

## Money in General Equilibrium\*

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This paper develops the Kiyotaki-Wright model of monetary general equilibrium in which trade is bilateral and enforced by requiring that transactions be *quid pro quo*, and studies which goods are chosen, and under what conditions, as media of exchange. We prove the existence of a rational expectations equilibrium in which agents' expectations concerning trading opportunities are realized in the present and all future periods. We also show that, exceptional cases aside, no rational expectations barter equilibrium exists; that an equilibrium generally supports multiple money goods; and that a fiat money (i.e., a good that is produced, has minimum storage costs, but is not consumed) cannot be traded in rational expectations equilibrium.

Keywords: monetary theory; general equilibrium; Markov models.

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## 1. Introduction

Walrasian general equilibrium models embody two assumptions that are ill-suited for the study of money. The first is that exchange can be treated as an interaction between agents on the one hand and an abstract 'market' (a.k.a. an 'auctioneer') on the other. By thus abstracting from the bilateral character of market exchange, these models cannot properly address the role of a medium of exchange (Ostroy, 1973). The second assumption is that the fulfillment of the terms of exchange is unproblematic: either agents never enter into contracts that they cannot or choose not to honor, or exchanges among agents are costlessly enforced by third parties (Williamson, 1985; Gintis, 1989; Bowles and Gintis, forthcoming). Thus a central attribute of money, the capacity to enforce budget constraints through *quid pro quo* transactions, is unrecognized in Walrasian models (Tsiang, 1966; Clower, 1967; Kohn, 1981).

This paper drops these assumptions in favor of treating trade as bilateral and contracts as enforced by requiring transactions to be *quid pro quo*. Our approach, generalizing a model of Nobuhiro Kiyotaki and Randall Wright, takes relative prices as given and addresses the question: which goods are chosen, and under what conditions, as media of exchange?<sup>1</sup> We show that considerable analytical simplification, as well as novel results, can be gained through a stochastic formulation in which agents' behavior depends on their expectations concerning trade opportunities.

By an *economy* we mean a finite non-empty set  $A$  of agents, and a finite

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1. Kiyotaki and Wright, 1988. See also Hahn, 1965; Feldman, 1973; Ostroy, 1973; Ostroy and Starr, 1974; Goldman and Starr, 1982; Iwai, 1988; Starr, 1989.

set  $G$  of goods with the following properties. Each good  $g \in G$  has production cost  $k_g > 0$ , and a storage cost  $s_g > 0$  of holding the good in inventory for one time period. To avoid degenerate cases, we assume all storage costs are distinct. Each agent  $a \in A$  produces a single good  $p_a$ , consumes some non-empty set of goods  $C_a \subset G$ , with  $p_a \notin C_a$ , and has a linear utility function in which  $u_c^a > k_p + s_p$  is the utility of consuming one unit  $c \in C_a$ .<sup>2</sup> We assume each good is indivisible and there are at least three goods. Unless otherwise stated, we assume also that for each  $p, c \in G$ , there multiple agents who produce  $p$  and consumes  $c$ , and no agent consumes her own production good. In addition, an agent can store in inventory any good she acquires through trade. We assume free disposal, so an agent can always discard unwanted inventory items.

At the start of a trading period, an agent has the option of producing a unit of her production good, and must do so if her inventory is empty.<sup>3</sup> She then chooses a good from her inventory to trade, and seeks an encounter with another agent. Encounters are random, with a probability distribution over the population of agents. Upon an encounter, if mutually acceptable, their offerings are exchanged (we assume all goods exchange one for one). At the end of the trading period, an agent has the option of consuming an acquired good.

We assume that agents have infinite time horizons and maximize their

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2. A list of symbols appears at the end of the paper. When the context is clear, we take the liberty of dropping the subscript on  $p_a$ ,  $C_a$ , and other references to a particular agent  $a$ .
  3. The requirement that an agent's inventory cannot be empty conveniently rules out the trivial rational expectations equilibrium where all agents have empty inventory, and neither produce nor consume.

discounted expected returns, including the utility of consuming, the cost of producing, and the per-period storage costs incurred by holding goods in inventory. We demonstrate the existence of a rational expectations equilibrium, in which agents' expectations concerning trading opportunities are realized in the present and all future periods. We also show that, exceptional cases aside, no rational expectations barter equilibrium exists; i.e., at least one agent accepts at least one good she does not consume. We also show that an equilibrium generally supports multiple money goods.<sup>4</sup> Finally, we prove that a fiat money (i.e., a good that is produced, has minimum storage cost, but is not consumed) cannot be traded in rational expectations equilibrium.

Our approach is to treat each agent as following a finite Markov chain, where the agent's state in each period is the agent's current inventory, and the transition probabilities are determined by the agent's trading strategy. The agent maximizes the expected payoff (a random variable defined over the path space of this Markov chain), by choosing an appropriate trading strategy, given her expectations concerning the probability of various trading opportunities. The existence of a rational expectations equilibrium is ensured by a fixed point argument (we use Nash's theorem on the existence of equilibrium in finite non-cooperative games), once we have a rational expectations criterion for aggregate trading fre-

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4. On multiple media of exchange, see Prescott (1987). Distinct national currencies are an obvious case. In addition, some transactions require fiat money rather than check, some require check but do not allow fiat money, or credit card but not check, etc. Moreover there are many *specialized* media of exchange, such as lines of credit extended to firms by their suppliers.

quencies. This criterion is derived from an ergodic theorem for finite Markov chains guaranteeing the existence of a time-invariant expected inventory for each agent.

## 2. The Conditions of Monetary Exchange

Since all goods are indivisible, we can identify inventories with the set of  $n$ -tuples over the non-negative integers. For agent  $a \in A$ , we call an inventory  $i$  *admissible* if its total storage costs  $\sum_{g \in G} s_g i_g$  are less than  $u_c$  for some  $c \in C_a$ . Since storage costs are strictly positive, the set  $I_a$  of admissible inventories for agent  $a$  is finite. Since an optimizing agent never voluntarily moves from an admissible to an inadmissible inventory, we lose no generality by restricting consideration to admissible inventories.

Consider an agent  $a \in A$  with production good  $p$  and consumption set  $C$ . Let  $\mu_t(g, h)$  be the agent's subjective probability at time 0 that a randomly chosen agent in the economy, in a single encounter, will accept good  $g$  in exchange for good  $h$  in trading period  $t \geq 0$ . If  $i_t$  is the agent's inventory in period  $t$  and if she offers  $g$  for sale in that period, her expected flow of utility  $v(i_t)$  is then

$$v(i_t) = \sum_{c \in C} \mu(g, c) u_c - \lambda_t k_p - \sum_{g \in G} s_g i_{gt}, \quad (1)$$

where  $i_{gt}$  is the number of units of good  $g$  in  $i_t$ , and  $\lambda_t = 1$  if the agent produces a unit of  $p$ , and  $\lambda_t = 0$  otherwise.<sup>5</sup>

A *trading strategy*  $\tau = [\hat{g}(), \hat{G}()]$  for the agent is a function  $\hat{g}: I \rightarrow G$  and a correspondence and  $\hat{G}: I \rightarrow 2^G$ , where  $\hat{g}(i)$  is the agent's choice of a

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5. Formula (1) assumes the agent actually consumes her consumption good, once acquired. As shown below, in some degenerate cases this assumption is not valid, and (1) must be adjusted accordingly.

good to sell from her inventory  $i$ , and  $\hat{G}(i)$  is the set of goods she is prepared to accept in exchange for  $\hat{g}(i)$ . We write  $i.g.h$  for the inventory obtained from inventory  $i$  by deleting  $g$  if  $h \in C$ , and by replacing  $g$  by  $h$  otherwise (i.e., if an agent holding inventory  $i$  trades  $g$  for  $h$ , her resulting inventory is  $i.g.h$ ). We call a trading strategy  $\tau = [\hat{g}(), \hat{G}()]$  *admissible* if, for any admissible inventory  $i$ ,  $i.\hat{g}(i).g$  is admissible for all  $g \in \hat{G}(i)$ . The set  $T^*$  of admissible trading strategies is finite. It is clear that admissible strategies exist, and that any optimal strategy is admissible. Thus restricting consideration to admissible strategies involves no loss in generality.

Each admissible trading strategy  $\tau = [\hat{g}(), \hat{G}()]$  and admissible initial inventory  $i_0$  for an agent  $a \in A$  give rise to a finite Markov chain with states consisting of the admissible inventories  $I_a$ , and transition probabilities  $p_{ij}$  equal to the probability of passing from inventory  $i$  to inventory  $j$  in one trading period using strategy  $\tau$ . We have

$$p_{ij} = \begin{cases} \mu(\hat{g}(i), h) & j = i.\hat{g}(i).h, h \in \hat{G}(i) \\ 1 - \sum_{h \in \hat{G}(i)} \mu(\hat{g}(i), h) & j = i \\ 0 & \text{otherwise.} \end{cases}$$

We denote by  $P[\tau]$  the Markov chain generated by the transition probabilities  $(p_{ij})$  for trading strategy  $\tau$ .

A *path* of  $P[\tau]$  is a sequence  $\omega = \{i_0, i_1, \dots\}$  of inventories. We write  $\omega_t = i_t$  for  $t = 0, 1, \dots$ , and we define a probability measure  $\pi[\tau]$  on the set  $\Omega$  of paths, such that if  $S$  is a measurable set of paths,  $\pi[\tau](S)$  is the probability that the lifetime trajectory of the agent lies in  $S$ , given that the agent uses strategy  $\tau$ . This measure is generated in the obvious way from the transition matrix  $(p_{ij})$ . Specifically, for any  $t > 0$ ,  $\pi_t: \Omega \rightarrow R$  is given by

$$\pi_t(\omega) = \prod_{k=1}^t p_{i_k i_{k-1}}.$$

Thus  $\pi_t(\omega)$  is the probability that an arbitrary path starting with  $i_0$  and with transitions defined by the trading process  $P[\tau]$ , agrees with  $\omega$  for the first  $t$  periods. Let  $F_k^*$  be the Borel field generated by the sets

$$\{\omega \in \Omega \mid \omega_0 = i_0, \omega_k = c_k \in I \text{ for } i = 1, \dots, k\}$$

for all  $k$ -tuples  $(c_1, \dots, c_k)$ . Thus  $F_k^*$  is the smallest Borel field containing sets consisting of all paths starting with some  $c_0, \dots, c_k$ , where  $c_0 = i_0$ . Clearly  $\pi_k$  is a probability measure on  $F_k^*$ . The  $F_k^*$  are increasing, and we denote by  $F^*$  the smallest Borel field over  $\Omega$  containing their union. Since the restriction of  $\pi_k$  to  $F_{k-1}^*$  coincides with  $\pi_{k-1}$  for all  $k > 0$ , the  $\{\pi_k\}$  extend uniquely to a probability measure  $\pi = \pi[\tau]$  on  $F^*$  (Kemeny, Snell, and Knapp, 1976).

We now define a  $\pi$ -measurable function  $V(\omega; i_0, \tau)$  as follows. First, let  $v_k(\omega)$  be the actual payoff from the  $k^{\text{th}}$  transaction. If we write  $g_k(\omega)$  for the good the agent takes away from the  $k^{\text{th}}$  transaction (i.e.,  $g_k(\omega) = \hat{g}(\omega_k)$  if no trade takes place, and  $g_k(\omega) =$  the good acquired otherwise), then

$$v_k(\omega) = \begin{cases} u_{g_k}(\omega) - \lambda k_p - s_{\omega_k} & \text{if } g_k(\omega) \in C \\ -s_{\omega_k} & \text{otherwise.} \end{cases}$$

where  $s_i$  is the storage cost associated with inventory  $i$ , and  $\lambda = 1$  if a unit of  $p$  is produced in the period, and  $\lambda = 0$  otherwise. We then define

$$V(\omega; i_0, \tau) = \sum_{t=0}^{\infty} \rho^t v_t(\omega), \quad (2)$$

where  $\rho \in (0, 1)$  is the agent's discount factor. Note that  $V(\omega; i_0, \tau)$  is finite,  $\pi$ -measurable (by the Dominated Convergence Theorem, since the  $v_k(\omega)$  are uniformly bounded), and has finite expected value. It follows from (1)

that the agent's problem is to maximize the expected value  $EV(\omega; i_0, \tau)$  by the proper choice of a trading strategy  $\tau \in T^*$ . Since the set of admissible trading strategies is finite, we have

**Theorem 1:** There exists an optimal trading strategy for any admissible initial inventory.

For any non-consumption good  $g$  we define the trading value of  $g$  as

$$\tilde{v}_g = \sum_{c \in C} \mu(g, c) (u_c - k_p) - s_g. \quad (3)$$

We say an agent has *stationary expectations* if  $\mu_t(g, h) = \mu_0(g, h)$  for all  $t$  and all  $g, h \in G$ , and we say  $m \in G$  is a *money good* for an agent if the agent does not consume  $m$  but accepts  $m$  in trade. We have

**Theorem 2:** If an agent starts with no inventory, has stationary expectations, and if there is a good  $m \in G$  such that  $\tilde{v}_m > \tilde{v}_p$  and  $\mu(p, m) > 0$ , then there exists a money good for the agent.

**Proof:** Suppose no money good exists for an agent  $a \in A$ . It is then clearly optimal for her to follow the *barter strategy*  $\tau_b$  of producing and offering to trade her production good  $p$ , accepting in trade only consumption goods  $c \in C$ ; i.e.,  $\hat{g}(\{p\}) = p$  and  $\hat{G}(\{p\}) = C$ . Since her initial inventory is empty, the return to  $\tau_b$  can be derived from the recursion relation

$$EV(\omega; \{p\}, \tau_b) = \tilde{v}_p + \rho EV(\omega; \{p\}, \tau_b).$$

Thus from (1) we get (with  $\lambda = 1$  since the inventory equals  $\{p\}$ )

$$EV(\omega; \{p\}, \tau_b) = \tilde{v}_p / (1 - \rho).$$

Here the expectation is taken over the probability distribution of next-period inventories, given the trading strategy  $\tau_b$ . Suppose now there is a good  $m \in G$  satisfying  $\tilde{v}_m > \tilde{v}_p$ . Consider the strategy  $\tau_m$  of accepting  $m$  in trade for  $p$ . In this case we have

$$EV(\omega; \{p\}, \tau_m) = \sum_{c \in C} \mu(p, c) [(u_c - k_p) + \rho EV(\omega; \{p\}, \tau_m)]$$



$$\begin{aligned}
& + \mu(p,m)\rho EV(\omega;\{m\},\tau_m) - s_p + \\
& [1 - \mu(p,c) - \mu(p,m)]\rho V(\omega;\{p\},\tau_m) \\
& = \frac{\sum_{c \in C} \mu(p,c)(u_c - k_p) + \mu(p,m)\rho V(\omega;\{m\},\tau_m) - s_p}{1 - \rho[1 - \mu(p,m)]},
\end{aligned}$$

where we have written  $\mu(g,C) = \sum_{c \in C} \mu(g,c)$ . Also, starting with inventory  $\{m\}$  we have

$$\begin{aligned}
EV(\omega;\{m\},\tau_m) &= \sum_{c \in C} \mu(m,c)[(u_c - k_p) + \rho EV(\omega;\{p\},\tau_m)] \\
&+ [1 - \mu(m,c)]\rho EV(\omega;\{m\},\tau_m) - s_m \\
&= \frac{\sum_{c \in C} \mu(m,c)[(u_c - k_p) + \rho EV(\omega;\{p\},\tau_m)] - s_m}{1 - \rho[1 - \mu(m,C)]}
\end{aligned}$$

Solving simultaneously, we have

$$\begin{aligned}
EV(\omega;\{p\},\tau_m) &= v_p + \\
&\mu(p,m) \frac{\bar{v}_m - \bar{v}_p}{r[1 - \rho(1 - \mu(p,m) - \mu(m,C))]}
\end{aligned}$$

Since  $EV(\omega;\{p\},\tau_m) > EV(\omega;\{p\},\tau_p)$ , the barter strategy was not optimal. This is a contradiction, which proves some money good exists for the agent. This proves Theorem 2.<sup>6</sup>

I conjecture that the following complete characterization of an optimal strategy is valid:

**Theorem 3:** If an agent begins with empty inventory, then in any period she holds in inventory only the good she is currently offering to trade.

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6. Note that the condition  $\bar{v}_m > \bar{v}_p$  is not a sufficient condition that  $m$  itself be accepted as money. To see this, suppose both  $m$  and  $p$  are poor candidates for being sold for  $c$ , but  $m$  is slightly better than  $p$  in this respect. Suppose, however, that  $m$  can be traded for nothing else but  $c$ , while  $p$  can be traded for another good  $m'$  that has a high probability of being traded for  $c$ . Then the agent may pass up a potential trade for  $m$ , and wait for a trade for  $m'$ .

This theorem is intuitively obvious, but I have not yet found a proof. Thus I shall not rely upon it in the remainder of the paper. Note that, if true, Theorem 3 implies that  $\lambda$  in (1) is always unity. We also then have the obvious corollary:

**Corollary:** Let  $i_0$  be an arbitrary initial inventory. Then the agent's optimal strategy consists of discarding a certain part of her inventory, selling off the remainder, and then following the strategy specified in Theorem 3.

### 3. The Existence of Rational Expectations Equilibria

By the state of the economy we mean an initial inventory  $i_0^a$ , a set of expectations  $\{\mu_a(g,h) | g,h \in G\}$ , and an admissible strategy  $\tau_a \in T_a^*$  for each agent  $a \in A$ . We call a state of the economy an *equilibrium* if  $\tau_a$  maximizes the payoff (2) to agent  $a$  with respect to these expectations for each  $a \in A$ . A *rational expectations equilibrium* is one in which each agent's expectations for all time periods equal the expected frequency of trading opportunities in that period, given the Markov processes generated by their trading strategies.

Suppose agents are permitted to choose mixed trading strategies in addition to the pure trading strategies analyzed above. To demonstrate the existence of a rational expectations equilibrium we construct an initial distribution of inventories among agents that is time-invariant when agents expect that it will be so. The intuition behind this approach is that a trading strategy gives rise to an initial period of inventory adjustment, after which trade follows an ergodic stochastic process, with constant average frequencies of recurrent inventories. If an agent could form a 'composite inventory' weighted by the expected frequency of each good in this steady state, this composite inventory, as well as the agent's

expected trading behavior, would be time-invariant. If other agents know this agent's utility and production functions but not her current inventory, her expected trading behavior is precisely given by this ergodic stochastic process. The existence of such steady state behavior is guaranteed by the ergodic theorem for finite Markov chains (Feller, 1950). We have

**Theorem 4:** There exists a rational expectations equilibrium in mixed strategies. If consumption utilities are sufficiently large, then any such equilibrium involves a positive level of trade with positive probability in each period.

Proof: By a theorem of Nash (1951), every non-cooperative game has a Nash equilibrium in mixed strategies. We will construct a non-cooperative game with players consisting of the agents  $A$  in our economy, with strategies consisting of the agents' trading strategies, and with payoffs corresponding to (2).

For agent  $a \in A$  with expectations  $\{\mu_a(g, h) | g, h \in G\}$  and admissible trading strategies  $T_a$ , consider the Markov chain  $P[\sigma]$  generated by an optimal pure trading strategy  $\sigma_a \in T_a$  corresponding to these expectations. Since  $P[\sigma_a]$  is a finite chain, it must enter a persistent state in finite time with probability one. The finiteness of  $P[\sigma_a]$  also implies that the states visited (i.e., the inventories attained) from this point form an irreducible chain of persistent states  $J[\sigma_a]$ . Since 'no trade' has positive probability in each period (a seller can always meet another holding the same good), all states in  $J[\sigma_a]$  are aperiodic, so all are ergodic. This implies that the transition matrix  $P[\sigma_a]$  has a unique left eigenvector  $u[\sigma_a]$  corresponding to the unit eigenvalue,

$$u[\sigma_a] = u[\sigma_a]P[\sigma_a],$$

with

$$\sum_{i \in J[\sigma_a]} u[\sigma_a]_i = 1, u[\sigma_a]_i \geq 0 \text{ for } i \in J[\sigma_a],$$

where  $u[\sigma_a]_i$  is the component of  $u[\sigma_a]$  corresponding to inventory  $i$ . In effect,  $u[\sigma_a]_i$  is the fraction of the time the Markov process is in state  $i$  for agent  $a$  using strategy  $\sigma_a$  and the relationship  $u[\sigma_a] = u[\sigma_a]P[\sigma_a]$  means that if the agent begins with extended inventory  $u[\sigma_a]$ , then the expected extended inventory in the next period is again equal to  $u[\sigma_a]$ . We call  $u[\sigma_a]$  a *stationary extended inventory* corresponding to  $\sigma_a$ . Note that an extended inventory is not an inventory, since  $u[\sigma_a]$  is not an integral vector of goods.

We form a non-cooperative game with players  $a \in A$  as follows. For each player  $a \in A$ , we identify the pure game strategies with the agent's set of admissible trading strategies  $T_a$ , and we let  $T_a^M$  be the set of mixed strategies based on  $T_a$ . We further define the stationary extended inventory of  $\tau_a \in T_a^M$  as a linear combination of the stationary extended inventories of its underlying pure strategies  $\sigma \in T_a$ , weighted by their contribution  $\pi_\sigma(\tau_a)$  to  $\tau_a$ . Finally, let  $T^* = \times_{a \in A} T_a^M$  be the strategy set for the game.

We define payoffs as follows. Given strategy  $\tau^* = \{\tau_a | a \in A\} \in T^*$ , for each  $g, h \in G$ , let  $f_{\tau^*}(g, h)$  be the expected frequency of agents selling  $g$  and willing to accept  $h$  in trade for  $g$  with strategy  $\tau^*$ :

$$f_{\tau^*}(g, h) = \sum_{a \in A} \sum_{\sigma \in T_a} \{\pi_\sigma(\tau_a) u[\sigma_a]_i | (\sigma, i) \in T_a(g, h)\} / N, \quad (4)$$

where  $N$  is the number of agents,  $u[\sigma_a]$  is the stationary extended inventory corresponding to pure trading strategy  $\sigma$ , and  $T_a(g, h)$  is the set of ordered pairs  $(\sigma, i)$  of trading strategies  $\sigma \in T_a$ ,  $i$  persistent with respect to strategy  $\sigma = [\hat{g}(), \hat{G}()]$  for player  $a \in A$ , such that  $\hat{g}(i) = g$  and  $h \in \hat{G}(i)$ . Endow each player with stationary expectations  $\mu_a(g, h) = f_{\tau^*}(g, h)$  for all

$g, h \in G$ . For each agent  $a \in A$ , and for each pure strategy  $\sigma \in T_a$ , let  $H_a(\sigma) = \sum_i u[\sigma_a]_i EV(\omega; i, \sigma)$  as defined in (2); i.e.,  $H_a(\sigma)$  is the expected return to choosing pure trading strategy  $\sigma$ , given the trading frequencies  $f_T^*(g, h)$  generated by other agents' mixed trading strategies, and supposing the agent is endowed with initial inventory  $i$  with probability  $u[\sigma_a]_i$ . Finally, for agent  $a$ 's mixed trading strategy  $\tau_a$ ,

$$H_a(\tau_a) = \sum_{\sigma \in T_a} \pi_{\sigma}(\tau_a) H_a(\sigma),$$

is the expected return to agent  $a$  for the mixed trading strategy  $\tau_a$ .

By Nash's theorem there exists an equilibrium for this non-cooperative game. Then by the Nash property, (2) is maximized for each agent. Also, the expectations  $\{\mu_0^a(g, h) | g, h \in G\}$  equal the expected frequencies of agent behaviors by construction. In addition, the expected extended inventory for each agent at the end of the period is the same as at the beginning. Therefore the expected future trading probabilities are stationary.

It remains to show that if consumption utilities are sufficiently high trading occurs with positive probability in each period. Assume for all agents  $a \in A$ , if  $A$  produces  $p$  and consumes  $c$ , we have the two inequalities:

$$u_c > k_p + N[k_p(1-\rho) + s_p]/\rho \quad (5a)$$

$$u_c > k_p + s_p/(1-\rho), \quad (5b)$$

where  $N$  is the number of agents in the economy. Suppose, contrary to the assertion of the theorem, that there is a zero probability of trade in each period. Then each agent offers one good for sale in each period, and is certain not to encounter an agent with whom a mutually acceptable trade can be effected. Let  $g \in G$  be the good with least storage cost that is offered for sale. Suppose agent  $a \in A$  is a  $g$ -seller and encounters another agent selling  $c \in C_a$ . Then this other agent is willing to trade, since she ac-

quires a lower storage cost good. By (5b), agent  $a$  is willing to exchange, consume  $c$ , produce her production good  $p$ , and hold  $p$  forever. Thus no such encounter is possible, implying that no consumption good of a  $g$ -seller is offered for sale. We shall show that this is impossible.

Let  $\alpha$  be the frequency of  $g$ -sellers in the economy. We necessarily have  $\alpha \geq 1/N$ . Let  $c \in G$  be in  $C_a$ . Suppose agent  $b \in A$  is a  $c$ -producer,  $g$ -consumer selling a good  $h \in G$ ,  $h \neq c$  in rational expectations equilibrium. We know that such an agent exists. The expected return for agent  $b$  to holding  $h$  forever is  $-s_h/(1-\rho)$ . The cost of discarding  $h$  and producing  $c$  is  $k_c$ . Let  $V$  be the expected return to taking this action, and then holding  $c$  until an encounter with a  $g$ -seller occurs, after which  $g$  is consumed and the process is repeated. We have

$$\begin{aligned} V &= \rho\{\alpha[u_g + (V - k_c)] + (1-\alpha)V\} - s_c \\ &= [\rho\alpha(u_g - k_c) - s_c]/(1-\rho). \\ &\geq [\rho(u_g - k_c)/N - s_c]/(1-\rho). \end{aligned} \tag{6}$$

Substituting (5a) in (6), we find  $V > k_c$ . Thus this strategy dominates holding  $h$  forever. This contradicts the assumption that agent  $b$  sells some good other than  $c$ . This completes the proof of Theorem 4.

The assumption that consumption utilities are sufficiently high is necessary, although our inequalities (5) may be excessively strong. For a rational expectations equilibrium with no trade of the following type is possible: Endow each agent with a unit of the lowest storage cost good  $g \in G$ , and endow each agent with the expectation that all other agents offer  $g$  for sale in each period. Since a non- $g$ -consumer never encounters a consumption good, and cannot improve her position by discarding  $g$  and producing her production good, each such agent is satisfied to sell  $g$  forever. If  $g$ -consumers cannot sell their production goods, and if the storage costs as-

sociated with these goods are sufficiently high in comparison with their consumption utility, they also prefer to offer  $g$  for sale to consuming  $g$  and producing their production good. Thus we have a Nash equilibrium with no trade.

I suspect that whenever the barter equilibrium (which of course is not in general a rational expectations equilibrium) offers all agents a positive expected return, there is a rational expectations equilibrium with a positive probability of trade. I have not attempted to demonstrate this proposition, however.

#### 4. The Existence of Money

Is there always a money good? The following shows that there almost always is. Let  $f_{cp}$  be the frequency of agents consuming  $c$  and producing  $p$ . We call an economy *generic* if all frequencies  $\{f_{cp} | c, p \in G\}$ , all production costs  $\{k_p | p \in G\}$ , and all storage costs  $\{s_g | g \in G\}$  are distinct and non-zero. We have

**Theorem 5:** If the economy is generic, and if there are two goods  $p$  and  $q$  such that  $f_{cp} > f_{cq}$  does not hold for all  $c \in G$ ,  $c \neq p, q$ , then there exists a money good in rational expectations equilibrium.

**Proof:** We need only show that a barter equilibrium is not Nash. Suppose it were. Then for all  $g, h \in G$ , the probability  $\mu(g, h)$  of acquiring  $h$  in exchange equals  $f_{gh}$ , the frequency of agents producing  $h$  and consuming  $g$ . Consider an agent producing  $p$  and consuming  $c$ . Since this agent uses a barter strategy, we must have  $\tilde{v}_p \geq \tilde{v}_g$  for any  $g \in G$ ,  $g \neq p, c$ , where  $\tilde{v}_g$  is the trading value of  $g$ , defined in (3) above. This implies

$$[\mu(p, c) - \mu(g, c)](u_c - k_p) \geq s_p - s_g. \quad (7)$$

Fix a consumption good  $c$ , and suppose  $p$  and  $q \in G$  satisfy  $f_{cp} > f_{cq}$ . Then  $\mu(p, c) > \mu(q, c)$ . Applying (7) with  $g = q$ , we have

$$[\mu(p, c) - \mu(q, c)](u_c - k_p) \geq s_p - s_q,$$

since the agent producing  $p$  does not accept  $q$  in trade. Inequality (7) must also hold if we interchange substitute  $q$  for  $p$  and  $p$  for  $q$ , so

$$[\mu(q, c) - \mu(p, c)](u_c - k_q) \geq s_q - s_p,$$

which gives

$$\begin{aligned} [\mu(p, c) - \mu(q, c)](u_c - k_p) &\geq s_p - s_q \\ &\geq [\mu(p, c) - \mu(q, c)](u_c - k_q). \end{aligned}$$

This implies  $k_q < k_p$ . We thus have

$$f_{cp} > f_{cq} \Rightarrow k_p > k_q \text{ for all } p, q \in G, c \neq p, q. \quad (8)$$

This series of implications in (8) is independent of the consumption good  $c$ . Thus if there is another good  $c'$  with  $f_{c'q} > f_{c'p}$ , we have the additional inequality  $k_q > k_p$ , which is a contradiction. Therefore at least one agent does not follow a barter strategy. This prove Theorem 5.

We call a good  $m$  a *universal medium of exchange* if  $m$  is accepted by all agents. We have

**Theorem 6:** Let  $m$  be the good with lowest storage costs. Then if the frequency of  $m$ -consumers in the economy is sufficiently large, there is a rational expectations equilibrium in which  $m$  is a universal medium of exchange.

**Proof:** Let  $m \in G$  be the good with lowest storage costs. In the proof of Theorem 4, we may restrict the set of admissible strategies available to an agent to those in which the following  $m$ -constraint is satisfied: If the agent follows strategy  $\tau_a$ , for any inventory  $i \in J[\tau_a]$ ,  $\hat{g}(i) \neq m$  implies  $m \in \hat{G}(i)$ ; i.e., each agent is forced to accept  $m$  in exchange for any other good. Since  $s_m$  is minimal among the  $\{s_g | g \in G\}$ , an admissible strategy remains admissible when the  $\{G^-(i) | i \in J[\tau_a]\}$  are augmented to satisfy the  $m$ -constraint. To prove that the resulting configuration of strategies is



Nash, we need only show that each agent will not change her optimal strategy when the  $m$ -constraint is lifted for her, while it remains for all other agents.

Consider an agent producing  $p$  and consuming a set of goods  $C$ . The assertion is trivially true when  $p = m$  or  $m \in C$ , so we assume  $m \notin C \cup \{p\}$ . Suppose the agent has admissible inventory  $i$  and  $g = \hat{g}(i)$  with  $g \neq m$ . It suffices to prove that  $m \in \hat{G}(i)$ . We denote by  $\mu(g_1, g_2 | E)$  the probability that a random agent in set  $E$  will accept  $g_1$  in exchange for  $g_2$ . Let  $E = \{a \in A | a \text{ is not currently selling } m\}$ . Since any agent who accepts  $g$  accepts  $m$  unless that agent is currently selling  $m$ , and since there is some agent who accepts  $m$  but not  $g$  (e.g., a  $g$ -producer not currently selling  $m$ ) we have  $\mu(m, h | E) > \mu(g, h | E)$  for any  $h \in G$ . Since

$$\mu(g_1, g_2) = \mu(g_1, g_2 | E)P(E) + \mu(g_1, g_2 | A - E)P(A - E),$$

where  $A - E$  is the complement of  $E$  in  $A$  and  $P(X)$  is the probability that a random agent belongs to set  $X$ , it is clear that for  $P(E)$  sufficiently large, we must have  $\mu(m, h) > \mu(g, h)$ . But all consumers of  $m$  are in  $E$ , so there is some frequency  $f_m(g, h)$  of  $m$ -consumers in the economy that is both less than unity and sufficiently large that  $\mu(m, h) > \mu(g, h)$ . Let  $f_m$  be the maximum of  $\{f_m(g, h) | g, h \in G, g, h \neq m\}$ . Then  $f_m < 1$  and if the frequency of  $m$ -consumers is at least  $f_m$ , we have  $\mu(m, h) > \mu(g, h)$  for all  $g, h \neq m$ . Since  $s_m < s_g$ , offering  $m$  for sale strictly dominates offering  $g$  for sale under these conditions. Thus  $m \in \hat{G}(i)$ . This completes the proof of Theorem 6.

It is important to note that the condition that the frequency of  $m$ -consumers in the economy be sufficiently large cannot be dropped from Theorem 6. The next theorem illustrates this point. Let us define a *fiat money* a medium of exchange that is produced, has least storage costs, but is consumed by no agent. We have

**Theorem 7:** Suppose all goods are produced and consumed with the exception of  $m \in G$ , which is produced but not consumed. Then there is no rational expectations equilibrium with fiat money  $m$  in which there is a positive probability of trade in any period.

**Proof:** Suppose the contrary, and let  $m$  be a fiat money that is circulated in rational expectations equilibrium. By the rational expectations assumption, the expected frequency of agents holding  $m$  in inventory must be constant. But since  $m$  is not consumed, the expected frequency of agents holding  $m$  in inventory must strictly increase each time a unit of  $m$  is produced. Hence  $m$  is not produced, implying that  $m$ -producers must have a zero probability of consuming. However an  $m$ -producer will always accept her consumption good in exchange for whatever she is currently selling. But then an  $m$ -producer can never encounter an agent offering to sell her consumption good  $c$ . But for every  $g \in G$ ,  $g \neq m$ , there is an  $m$ -producer that consumes  $g$ , all agents must be selling  $m$ . Hence no trade takes place. This proves Theorem 7.

We conclude from Theorem 7 that a theory of fiat money requires more institutional structure than that of decentralized reciprocal trade. This result is not surprising, since fiat currencies the real world certainly take this form; i.e., they are provided by states or other actors with a considerable, non-infinitesimal impact on the economy.

There are generally multiple money goods. We prove only the case of equal relative storage costs, by which we mean that there exist  $\{\alpha_g | g \in G\}$  such that for any  $g \in G$ , and for any agent producing  $p$  and consuming  $c \neq g$ , we have  $\alpha_g = (u_c - k_p)/s_g$ .

**Theorem 8:** In a rational expectations equilibrium with one money good and equal relative storage costs, there are multiple money goods.

Proof: Consider a rational expectations equilibrium with a unique money good  $m$ . For simplicity of exposition, assume all agents consume only one good. Consider an agent  $a$  who neither produces nor consumes  $m$ , and let  $V_a(g)$  be the present value to agent  $a$  of currently holding inventory  $\{g\}$  and offering  $g$  for sale. Also write  $u_{ac}$  and  $k_{ap}$  for the utility of consumption and the cost of production to agent  $a$ . Then we have

$$V_a(p) = \mu(p,c)[(u_{ac}-k_{ap}) + \rho V_a(p)] + \mu(p,m)\rho V_a(m) \\ + [1 - \mu(p,c) - \mu(p,m)]\rho V_a(p) - s_p.$$

Thus

$$V_a(p) = \frac{\nu_{ap} + \mu(p,m)V_a(m)}{r + \mu(p,m)}, \quad (9)$$

where for any  $g \in G$ , we define

$$\nu_{ag} = [\mu(g,c)(u_{ac}-k_{ap}) - s_g](1+r) = s_g[\alpha_g\mu(g,c) - 1](1+r) \quad (10)$$

Similarly,

$$V_a(m) = \frac{a\nu_m + \mu(m,c)V_a(p)}{r + \mu(m,c)}. \quad (11)$$

Now since  $m$  is the only money good, we must have  $V_a(g) < V_a(p)$  for any good  $g \in G$  such that  $g \notin \{p,m,c\}$ . But

$$V_a(g) = \mu(g,c)[(u_{ac}-k_{ap}) + \rho V_a(p)] + \mu(g,m)\rho V_a(m) \\ + [1 - \mu(g,m) - \mu(g,c)]\rho V_a(p) - s_g,$$

so

$$V_a(g) = \frac{\nu_{ag} + \mu(g,c)V_a(p) + \mu(g,m)V_a(m)}{r + \mu(g,m)}.$$

Then

$$V_a(g) - V_a(p) = \frac{\nu_{ag} - \nu_{ap} + [\mu(g,m)-\mu(p,m)][V_a(m)-V_a(p)]}{r + \mu(g,m)} < 0. \quad (12)$$

Using (9), we find that a necessary condition for (12) is

$$\frac{\nu_{ag}}{r + \mu(g,m)} < \frac{\nu_{ap}}{r + \mu(p,m)} \quad \text{for all } g \notin \{p,m,c\}. \quad (13)$$

Suppose (13) holds and consider another agent b consuming c but producing g. By assumption agent b does not accept p in trade for g, so from (13)

$$\frac{\nu_{bp}}{r + \mu(p,m)} < \frac{\nu_{bg}}{r + \mu(g,m)} \quad \text{for all } g \notin \{p,m,c\}. \quad (14)$$

But from (10) (the assumption of equal relative storage costs),  $\nu_{ap} = \nu_{bp}$  and  $\nu_{ag} = \nu_{bg}$ . Thus (13) and (14) are inconsistent. It follows that the assumption that both agent a and agent b accept only m and the their own consumption good is incorrect. This proves Theorem 8.

The sufficient condition of Theorem 8 is doubtless excessively strong. Any assumption that leads to (13) and (14) will give the same result. These equations have a straightforward interpretation. The expression  $\nu_{ag}/[r+\mu(g,m)]$  represents the present value to agent a of offering g for sale from the current period until g is traded for money. Equation (13) says that this must be maximized by  $g = p$  for each agent. Only by coincidence will this be the case.

Of course we could obtain uniqueness by postulating that storage costs are non-uniform across agents. In particular, if each agent can store her own production good at lower cost than any other good, (13) becomes reasonable. For instance, suppose that except for m, all goods are 'services' that can be used only at the moment they are produced. Then for each agent  $s_p = 0$  and  $s_g = \infty$  for  $g \neq p,m$ .

## 5. Further Considerations

A number of interesting questions emerge from this model of monetary equilibrium.

**The Spontaneous Emergence of Money:** Suppose we start with an economy in which each agent has only her production good in inventory, and all agents know the frequency of various types of agents in the economy. What types of media of exchange emerge spontaneously from such an initial state? We have proven the existence of rational expectations equilibria under plausible conditions, but we have not proven the existence of a locally stable equilibrium, and a *fortiori*, have not demonstrated that an equilibrium will be attained from an arbitrary initial distribution of inventories.

That this problem is likely to be difficult should be evident from our experience with general equilibrium models of the Arrow-Debreu type, where no satisfactory theory has emerged despite extensive efforts (Fisher, 1983). If the Kiyotaki-Wright model can be implemented as 'face-to-face' interactions of agents who adjust their expectations dynamically in light of their exchange experience, there is some hope for a microanalytically satisfying solution to the problem (Carlson, 1987).

It appears likely that if agents update their subjective priors concerning the  $\{\mu(g,h) | g,h \in G\}$  in a 'reasonable' manner, a stable equilibrium will emerge from any initial distribution of inventories. Theorem 2 tells us that this equilibrium is likely to involve media of exchange. However unless we impose a good deal of symmetry on agents, it is unlikely that a universal medium of exchange will emerge. Rather, groups of agents are likely 'cluster' in the use of each others' goods as money.

I have attacked this problem through simulation using genetic algorithms (Langton, 1986; Goldman, 1989).<sup>7</sup> My approach (Gintis, 1990) has been to treat the agents in the economy as 'artificial life forms' whose trading strategies are 'genetically programmed' and who develop evol-

utionarily stable strategies through selection, reproduction, and mutation (Maynard Smith, 1982). These models exhibit both global stability and the emergence of a universal medium of exchange when agents are sufficiently symmetrically situated in terms of production, consumption, and storage costs.<sup>8</sup>

**Price Formation in a Monetary Economy:** We have assumed prices have already been determined in the economy, and we have normalized these prices to unity. It should not be difficult to prove the existence of a rational expectations equilibrium where relative prices are allowed to emerge from agents' knowledge of the technical conditions of production and the frequency of demand for various goods. However the dynamics of such a model are doubtless considerably complicated by relaxing the assumption of fixed prices.

**The Political Economy of Fiat Money:** Theorem 7 shows that we cannot derive a theory of fiat money as a straightforward extension of the theory of commodity money. If we assume the money producer does not consume, the conclusions of Theorem 7 are no longer valid: *a good that is neither produced nor consumed in equilibrium can serve as a universal medium of exchange.* We could of course arbitrarily postulate the existence of such a

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7. Previous research in this area, using learning algorithms based on the work of John Holland (Holland, 1986; Marimon, McGrattan, and Sargent, 1989) supports this conclusion.

8. It may be argued that a theory of money cannot be developed using simple life forms, since organisms with limited cognitive capacity at best engage in barter exchange (i.e., symbiosis). While a general biological theory of trade has not been attempted, biologists have described many examples. For instance, a flower 'trades' nectar for the pollen adhering to the bee's body. The immune and other defense systems of the flower involve a 'refusal to trade' with other insects, and similarly for the bee.

good *ex cathedra*, provided (and enforced) for instance by a benevolent despot. A remote real world analogue to such a benevolent despot is the modern state, which does consume and which fails normally to be benevolent. Thus our model leads us naturally into the realm of the political economy of money.

# List of Symbols

$A$	Set of agents
$c_a$	Good consumed by agent $a \in A$
$C_a$	Set of goods consumed by agent $a \in A$
$\hat{g}(i)$	Good offered for sale when inventory is $i$
$\hat{G}(i)$	Goods accepted in trade when inventory is $i$
$G$	Set of goods
$i.g.h$	Inventory obtained from inventory $i$ by trading $g$ for $h$
$i_t$	Inventory in period $t$
$I_a$	Set of admissible inventories for agent $a \in A$ .
$J[\sigma_a]$	Set of persistent inventories when trading strategy is $[\sigma_a]$
$k_p$	Cost of production of good $p$
$p$	Good produced by an agent
$P_{ij}$	Probability of moving from inventory $i$ to inventory $j$ in one period
$P[\tau]$	Markov transition matrix generated by trading strategy $\tau \in T$
$s_g$	Cost of storing good $g \in G$ for one period
$T_a$	The set of trading strategies $\{[\hat{g}(), \hat{G}()]\}$ for agent $a \in A$
$\tau_b$	The barter strategy
$\tau_a$	A trading strategy of agent $a$
$T_a^*$	The set of admissible trading strategies for agent $a$
$T_a^M$	The set of mixed trading strategies for agent $a$
$u_c$	Utility of consumption of good $c$
$u[\sigma_a]$	Stationary extended inventory of agent with trading strategy $\sigma_a$
$\tilde{v}_g$	Trading value of good $g$
$v(i)$	Expected flow of utility when holding inventory $i$



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