_i \lambda^i x^i$. Convexity of X implies $y \in X$. So $[(x^i),y]$ being feasible implies $x^i \in X$ for all i and $y \in X \cap Y$. Y is closed, X is compact; hence, $X \cap Y$ is compact. Applying Tychonov's Theorem again, the set of feasible allocations i.e., $\{X^I \times X \cap Y\}$ is compact in the topology being used.

Lemma 2 shows u^i is a continuous function. Then $\sum_i \mu^i \lambda^i u^i(x^i)$ is also a continuous function since $\mu, \lambda \in \mathbf{R}^I$.

Continuous real functions over compact sets achieve a maximum. □

5. Optimality of Competitive Equilibria

A competitive equilibrium is a feasible allocation $[(\bar{x}^i),\bar{y}]$ together with a price system v (i.e., a nontrivial linear functional on S) for which:

(i) For all i, $x \in X$ and $u^i(x) > u^i(\bar{x}^i)$ implies $v(x) > v(\bar{x}^i)$.

(ii) $y \in Y$ implies $v(y) \leq v(\bar{y})$.

Condition (i) is utility maximization subject to a budget constraint while condition (ii) is profit maximization subject to a technology constraint.

<u>Theorem 2</u>: If the allocation $[(\bar{x}^i), \bar{y}]$, for $i = 1, \ldots, I$, together with the price functional v, is a competitive equilibrium then it is a Pareto optimal allocation.

<u>Proof</u>: To establish optimality of competitive equilibria we apply Theorem 1 of Debreu (1954). The theorem is that competitive equilibria are necessarily Pareto optima if the X are convex, no x^i is a satiation point and preferences are convex. By Lemma 1 the X are convex. The point x^s which places mass one on the point set $\{(0,\bar{c})\}$ for all $(\ell,t,h_{\ell t})$ is the only satiation point. Given our assumptions no allocation that places mass one on $\bar{c}$ for any type for any $(\ell,t,h_{\ell t})$ is feasible. Consequently feasibility rules out satiation for any type. The convexity of preferences property is that x', $x'' \in X$ and $u^i(x') > u^i(x'')$ implies $u^i[\alpha x'+(1-\alpha)x''] > u^i(x'')$ for any $\alpha \in (0,1)$. This is an immediate result of Lemma 2. □

Debreu's Theorem does not require S to be a topological space and therefore price system v to be continuous. An alternative approach would be to introduce a topology, modify Debreu's definition of competitive equilibrium to require continuity of the valuation function, and then establish local nonsatiation. Optimality of competitive equilibrium then follows by the standard argument.

For technical reasons the underlying consumption set C was constrained to be bounded. Our theory, however, can also be applied to some environments in which this set is not bounded. Whenever the utility possibility frontier for the n types is not reduced by introducing some sufficiently large bound on consumption, the theory is applicable. For example, if $C = [-1,0] \times [0,\infty)$ and for fixed c_1 the U^i are strictly concave in c_2, any

$$\bar{c} > \sup_{z,k,i} \frac{zf(k,1)}{\lambda_i} < \infty$$

is an appropriate bound.

To see why this is the case, note consumption $\bar{c}$ is a bigger number than that which can be feasibly consumed by all type i people. Still it is possible for agents of type i to consume an x that at some $(\ell,t,h_{\ell t})$ puts some positive probability on an amount bigger than $\bar{c}$, and hence necessarily also puts positive probability on an amount smaller than $\bar{c}$. But given the fact that agents are risk averse, i.e., the U^i are strictly concave in c_2, that allocation will not be Pareto optimal since it is dominated by another that puts probability one on the first moment of the former. This new allocation is feasible (first moments do not change). All economies with a $\bar{c}$ that satisfies the above property will share Pareto optimal allocations and, as Theorem 2 shows, competitive equilibria.

6. The Second Welfare Theorem

Debreu (1954, Theorem 2) establishes that if S is a linear topological space, Pareto optimum allocations $[(\bar{x}^i),\bar{y}]$ for which no $\bar{x}^i$ is a satiation point can be supported as a quasi

competitive equilibrium if the following five conditions are satisfied:

I. X is a convex set.

II. For x', $x'' \in X$, and for all i

$$u^i(x') < u^i(x'') \text{ implies } u^i(x') < u^i(x^\alpha)$$

where $x^\alpha = \alpha x' + (1-\alpha)x''$ for $\alpha \in (0,1)$.

III. For all x, x', $x'' \in X$ and for all $\alpha \in [0,1]$ the set $\{\alpha \in [0,1]: u^i(x^\alpha) \leq u^i(x)\}$ is closed, where x^α is as before.

IV. Y is a convex set.

V. Y has an interior point.

Theorem 3: Any Pareto optimal allocation $[(\bar{x}^i),\bar{y}]$ can be supported as a quasi competitive equilibrium; that is, there exists a nontrivial continuous linear functional v such that:

(i) For all i, $x \in X$ and $u^i(x) \geq u^i(\bar{x}^i)$ implies $v(x) \geq v(\bar{x}^i)$.

(ii) $y \in Y$ implies $v(y) \leq v(\bar{y})$.

Proof: With the product topology which was used to establish the existence of optima, Y has an empty interior. Hence another topology is needed for application of Debreu's Theorem 2. The topology used in the one induced by the following norm:

$$\|s\| = \sup_{h \in H}\{\|s_h\|_M\} = \sup_{h \in H}\{\sup_{\substack{g \in \mathbf{C}(C) \\ \|g\|_\infty = 1}} |\int_C g \, ds_h|\}.$$

Note that the term in braces is the usual norm for signed measures, with C(C) being the space of continuous bounded functions on C. The topology induced by this norm is the topology of uniform convergence of sequences of signed measures.

Conditions I and IV are Lemma 1. Condition II is the immediate consequence of the linearity of u^i as established in Lemma 2. Continuity of u^i with respect to a weaker topology as established in Lemma 2, implies continuity with respect to this stronger topology. Continuity of the u^i is a stronger condition than condition III.

To prove condition V, we will show a point y^0[5] is in the interior of Y, where for all h

$$y_h^0(A) = 1 \text{ if } (-1/2,0) \in A$$

and zero otherwise. To show $y^0 \in Y$, let a^0 be identically zero and k^0 be such that $k_{\ell t}^0(h_{\ell t}) = k_{\ell 0}$ for all h. Since $(y^0,k^0,a^0) \in T_2$, $y^0 \in Y$.

Let

$$N_1 \equiv \{s_h \in M(C): |\int c_1 ds_h - \int c_1 dy_h^0| < 1/4\}$$

$$N_2 \equiv \{s_h \in M(C): |\int c_2 ds_h - \int c_2 dy_h^0| < \varepsilon\}$$

where

$$\varepsilon = \inf_{\ell,z} z\, f(k_{\ell 0}, 1/4).$$

Sets N_1 and N_2 are weak* neighborhoods of y_h^0. Since the topology induced by $\| \ \|_M$ is a stronger one, there exists an $\varepsilon' > 0$ such that

$$O_h(y_h^0) = \{s_h \in M(C): \|s_h - y_h^0\|_M < \varepsilon'\} \subset N_1 \cap N_2.$$

This ε' is the same for all h. We will show that the open set

$$O(y^0) = \{s \in S: \|s - y^0\| < \varepsilon'\}$$

is contained in Y.

If $s \in O(y^0)$ then for all h, $s_h \in O_h(y_h^0)$ which implies $s_h \in N_1 \cap N_2$. Thus

$$\int -c_1 ds_h \in (1/4, 3/4)$$

and

$$|\int c_2 ds_h| < \varepsilon.$$

This, in turn, implies (s, k^0, a^0) satisfies (5)-(7) and hence $s \in Y$. This completes the proof. □

<u>Comment</u>: To support the optimum, the transfer must be

$$\psi^i = v(\bar{x}^i) - \theta^i v(\bar{y})$$

where θ^i is the share of the firm owned by type i (note $\lambda \cdot \theta = 1$). A household of type i maximizes $u^i(x^i)$ subject to $x^i \in X$ and to

$$v(x^i) \leq \psi^i + \theta^i v(\bar{y}).$$

Market clearing insures $\lambda \cdot \psi = 0$.

Theorem 3 guarantees the existence of a quasi competitive equilibrium with transfer payments. In the next theorem we show that a quasi competitive equilibrium with zero transfers also exists. The argument proceeds by constructing a correspondence χ: $M \to M$, where $M \subset R^I$ is compact, and χ is convex, upper hemi-

continuous (uhc), and compact, and then applying Kakutani's Fixed Point Theorem. The argument is in the spirit of Bewley (1969), Magill (1981), and Mas-Colell (1986).[6]

Theorem 4: For the class of economies studied, there exists a quasi equilibrium, i.e., a feasible allocation $[(\bar{x}^i),\bar{y}]$, and price system v_1 such that:

$$u^i(x) \geq u^i(\bar{x}^i) \text{ implies } v(x) \geq \theta^i v(\bar{y}) \text{ for all } i,$$

and

$$y \in Y \text{ implies } v(y) \leq v(\bar{y}).$$

Proof: See Appendix. □

We still have not shown the existence of a competitive equilibrium. We will show its existence by applying the well-known result (see Debreu (1954)) that if every agent has a cheaper point in his budget set, then the quasi competitive equilibrium is a competitive one. In the following theorem we give sufficient conditions on the Pareto optima to guarantee that every agent has a cheaper point. These conditions are that agents do not consume the least desirable point in the underlying consumption set C for all $(\ell,t,h_{\ell t})$. The argument establishes a contradiction between profit maximization and lack of existence of a cheaper point.

Define X^ε as

$$X^\varepsilon \equiv \{x \in X\colon \int (1+c_1)dx_h<\varepsilon \text{ and } \int c_2 dx_h<\varepsilon \text{ for some } h\in H\}.$$

Theorem 5: If allocation $[(\bar{x}^i),\bar{y}]$ and nontrivial continuous linear functional v is a quasi equilibrium such that there exists

$\varepsilon > 0$ with the property that for all i, $\bar{x}^i \notin X^\varepsilon$, then $\{[(\bar{x}^i),\bar{y}],v\}$ is a competitive equilibrium.

<u>Proof</u>: Let $x_h^*(A) = 1/L$ if $(-1,0) \in A$ and zero otherwise for all h. Note $x^* \in X$. Consider the allocation $[(x^i),y^*]$ where $x^i = x^*$ for some i, $x^j = \bar{x}^j$ for $j \neq i$ and $y^* = \sum_{j\neq i} \lambda^j\bar{x}^j + \lambda^i x^*$. Since $\bar{x}^i \notin X^\varepsilon$ for some $\varepsilon > 0$, type i are consuming less per capita and supplying more labor per capita for x^* than for $\bar{x}^i$ for all h. Consequently $y^* \in Y$. Profit maximization implies

$$v(y^*) \leq v(\bar{y}).$$

Linearity of v implies

$$v(y^*) = \lambda^i v(x^*) + \sum_{j\neq i} \lambda^j v(\bar{x}^j).$$

Hence $v(x^*) \leq v(\bar{x}^i)$.

Note $v(x^*) = v(\bar{x}^i)$ implies $v(y^*) = v(\bar{y})$, which will be shown to be inconsistent with profit maximization. The nature of the argument is to show y^* is an interior point of Y. If $v(y^*) = v(\bar{y})$, given v is nontrivial, there would be some point y in every neighborhood of y^* with the property that $v(y) > v(y^*)$.

A neighborhood of y^*, $O(y^*) \subset Y$ is constructed as follows:

Two weak* neighborhoods of y_h^* are those associated with continuous functions c_1 and c_2 and bounds $\varepsilon\lambda^i/2L$. By the same reasoning as in the proof of condition V of Theorem 3, there exists an open set $O(y^*)$ in the topology induced by the norm on S such that $y \in O(y^*)$ implies that for all h,

$$|\int c_1 dy_h - \int c_1 dy_h^*| < \frac{\lambda^i \varepsilon}{2L}$$

and

$$|\int c_2 dy_h - \int c_2 dy_h^*| < \frac{\lambda^i \varepsilon}{2L}.$$

Since

$$\int c_1 d\bar{y}_h - \int c_1 dy_h^* > \frac{\lambda^i \varepsilon}{L}$$

and

$$\int c_2 d\bar{y}_h - \int c_2 dy_h^* > \frac{\lambda^i \varepsilon}{L},$$

and

$$\int c_1 d\bar{y}_h - \int c_1 dy_h > \frac{\lambda^i \varepsilon}{2L}$$

and for all h

$$\int c_2\ d\bar{y}_h - \int c_2\ dy_h > \frac{\lambda^i \varepsilon}{2L}.$$

Note that $(y,\bar{k},\bar{a}) \in T_2$, where $\bar{k}$, $\bar{a}$ are the functions associated with the quasi equilibrium allocation $\bar{y}$. Therefore $O(y^*) \subset Y$. Hence $v(x^*) < v(\bar{x}^i)$ for all i. This concludes the proof. □

7. Price Representation

The equilibrium price v lies in the dual of the commodity space S, a space difficult to characterize. In Theorem 6 we show that there exists a quasi equilibrium price functional p that lies in the predual i.e., the space of sequences of continuous bounded functions on C whose norms are summable. This price functional p has a dot product representation which agrees with our intuitions of what prices should be, namely they are rooted in marginal conditions.

Prescott and Lucas (1972) develop sufficient conditions for existence of price systems with dot product representations

for commodity spaces whose elements are sequences of members of normed linear spaces.

Their requirements are that any feasible point can be truncated; i.e., if a point is feasible so is any other that is equal to it in its first T components and zero thereafter. In addition they require that agents discount consumption in distant states. Since an allocation in our economy includes a set of probabilities for consumers, truncating with the zero measure is not feasible and we cannot apply their result. In the following result we generalize their Theorem 1 to permit truncation with other than the zero element. For this theorem, the aggregate endowment, ω, is not assumed to be zero, and there are a finite number of technologies indexed by j.

Let S be the linear space $\|s\| = \sup_h \|s_h\|_H$. For any $s \in S$, let $s_{\leq T}$ be such that its first T components are those of s and the latter zero. Let $s_{>T} \equiv s - s_{\leq T}$.

<u>Theorem 6</u>: Suppose allocation $[(\bar{x}^i),(\bar{y}^j)]$ and nontrivial continuous linear functional v is a competitive equilibrium for an economy for which no $\bar{x}^i$ is a satiation point. Suppose in addition to conditions I and II that there exist $\hat{x}^i \in X^i$ for all i, and $\hat{y}^j \in Y^j$ for all j such that:

VI. For all $x^i \in X^i$, all T, $x^i_T \equiv x^i_{\leq T} + \hat{x}^i_{>T} \in X^i$. For all $y^j \in Y^j$, all T, $y^j_T \equiv y^j_{\leq T} + \hat{y}^j_{>T} \in Y^j$.

VII. If x^i is strictly preferred to $x^{i'}$ by agent i, then there exists T' such that for T > T', x^i_T is strictly preferred to $x^{i'}$.

VIII. $\sum_i \lambda^i \hat{x}^i = \sum_j \hat{y}^j + \omega$.

Then

$$q(s) \equiv \lim_{T\to\infty} v(s_{\leq T}),$$

along with $[(\bar{x}^i),(\bar{y}^j)]$ is a quasi equilibrium for this economy.

Proof: See Appendix. □

The final theorem uses this result to establish existence of a specific type of representation of price system. First it allows for a price to be represented as a dot product, i.e., the value of a commodity point is the sum of the values of its date-location-event components. Second, within each date-location-event component the value depends only on the first moments of the signed measures with respect to both the consumption good and time, in a way that their relative values coincide with the marginal rates of substitution and transformation.

Theorem 7: Suppose allocation $[(\bar{x}^i),\bar{y}]$ and linear functional v are a competitive equilibrium then there exists a continuous linear functional p of the form

$$p(s) = \sum_h \int (p_{1h}c_1+p_{2h}c_2)ds_h$$

such that $[(\bar{x}^i),\bar{y}]$ together with p are a quasi competitive equilibrium.

Proof: Note that assumption VI holds for our economy for $\hat{x}^i_h$ = $(\{(-1,0)\})$ = 1/L for all i and all h, and $\hat{y}_h(\{(-1,0)\})$ = 1/L for all h. Discounting takes care of assumption VII and $\hat{x} = \hat{y}$ which

suffices for assumption VIII. Then Theorem 6 guarantees that a q exists and that it together with $[(\bar{x}^i),\bar{y}]$ is a quasi equilibrium.

A result used in this proof and proven in the Appendix is:

Lemma 3: Suppose v, p are continuous, nontrivial linear functionals on topological linear space S. Suppose $\bar{s} \in \text{Argmax } v(s)$ subject to $p(s) \leq p(\bar{s})$. Then there exists $\alpha > 0$ such that $\alpha p(s) = v(s)$, for all $s \in S$.

Define for all h the set

$$Y_h \equiv \{y \in Y: y_{h'} = \bar{y}_{h'} \text{ for all } h' \neq h\}.$$

Consider the following program:

$$\max_y q(y) \text{ subject to } y \in Y_h.$$

Element $\bar{y}$ solves it.

Let $s^{(h)} \equiv s_{\leq h} - s_{\leq h-1}$, then $q(s) = \sum_h v(s^{(h)})$. Define $v_h(s_h) \equiv v(s^{(h)})$. Since for no h agents put mass on the most preferred point in C, utility maximization guarantees v_h is nontrivial. Consider also the program:

$$\max_{s_h} v_h(s_h)$$

subject to:

$$\int_C (c_2 + w_h c_1) ds_h \leq \int_C (c_2 + w_h c_1) d\bar{y}_h$$

where $w_h = z_h f_2[\bar{k}_h, -\int c_1 d\bar{y}_h]$. Suppose $\bar{y}_h$ does not solve it. Then there exists s_h such that $v_h(s_h) > v_h(\bar{y}_h)$ and

$$\int_C (c_2 + w_h c_1) ds_h \le \int_C (c_2 + w_h c_1) d\bar{y}_h.$$

Then there exists a $\delta \in (0,1)$ such that $v_h(\delta s_h) > v_h(\bar{y}_h)$ and

$$\int_C (c_2 + w_h c_1) d(\delta s_h) < \int_C (c_2 + w_h c_1) d\bar{y}_h.$$

Given the value of w_h and the properties of the production function, there exists a $\gamma \in (0,1)$ such that for

$$s_h^\gamma \equiv \gamma \delta s_h + (1-\gamma)\bar{y}_h,$$

$v_h(s_h^\gamma) > v_h(\bar{y}_h)$. Furthermore, the set $\{y \in S\colon y_h^\gamma = s_h^\gamma$ and $y_{h'} = \bar{y}_{h'}$ for $h' \neq h\}$ is contained in Y_h. But this contradicts the fact that $\bar{y}$ solves the previous program, hence $\bar{y}_h$ solves this one.

Since v_h and the constraint are both nontrivial continuous linear functionals on a topological linear space, for all h, Lemma 3 implies that there exists $\alpha_h > 0$ such that

$$v_h(s_h) = \alpha_h \int (c_2 + w_h c_1) ds_h.$$

Since q(s) is well-defined,

$$p(s) = \sum_h \int (p_{1h} c_1 + p_{2h} c_2) ds_h$$

is also well-defined where $p_{1h} = \alpha_h w_h / \alpha_1$, $p_{2h} = \alpha_h / \alpha_1$. (Note that nontriviality of v_h implies $\alpha_h \neq 0$ all h.) Since q(s) was a quasi equilibrium, so is p(s). □

Recall that Theorem 4 gives sufficient conditions for a quasi equilibrium to be a competitive one regardless of whether or not the price function has a dot product representation.

Appendix

<u>Proof of Lemma 2</u>: Existence, continuity and linearity of the utility function defined. Let

$$u(x) \equiv E_0\{\sum_{t=0}^{\infty} \beta^t \int_C U(c)[\sum_{\ell} x_{\ell t}(h_{\ell t},dc)]\}.$$

We must show first the expectation operator is well-defined.

First define $g_T\colon \Omega \to R$ as:

$$g_T(\omega) \equiv \sum_{t=0}^{T} \beta^t \sum_{\ell} \int_C U(c)\ x_{\ell t}(h_{\ell t}(\omega),dc) > 0.$$

Now given that up to t the set of possible histories is a finite set, we have that:

$$u_{\leq T}(x) \equiv E_0\{g_T(\omega)\}$$

$$= \sum_{t=0}^{T} \beta^t \sum_{\ell} \sum_{h_{\ell t}\in H_{\ell t}} \pi(h_{\ell t}) \int_C U(c)\ x_{\ell t}(h_{\ell t}(\omega),dc) > 0$$

where again $H_{\ell t}$ is the set of possible histories up to (ℓ,t) and $\pi(h_{\ell t})$ is the probability of $h_{\ell t}$. Now for each ω, $g_T(\omega) \to g(\omega)$, where

$$g(\omega) = \sum_{t=0}^{\infty} \beta^t \sum_{\ell} \int_C U(c)\ x_{\ell t}(h_{\ell t}(\omega),dc) > 0,$$

since there is discounting, C is compact and U(c) is both non-negative and bounded, this limit exists.

Now since for all ω, $g_T(\omega)$ is bounded above by $\psi = \sum_{t=0}^{\infty} \beta^t \sum_{\ell} \sup_{c\in C} U(c)$, we apply Lebesgue Dominated Convergence Theorem (see Wheeden and Zygmund (1977), Theorem 5.19) to conclude that:

$$u(x) = E_0\left[\lim_{T\to\infty} g_T(\omega)\right]$$

$$= \lim_{T\to\infty} u_{\leq T}(x)$$

$$= E_0\left\{\sum_{t=0}^{\infty} \beta^t \int_C U(c) \sum_{\ell} x_{\ell t}[h_{\ell t}, dc]\right\}.$$

Hence, $u(x)$ is a well-defined function.

We now turn to continuity. Recall that for each $(\ell, t, h_{\ell t})$ the topology used is the weak* while over the product across dates, locations and histories the topology is the product one. So $x^n \to x^0$ means the convergence is point wise.

The argument of the proof is to show that for all $\varepsilon > 0$ there exists an N such that for any $n > N$:

$$|u(x^n) - u(x^0)| < \varepsilon.$$

Now let $u_{>T}(x) = u(x) - u_{\leq T}(x)$. By the triangle inequality it follows that

$$|u(x^n) - u(x^0)| \leq |u_{\leq T}(x^n) - u_{\leq T}(x^0)|$$

$$+ |u_{>T}(x^n) - u_{>T}(x^0)|.$$

But notice that for any $x \in X$,

$$0 \leq u_{>T}(x) \leq \frac{\beta^{T+1}}{1 - \beta} \max_{c \in C} U(c).$$

Consequently, for any $\epsilon > 0$ there exists T such that for all $x \in X$ and hence for x^0 and the x^n,

$$u_{>T}(x) < \frac{\varepsilon}{2}.$$

This implies $|u_{>T}(x^0) - u_{>T})x^n| < \varepsilon/2$, for all n.

Now, fixed T, $u_{\leq T}(x)$ is a finite sum of continuous functions, so we can always choose N_ϵ such that for any $n > N_\epsilon$

$$|u_{\leq T}(x^0) - u_{\leq T}(x^n)| < \frac{\varepsilon}{2}.$$

Hence $|u(x^0)-u(x^n)| < \varepsilon$ for $n \geq N$. This establishes continuity of the u functions.

Linearity of u(x) follows trivially. □

Proof of Theorem 4: In Step 1 we show that any allocation in $\Gamma(\mu)$, the set of solutions to the social planner problem, can be supported as a quasi equilibrium with transfers (QET) by all the members of certain convex, nonempty set of valuation functions $V(\mu)$. In Step 2 we construct a function that maps the unit simplex into the real line and show it is continuous. Associated with any point in the range of this function it will be shown there is a Pareto optimal allocation. Since every Pareto optimal allocation is the solution to some social planner's problem, we can support it as a QET. In Step 3 we construct a transfers correspondence and show it is compact, nonempty and convex. In Step 4 we show it is uhc and finally in Step 5 that a suitable transformation of it has a fixed point that corresponds to a quasi equilibrium with zero transfers.

Step 1. Consider the following program

$$\max_{x^i \in X^i} \sum_i \mu^i \lambda^i u^i(x^i)$$

subject to

$\sum_i \lambda^i x^i = x.$

This problem has a solution (by Theorem 1) and the value of the objective is a continuous function $U(\mu,x)$. We define an economy with one agent and with preferences $U(\mu,x)$ and technology set Y. Its set of Pareto optima includes the solutions to

$$\max_{x \in X \cap Y} U(\mu,x).$$

Any solution to the social planner's problem of the original economy implies a solution to this one where $x = y$. The following lemma is an important intermediate step:

<u>Lemma</u>: Let $\bar{x}$, $\bar{\bar{x}}$ be two different solutions to the one agent economy social planner problem and let $\bar{v}$ support $\bar{x}$ as a QET, then $\bar{v}$ also supports $\bar{\bar{x}}$.

<u>Proof</u>: Since $\bar{x}$, $\bar{\bar{x}}$ solve the problem, $U(\mu,\bar{x}) = U(\mu,\bar{\bar{x}})$, $\bar{x}$, $\bar{\bar{x}} \in Y$, and $\bar{v}$ supports $\bar{x}$, $\bar{v}(\bar{x}) = \bar{v}(\bar{\bar{x}})$. Then $U(\mu,x) \geq U(\mu,\bar{x})$ implies $\bar{v}(x) \geq \bar{v}(\bar{x}) = \bar{v}(\bar{\bar{x}})$ and $y \in Y$ implies $\bar{v}(y) \leq \bar{v}(\bar{x}) = \bar{v}(\bar{\bar{x}})$, hence $\bar{v}$ supports $\bar{\bar{x}}$. $\square$

Arbitrarily select $[(\bar{x}^i),\bar{y}] \in \Gamma(\mu)$ with associated allocation in the one agent economy $\bar{x} = \bar{y}$. Theorem 3 guarantees the existence of a nontrivial $\bar{v} \in S^*$ that supports it as a QET in the one agent economy, i.e., $U(\mu,x) \geq U(\mu,\bar{x}) = \sum_i \mu^i \lambda^i u^i(\bar{x}^i)$ implies $v(x) \geq v(\bar{x}^i)$ and $y \in Y$ implies $v(y) \leq v(\bar{x})$. Pick any $(\tilde{x}^i)_{i \in I}$ with the property that for $i \neq j$, $\tilde{x}^i = \bar{x}^i$ and $\tilde{x}^j$ is such that $u^j(\tilde{x}^j) \geq u^j(\bar{x}^j)$. Let $\tilde{x} = \sum_i \lambda^i \tilde{x}^i$. As

$$\sum_i \lambda^i \mu^i u^i(\tilde{x}^i) \geq \sum_i \lambda^i \mu^i u^i(\bar{x}^i)$$

it follows that $U(\mu,\tilde{x}) \geq U(\mu,\bar{x})$ and this in turn that $\bar{v}(\tilde{x}) \geq \bar{v}(\bar{x})$, $\sum_i \lambda^i \bar{v}(\tilde{x}^i) \geq \sum_i \lambda^i \bar{v}(\bar{x}^i)$ and $\bar{v}(\tilde{x}^j) \geq \bar{v}(\bar{x}^j)$. Since the technology sets are the same, $\bar{v}$ supports $[(\bar{x}^i),\bar{y}]$ as QET in the original economy.

Now define $V(\mu) = \{v \in S^*: v(\hat{x})=1$ ($\hat{x}$ the most preferred point in X) v supports $\bar{x}$ as QET for the one agent economy where $\bar{x}$ is associated with some $[(\bar{x}^i),\bar{y}] \in \Gamma(\mu)$, i.e., $\bar{x}=\bar{y}\}$. Note that, as we have shown, if $v \in V(\mu)$ supports some solution to the social planner problem, it supports them all. The sets $V(\mu)$ are closed and convex. To show it is nonempty, note that Theorem 3 guarantees the existence of a nontrivial supporting v. As αv also supports for $\alpha > 0$, we only have to show $v(\hat{x}) > 0$. In order to do this consider $\underset{\wedge}{x}$, the worst possible point. It is the one that puts all measure on $(-1,0) \in C$, $\underset{\wedge}{x} \in Y$. Since $0 \in Y$ and $\bar{y}$ maximizes profits, $v(\bar{y}) \geq 0$. Note that $(\bar{y}+\alpha\underset{\wedge}{x}) \in \text{Int } Y$ (to show that, it is easy to construct the right neighborhood) and profit maximization precludes then $v(\underset{\wedge}{x}) \geq 0$. Consider $x^\alpha = \alpha\underset{\wedge}{x} + (1-\alpha)\hat{x}$. There exists $\alpha \in (0,1)$ with the property $U(\mu,x^\alpha) = U(\mu,\bar{x})$ which implies

$$\alpha v(\underset{\wedge}{x}) + (1-\alpha)v(\hat{x}) = v(x^\alpha) \geq v(\bar{x}) = v(\bar{y}) \geq 0.$$

But, since $v(\underset{\wedge}{x}) < 0$ it is true that $v(\hat{x}) > 0$. Normalizing v so that $v(\hat{x}) = 1$, the nonemptiness of $V(\mu)$ is established.

<u>Step 2</u>. Next we construct the following function α: $\Delta^I \to R_+$ where $\alpha(\mu) = \max\{\alpha \in \mathbf{R}: \alpha\mu \in U\}$. Here U denotes the set of feasible utilities, that have been normalized so that $u^i(\underset{\wedge}{x}) = 0$ all i and $u^i(\hat{x}) < 1$.

Note that associated with a point $u \in U$ there exists a feasible allocation. Convexity of X, Y, and preferences imply U is convex. U is trivially bounded. To show it is closed, pick any convergent sequence $\{u_n\}$, $u_n \in U$. Associated with it there is a sequence of allocations that belong to a weak* compact set and hence a limit of a subsequence exist. Continuity of the utility functions in this topology guarantees that the limit of $\{u_n\}$ is in U. Hence U is closed. This implies $\alpha(\mu)$ is well-defined. The fact that $\alpha(\mu) > 0$ is obvious from the fact that for all i, $u^i(x_0) > u^i(\hat{x})$, where x_0 puts 1/L mass on the point $(0,0) \in C$.

To show $\alpha(\mu)$ is continuous note that $\alpha(\mu)$ is the unique solution to

$$\arg \min_{\alpha,u} \|\mu - u\|$$

subject to $u \in U$ and $\alpha\mu = u$. The theorem of the maximum implies $\alpha(\mu)$ is a continuous function. Furthermore note that associated with $u(\mu)$ (also part of the solution of the program) there is a Pareto optimum allocation, $[(x^i),y]$. Otherwise, given strict monotonicity of preferences with respect to the stochastic ordering, a proper reallocation can be made which yields a higher utility for everybody.

By Lemma 3.1 of Prescott and Townsend (1984a) any Pareto optimal allocation is the solution to some social planner's problem; i.e., there exists a $\mu'(\mu) \in \Delta^I$ such that $[(x^i),y] \in$ Arg max $\sum \mu'^i \lambda^i u^i(x)$ subject to feasibility. By Theorem 3 and Step 1 it can be supported by any $v \in V(\mu')$.

Step 3: Let ϕ: $\Delta^I \to R^I$ be

$$\phi(\mu) = \{z \in \mathbf{R}^I: \textstyle\sum_i z^i = 0,\ z^i = \lambda^i v(x^i) - \lambda^i \theta^i v(y) \text{ where } [(x^i), y]$$
$$\text{is some arbitrarily chosen allocation that gives}$$
$$u(\mu) \text{ and } v \in V(\mu'(\mu))\}.$$

Clearly z^i/λ^i is the set of transfers for QET $\{[(x^i),y],v\}$. As shown, ϕ is nonempty. Convexity of $V(\mu)$ implies ϕ is convex. It follows that a zero of ϕ implies a quasi equilibrium with no transfer.

Step 4. It is to show that ϕ is uhc i.e., if $z_n \to z$, $\mu_n \to \mu$ and $z_n \in \phi(\mu_n)$ all n, then $z \in \phi(\mu)$. We will start by finding a neighborhood of $0 \in S$, $N(0)$ such that for any μ, $v \in V(\mu)$, and $|v(s)| \le 1$ for all $s \in N(0)$. Recall that $1 = v(\hat{x}) \ge v(\bar{x}) = v(\bar{y}) \ge v(\underset{\wedge}{x})$ and $v(\bar{y}) \ge 0$. There exists $\alpha \in [\varepsilon, 1)$ such that $U(\mu, x^\alpha) = U(\mu, \bar{x})$, (because by compactness of the simplex, $U(\mu,\bar{x}) \ge U(\mu, x_0) > U(\mu, \underset{\wedge}{x})$ where x_0 is atomic measure on $(0,0) \in C$ and in turn $\alpha v(\underset{\wedge}{x}) + (1-\alpha)v(\hat{x}) \ge v(\bar{x}) \ge 0$ which implies $v(\hat{x}) \ge -((1-\varepsilon)/\varepsilon)$. Note that there exists $N \subset Y$ open set such that $v(s) > v(\underset{\wedge}{x})$ all $s \in N$. Just pick a $y \in Y$ such that for all h, $\int (1+c_1)dy_h > 0$, $\int c_2 dy_h > 0$, and $\int c_2 dy_h < \underset{=}{z}\, f[k_h, \int (1+c_1)dy_h]$ and construct a neighborhood around it. Since for all $s \in N$, $\alpha(\underset{\wedge}{x}-s) \in Y$, $\alpha > 0$, it is clear that $v(s) \ge v(\underset{\wedge}{x})$. Now, define $N(0) = \{s \in S: s = (1/2\gamma)(s'-y),$ all $s' \in N, \gamma = \max[1, 1-\varepsilon/\varepsilon]\}$. $N(0)$ is open and $0 \in N(0)$. For any $s \in N(0)$, $|v(s)| \le 1$ for any $v \in V(\mu)$. Define the set $K = \{v \in S^*: |v(s)| \le 1, s \in N(0)\}$. By the ##?Banach-Alaognu Theorem, this set is weak* compact, and by construction, for any μ, $V(\mu) \subset K$.

Now associated with z_n, there is a $\{[(x_n^i),y_n],v_n\} \in \Gamma(\mu) \times V(\mu)$. Since $v_n \in K$ for all n and S* is complete, a subsequence of it has a weak* limit v. Rename the sequence. Let $w^i \in X^i$ be such that $u^i(w^i) \geq u^i(x^i)$ where x^i is in $\Gamma(\mu)$. Pick $w_m^i \to w^i$ with the property $u^i(w_m^i) > u^i(x^i)$. Let $u_n^i = u^i(x_n^i)$. Since for all $x \in X^i$, $u^i(x)$ is bounded above and below, there is a convergent subsequence of u_n^i. We again rename the sequence. The utility functions and $\alpha(\mu)$ are continuous. Consequently $u_n^i \to \alpha(\mu)\mu^i$, with which some allocation is associated. This implies that for all m, there exists $n_0(m)$ such that $n > n_0(m)$ implies $u^i(w_m^i) \geq u^i(x_n^i)$, which in turn implies $v_n(w_m^i) \geq v_n(x_n^i)$. By definition of ϕ, $\lambda^i v_n(x_n^i) = z_n^i + \lambda^i\theta^i v_n(y_n)$ and by profit maximization, for all $y \in Y$, $\lambda^i v_n(x_n^i) \geq z_n^i + \lambda^i\theta^i v_n(y)$. These three facts imply that for $n > n_0(m)$. $\lambda^i v_n(w_m^i) \geq z_n^i + \lambda^i\theta^i v_n(y)$.

Taking limits on n, $\lambda^i v(w_m^i) \geq z^i \lambda^i\theta^i v(y)$. Taking limits on m, $\lambda^i v(w^i) \geq z^i + \lambda^i\theta^i v(y)$. But, for any w^i, $u^i(w^i) \geq u^i(x^i)$, $[(x^i),y] \in \Gamma(\mu)$, and in particular for $w^i = x^i = \bar{x}^i$, since $[(\bar{x}^i),\bar{y}] \in \Gamma(\mu)$ where $\bar{y} = \sum \lambda^i\bar{x}^i$. So $\lambda^i v(\bar{x}^i) \geq z^i + \lambda^i\theta^i v(y)$ for any $y \in Y$, and so also for $\bar{y}$. By aggregation we conclude that $v(\bar{x}^i) = z^i/\lambda^i + \theta^i v(\bar{y})$. To show that v supports, note that for $u^i(w^i) \geq u^i(\bar{x}^i)$ which implies $v(w^i) \geq v(\bar{x}^i)$. Finally, $v(\hat{x}) = 1$ because it is the weak* limit of v_n. This shows that $z \in \phi(\mu)$ and hence uhc of ϕ.

Step 5. In order to apply Kakutani's fixed point theorem we must verify the correspondence maps a set into itself. First consider $\phi(\bar{\mu})$ where $\bar{\mu}^i = 0$ for some i. Then $x^i = \hat{x}$ and $z^i < 0$. In the same fashion when $\bar{\mu}^i = 1$ for some i, $z^i < 1$ since $\hat{x}$ is infeasible and $v(\hat{x}) = 1$. Extend now ϕ to the set

$$M = \{s \in R^I: \textstyle\sum_i s^i = 1,\ s^i \geq -\max_j(1/\lambda^j)\}$$

where $\phi_i(\mu) = \phi_i(\bar{\mu})$, when $\mu^i < 0$ and $\bar{\mu}^i = 0$. Obviously $\phi(M) \subset M$. A zero of ϕ, cannot exist outside the unit simplex. Finally consider χ: $M \to M$, $\chi(\mu) = \mu - \phi(\mu)$. By Kakutani's fixed point theorem, a μ exists such that $\chi(\mu) = \mu$ and hence $\phi(\mu) = 0$ which implies existence of a quasi equilibrium with zero transfers. □

<u>Proof of Theorem 6</u>: Let $q(s) \equiv \lim v(s_{\leq T})$. This limit exists (Lemma 1 of Prescott and Lucas (1972)). Linearity of v and the fact that $x_T = x_{\leq T} + \hat{x} - \hat{x}_{\leq T}$ implies

$$\text{(A.1)} \qquad \lim_{T\to\infty} v(x_T) = \lim_{T\to\infty} v(x_{\leq T} + \hat{x} - \hat{x}_{\leq T}) = q(x) + v(\hat{x}) - q(\hat{x}).$$

<u>Step 1</u>: It is to show that if x is as good as $\bar{x}^i$ for agent i, then

$$\text{(A.2)} \qquad q(x) \geq v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i).$$

Start by selecting x^i, $x'^i \in X^i$ such that x'^i is strictly preferred to $\bar{x}^i$ and x is as desirable as $\bar{x}^i$. Let $x^{\alpha i} = \alpha x^i + (1-\alpha)x'^i$. By I, $x^{\alpha i} \in X^i$ and by II $x^{\alpha i}$ is strictly preferred to $\bar{x}^i$. By VII there exists T' such that for all T > T', $x_T^{\alpha i}$ is strictly preferred to x'^i. Since v is an equilibrium price, this implies

$$v(\bar{x}^i) < v(x_T^{\alpha i}).$$

Taking limits when $T \to \infty$ and $\alpha \to 1$,

$$v(\bar{x}^i) \leq \lim_{T\to\infty} v(x_T^i)$$

which by (A.1) implies

$$v(\bar{x}^i) \leq q(x) + v(\hat{x}^i) - q(\hat{x}^i).$$

Rearranging we obtain (A.2).

Step 2: It is to show that if $y \in Y^j$ then

$$\text{(A.3)} \qquad q(y) \leq v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j).$$

To show this step note that v being an equilibrium and by VI

$$v(y^j_T) \leq v(\bar{y}^j).$$

Taking limits and by (A.1)

$$v(\bar{y}^j) \geq q(y) + v(\hat{y}^j) - q(\hat{y}^j).$$

Rearranging we get (A.3).

Step 3: It is to show that profit maximization implies that for all i

$$\text{(A.4)} \qquad q(\bar{x}^i) = v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i)$$

and for all j

$$\text{(A.5)} \qquad q(\bar{y}^j) = v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j).$$

Let

$$\bar{x} \equiv \sum_i \lambda^i \bar{x}^i, \quad \hat{x} \equiv \sum_i \lambda^i \hat{x}^i, \quad \bar{y} \equiv \sum_j \bar{y}^j, \quad \hat{y} \equiv \sum_j \hat{y}^j.$$

Now note (A.2) and (A.3) also apply for $\bar{x}^i$ and $\bar{y}^j$. By linearity of v and aggregating over consumers,

(A.6) $\quad q(\bar{x}) \geq v(\bar{x}) + q(\hat{x}) - v(\hat{x}).$

Aggregating over producers,

(A.7) $\quad q(\bar{y}) \leq v(\bar{y}) + q(\hat{y}) - v(\hat{y}).$

By VIII $\hat{y} + \omega = \hat{x}$. By feasibility of the equilibrium allocation $\omega + \bar{y} = \bar{x}$. Then, from (A.5) we have

$$q(\bar{y}+\omega) \geq v(\bar{y}+\omega) + q(\hat{y}+\omega) - v(\hat{y}+\omega).$$

Linearity of v implies

$$q(\bar{y}) \geq v(\bar{y}) + q(\hat{y}) - v(\hat{y}).$$

This, together with (A.7), implies

$$q(\bar{y}) = v(\bar{y}) + q(\hat{y}) - v(\hat{y}).$$

Since for all j,

$$q(\bar{y}^j) \geq v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j)$$

it follows

$$q(\bar{y}^j) = v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j).$$

By the same reasoning, for all i

$$q(\bar{x}^i) = v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i).$$

<u>Step 4</u>: It is to show that q is a quasi equilibrium. But this is immediate from (A.2), (A.3), (A.4), and (A.5). Just substitute $v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i)$ for $q(\bar{x}^i)$ in (A.2) and $v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j)$ for $q(\bar{y}^j)$ in (A.3) and the results are the conditions that define a quasi equilibrium. This completes the proof. □

Proof of Lemma 3: Pick any $s_0 \in S$ such that $v(s_0) \neq 0$ and $p(s_0) \neq 0$. Define $\alpha \equiv v(s_0)/p(s_0)$. If $\alpha < 0$ then $\bar{s}$ does not solve the program. Suppose there exists $s_1 \in S$ such that $v(s_1) \neq \alpha p(s_1)$. Pick $\gamma_0 \in R$ and $\gamma_1 \in R_+$ such that $\gamma_0 p(s_0) + \gamma_1 p(s_1) = 0$.

The first case is $v(s_1) > \alpha p(s_1)$. Let $y = \bar{s} + \gamma_0 s_0 + \gamma s_1$. Then

$$
\begin{aligned}
p(y) &= p(\bar{s}) + \gamma_0 p(s_0) + \gamma_1 p(s_1) \\
&= p(\bar{s}) v(y) \\
&= v(\bar{s}) + \gamma_0 v(s_0) + \gamma_1 v(s_1) \\
&> v(\bar{s}) + \alpha\left[\gamma_0 p(s_0) + \gamma_1 p(s_1)\right] = v(\bar{s}).
\end{aligned}
$$

This contradicts the fact that $\bar{s}$ solves the program. The second alternative is $v(s_1) < \alpha p(s_1)$. For this case let $y = \bar{s} - \gamma_0 s_0 - \gamma_1 s_1$. This implies $v(y) > v(\bar{s})$ and hence a contradiction. □

Footnotes

[1]In a recent paper, R. Rogerson (1987) has developed a model for locational decisions within the Arrow-Debreu-McKenzie paradigm. His approach is to study certain reasons for appearance of unemployment. He deals with the nonconvexity problem in pretty much the same way we do. However, his approach is very specific, and not easily generalizes to incorporate search. He contemplates a disutility of moving, that we think is easy to include in our model.

[2]Other notable examples of search environments include those of Diamond (1984) and Mortensen (1982) who use the Nash rather than the competitive equilibrium concept.

Also, Starrett in a famous paper (1972) discusses but not analyzes the possibility for people to move in order to avoid the disutility associated with pollution. This problem disappears in our framework. Another example is Topel's local labor market model (1986).

[3]There is another way of handling this problem of the relation between ex ante probabilities and ex post measures. It is by generating an artificial stochastic process for all $(\ell, t, h_{\ell t})$ upon where each agents makes contingent his lotteries. This would not only increase a lot the dimensionality of the problem but would also oblige us to keep track of the agents names. We think that the use of Uhlig's theorem makes things much easier.

[4]Given the way the commodity space is constructed, we do not require the underlying consumption possibility set to be

convex. This is important because indivisibilities and specialization, which often are essential parts of applied general equilibrium analyses, result in this set being nonconvex.

[5]It is in this step of the proof (as well as in a similar argument in the proof of Theorem 5) where the assumption of nondepreciation of capital becomes handy. Its exclusion would make the statement of point y^0 much messier without adding any interesting feature.

[6]There are other strategies for proving this type of result. Jones (1987) uses certain properties of asymptotic cones to show existence and Bewley (1969) uses approximation of economies in the right topology. The choice of our approach was influenced by its strong links to the second welfare theorem, which plays a central role in our analysis.

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