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NOTES ON MODELS OF OVERLAPPING GENERATIONS

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Contents

0. Introduction

I. A model of pure exchange

II. Equilibrium in the pure exchange model

III. Long-term assets, expectations, and perfect foresight equilibrium

IV. The pricing of an infinitely long-lived asset under perfect foresight

V. Intertemporal production

0. Introduction

The purpose of these notes is to acquaint students with a class of general equilibrium models that can be used to study various topics in intertemporal economics. These notes deal only with "real" (as opposed to "monetary") topics, but the models studied are useful background for those that are applicable to "monetary" topics. Here, we will study saving, investment, and the determination of interest rates and asset prices. We will also study how these depend on the government policy in effect; for example, on the extent to which the government borrows rather than taxes, and on the kind of social security program in effect.

Our approach is a microeconomic general equilibrium approach. We begin with a description of an environment--a description of people in terms of their preferences and what they own or start with prior to engaging in any activity, and a description of the total resources and the production technologies available. Aside from assuming that the outcome of interaction among individuals is a competitive equilibrium--a perfect foresight or rational expectations competitive equilibrium in versions where expectations matter--we make no other assumptions. This approach has two important virtues. First, when we study the effects of a policy, we can easily appraise the credibility of our assumptions about what in the economy is invariant to, or independent of, whether that policy or some other is in effect. Second, we can judge the effects of a policy using the tools of traditional welfare economics. Thus, for example, our models allow us to study the effects of a social security system on the utility people realize or the effects of a government deficit on the utility people realize.

The models we use are called models of overlapping generations or OG models. They were invented, independently, by Edmond Malinvaud [1954] and

Paul Samuelson [1958]. OG models are different from standard microeconomic models because they contain a double infinity. Since new generations are always appearing, there is an infinite number of people to be considered, even though only a finite number are alive at any time. And, since there is an infinite number of dates stretching into the future and at least one consumption good per date, there is, in addition, an infinite number of goods. This double infinity is responsible for several special features, one of which is the failure of Adam Smith's invisible hand proposition to hold in general in these models; that is, in OG models a competitive equilibrium is not necessarily Pareto optimal. As we will see, some of what we do depends on the double infinity in an essential way and some does not.

I. A model of pure exchange

We start out using a model without production. Such models are called pure exchange models. In them, the total resources consist of amounts of goods and the only market activity is trading or exchanging. It turns out that much of what we say about pure exchange will carry over to other models we study. In this section, we describe the physical aspects of the model.

Discrete time

All actions and events occur at points in time which we denote by an index t that takes on integer values. We will always be concerned with how an economy evolves from some initial or current date on into the indefinite future. For convenience, we label the initial or current date $t = 1$. This does not mean that the economy has no past; the effects of the past determine initial conditions at $t = 1$. Our point of view is, in part, motivated by the following sort of question: Given the past, how would things evolve from this time on under different policy rules?

Here are some questions we will return to. Why not ask only about what happens at $t = 1$? What does "rule" mean? Why ask about rules? Why, for example, not ask about different policy actions at $t = 1$ only?

Discrete time is to be contrasted with and is a special case of continuous time. In choosing discrete time, we are sacrificing generality for expositional and mathematical simplicity.

The population

Our models, overlapping generations models, get their name from the way population evolves. We take the time path of population as a given.

At each date t, a new generation appears; we call them generation t. Each member of generation t is alive at t and t+1 only. We consider two-

period lived individuals because that is the simplest case of interest. The size of generation t is denoted N(t). Thus, at any date t, there are members of two generations present: members of generation t-1 (the old) and members of generation t (the young). The number of people alive at time t is, therefore, N(t-1) + N(t).

Total resources

We assume that there is only one consumption good at each date. We call the time t consumption good, time t good. (You may want to think of the goods as being "time t bread," "time t+1 bread," and so on.) For now, we assume that there is no technology for converting time t good into time t+i good for $i \neq 0$. (Later we will drop that assumption and assume there is a "storage" technology that permits conversion of time t good into time t+1 good, but not necessarily one-for-one.) We let Y(t) be the total amount of time t good available to the economy. (We are not concerned with where this comes from. We are concerned only with how it gets used.)

Feasible consumption allocations

A consumption allocation describes who consumes what. A feasible consumption allocation--or, more simply, a feasible allocation--is a consumption allocation that is consistent with total resources and the technology. It is helpful to have some notation.

Let $c_i^h(t)$ be consumption of time t good of member h of generation i. Thus, $c_t^h(t)$ is consumption of time t good of h in generation t (consumption of h when young) and $c_t^h(t+1)$ is consumption of time t+1 good (consumption when old) of the same person. Also, let C(t) be total consumption of time t good. We require that consumption be nonnegative.

Definition. A consumption allocation is feasible if the implied total consumption path satisfies $C(t) \leq Y(t)$ for all $t \geq 1$.

Here are some exercises.

1. Let $N(t) = N > 0$ and let $Y(t) = yN > 0$ for all t. Prove that if $0 < \alpha < 1$, then $c_t^h(t) = \alpha y$, $c_{t-1}^h(t) = (1-\alpha)y$ for all h and $t \geq 1$ is feasible.
2. Let $N(t+1)/N(t) = n$ and let $Y(t) = yN(t)$ for all t. Prove that if $0 < \alpha < 1$, then $c_t^h(t) = \alpha y$, $c_{t-1}^h(t) = n(1-\alpha)y$ for all h and $t \geq 1$ is feasible.
3. Let $N(t-1) = 2$ and $Y(t) = 2$ for all $t \geq 1$. Prove that the following allocation is feasible: $c_0^h(1) = 1/2$ for $h = 1, 2$; $c_1^1(1) = 1/4$, $c_1^1(2) = 3/4$; $c_1^2(1) = 3/4$, $c_1^2(2) = 1/4$; $c_t^h(t) = c_t^h(t+1) = 1/2$ for $h = 1, 2$ and all $t \geq 2$.
4. Let $N(t-1) = 1$ and $Y(t) = 1$ for all $t \geq 1$. Show that the following is a feasible allocation: $c_{t-1}(t) = (1/2) - (1/2)^{t+1}$, $c_t(t) = (1/2) + (1/2)^{t+1}$, all $t \geq 1$.

Efficient consumption allocations

We use the term efficiency in what is a fairly standard way. A consumption allocation is efficient if there is no alternative feasible allocation which implies more total consumption of some good and no less of any good.

Here is an exercise.

5. Prove the following. A consumption allocation is efficient if and only if it satisfies $C(t) = Y(t)$ for all $t \geq 1$.

As this exercise shows, in pure exchange models (models without production), efficiency is not a very interesting concept.

Preferences

We will represent preferences by utility functions and their implied indifference curve maps. We assume that h in generation t for $t \geq 1$ has a utility function u_t^h which is a function of $c_t^h(t)$ and $c_t^h(t+1)$, and is defined for all positive values of its arguments. We assume that u_t^h is strictly increasing in each of its arguments and has continuous partial derivatives. Moreover, preferences are strictly convex in that if $c' \neq c''$, $u(c') \geq u(c'')$, and $1 > a > 0$, then $u_t^h[ac'+(1-a)c''] > u_t^h(c'')$. Note that c' and c" are each two-element vectors.

We assume that each member of generation 0, who is in the second period of life at t = 1, prefers more time 1 consumption to less.

Pareto optimal allocations

We begin with a definition of Pareto superiority.

Consumption allocation A is Pareto superior to consumption allocation B if no one prefers B to A and if at least someone prefers A to B. If A is not superior to B and B is not superior to A, then we say they are noncomparable. (In applying this definition, take into account all members of all generations including those of generation 0, the current old.)

Here are some exercises.

6. Let $u[c_t^h(t),c_t^h(t+1)] = c_t^h(t)^{1/2} + c_t^h(t+1)^{1/2}$ be the utility function for all members of all generations (except the current old) and consider the exercise 1 allocations. Describe two, one of which is Pareto superior to the other. Describe two which are noncomparable to each other.

We can now define Pareto optimality. A consumption allocation is Pareto optimal is there does not exist a feasible consumption allocation that is Pareto superior to it.

Here are some exercises.

7. Prove the following. If an allocation is Pareto optimal, then it is efficient. (In other words, efficiency is necessary for Pareto optimality.)
8. Consider the resources, technology, and allocations of exercise 1 and the preferences of exercise 6. Prove that the allocation determined by $\alpha = 3/4$ is <u>not</u> Pareto optimal.
9. Consider the setup and allocations of exercise 2 and the preferences of exercise 6. Show that if $n = 4$, then the allocation determined by $\alpha = 1/2$ is not Pareto optimal.
10. Consider the setup and allocation of exercise 3. Show that this allocation is not Pareto optimal if preferences are those described in exercise 6.
11. This is a generalization of the last exercise. Suppose $u_t^h[c_t^h(t), c_t^h(t+1)]$ is the utility function of person h in generation t, $t \geq 1$. Let the marginal rate of substitution (MRS) for this person be defined by $u_{t1}^h[c_t^h(t), c_t^h(t+1)] / u_{t2}^h[c_t^h(t), c_t^h(t+1)]$, where $u_{tj}^h[c_t^h(t), c_t^h(t+1)]$ is the partial derivative of u_t^h with respect to (w.r.t.) its jth argument. Suppose h and h' are two members of generation t, for some $t \geq 1$. Prove the following: a feasible allocation which implies different values of the MRS for h and h' is not Pareto optimal.

The last exercise establishes that a <u>necessary</u> condition for Pareto optimality is that all members of generation t have the same MRS. We will call this the <u>equality of MRS</u> condition. Another is efficiency (see exercise 7).

12. Prove the following. These conditions are not <u>sufficient</u> for Pareto optimality.

Our physical environments are such that it is difficult to derive conditions which are necessary and sufficient for Pareto optimality. We can, however, get some appreciation of the nature of the conditions.

13. Consider the setup of exercise 1 and the preferences of exercise 6. Prove that no allocation in the class described in exercise 1 is Pareto superior to the allocation given by $\alpha = 1/4$.
14. Within the same setup, describe all α's for which there does not exist a Pareto superior allocation among the class of allocations described in exercise 1.
15. Now consider the exercise 2 setup and the preferences of exercise 6. Describe all α's for which there does not exist a Pareto-superior allocation among the class of allocations described in exercise 2.

From these exercises, it seems that another necessary condition for Pareto optimality is that MRS's as defined in exercise 11 be sufficiently high. This is the condition that appears because of the double infinity in the model. There is no analogue of it in finite models.

Although I will not describe in any precise way how high MRS's must be in order to insure Pareto optimality, I can cite some results and also explain in a loose way how they are obtained.

To begin, assume the exercise 1 setup and the exercise 6 preferences. Consider the exercise 1 allocation with $\alpha = 3/4$. This allocation is not Pareto optimal. Starting with this allocation, consider how we should shift goods among people in order to produce a Pareto superior allocation. Suppose we take a little bit of time t good from each young person at each date and give it to each old person. Since there is an equal number of young and old at each date, this is feasible in the sense that it does not use up

more of any good than is available. Moreover, everyone prefers the resulting allocation to the one we started with. Certainly, the people who are old at $t = 1$ prefer it; they get more time 1 good, which is all they care about. To see that everyone else prefers it, sketch the indifference curve map and the initial and new allocation. It should be clear that the proposed alternative allocation puts each young person on a higher indifference curve if the "little bit" is not too big. Note that there exists a little bit which is not too big if the MRS at the initial allocation is smaller than unity. Unity shows up in this example because of feasibility. Given the setup of exercise 1, if we take a little bit from each young person, the most we can give to each old person is the same amount. If we were working with the setup of exercise 4, then the role of unity would be taken by the parameter n.

We can use the example just discussed to point out the crucial role played by the double infinity; that is, by the absence of a last date. Suppose, instead, that there is a last date. In particular, suppose the setup of exercise 1 applies only for dates 1, 2, ..., T, and suppose that the young people at T will not live to be old. Then the natural assumption to make about their preferences is that they care only about how much time T good they consume. In this case, then, we cannot produce a Pareto superior allocation by taking some time T good away from them.

Next, in the original infinite exercise 1 setup, consider the exercise 1 allocation with $\alpha = 1/4$. Let us consider the consequences of shifting some goods between young and old. If we take a little bit from each young person at each date and give that amount to each old person, then, as you should be able to show, only the old at $t = 1$ benefit; everyone else is made worse off. Suppose, instead, that we take a little bit from each old person and give it to each young person. This makes everyone but the old at $t = 1$

better off, but it makes them worse off. Thus, neither kind of shift produces a Pareto superior allocation. (Note that I cannot propose taking a little bit from each young person and giving, say, 100 times that little bit to each old person; that is not feasible in the exercise 1 setup.)

II. Equilibrium in the pure exchange model

Here we will study exchange in a private ownership version of our model in which we assume that individuals behave competitively. Competitive behavior means that individuals view themselves as being able to buy and sell any amounts at ruling prices without affecting those prices; that is, they treat prices parametrically.

Prices as always are market rates of exchange. In the present version of the model, we are interested in rates of exchange between time t good and time t+1 good for all $t \geq 1$. As is usual, we have to make some decision about how to express prices. One scheme, perhaps the most commonly used, is to express all prices in terms of time 1 good; for example, by letting, say, $\gamma(t)$ be the price of time t good in units of time 1 good with $\gamma(1) = 1$. We will start out with a different scheme, one that expresses exchange ratios in terms of one-period gross interest rates. Thus, we let $r(t)$ be the price of time t good in units of time t+1 good. (The correspondence between the two ways of expressing prices is $r(t) = \gamma(t)/\gamma(t+1)$.)

In order to describe competitive behavior and equilibrium, we need a description of who owns what; i.e., a description of what individuals start out with prior to trade. We use the word "endowment" to describe this and let $w_i^h(t)$ be the part of the endowment of member h of generation i that consists of time t good. In general, we assume that any member of generation t has an endowment consisting of some time t good and some time t+1 good and nothing of any other good.

Before we describe competitive trade formally, it may help to think in a general way about what kind of trade can occur in this economy. Consider the situation at some date t, when there are members of generation t-1 present (the old) and members of generation t present (the young). First note that

because there is only one good at each date, there is no potential for intergeneration (between generation) trade. No member of generation t-1 (no old person) has anything to give a member of generation t (a young person) in exchange for time t good; no young person has anything to give an old person in exchange for time t good which is of value to the old person. There may, however, be potential for intrageneration (within generation) trade among members of generation t. (As we will see later in a model with more objects in it, intergeneration trade is a possibility.)

In general, a competitive equilibrium is a set of prices and quantities that satisfy two conditions: (a) the quantities that are relevant for a particular person maximize that person's utility in the set of all quantities that are affordable given the prices and the person's endowment, and (b) the quantities clear markets. For the application now being considered the prices are an r(t) sequence (that is, r(1), r(2), r(3), ..., r(t)), while the quantities are a consumption allocation and, perhaps, quantities of loans granted or received. Roughly speaking, we will proceed as follows. Condition (a) will give us relationships between utility maximizing quantities and prices--relationships that are usually labeled demand or supply functions. And (b) picks out those prices consistent with the utility maximizing quantities being market clearing quantities.

Our first task is to find the relationship between utility maximizing quantities and prices. To do this, we first describe the set of affordable lifetime consumption bundles for some member h of generation t, $t \geq 1$. We suppose that person h faces some positive price r(t), a gross interest rate, at which h can borrow or lend. One way to describe the set of affordable consumption bundles is as follows:

$$(1) \qquad c_t^h(t) \leq w_t^h(t) - \ell^h(t)$$

(2) $$c_t^h(t+1) \le w_t^h(t+1) + r(t)\ell^h(t)$$

where $\ell^h(t)$ is the loans that h chooses to grant. It is crucial to bear in mind that a negative value of $\ell^h(t)$ means that h borrows.

We also find it convenient to describe the set of affordable consumption bundles for h by

(3) $$c_t^h(t) + c_t^h(t+1)/r(t) \le w_t^h(t) + w_t^h(t+1)/r(t).$$

As an exercise, prove that the pair of constraints, (1) and (2), is equivalent to (3); that is, prove the following two statements:

(i) If $(\bar{c}_t(t),\bar{c}_t(t+1))$ is a consumption bundle that satisfies (3), then there exists some value of $\ell^h(t)$ for which $(\bar{c}_t(t),\bar{c}_t(t+1))$ satisfies (1) and (2).

(ii) If $(\hat{c}_t^h(t),\hat{c}_t^h(t+1),\hat{\ell}^h(t))$ satisfies (1) and (2), then $(\hat{c}_t^h(t),\hat{c}_t^h(t+1))$ satisfies (3).

We now describe the relationship between utility maximizing choices and $w_t^h(t)$, $w_t^h(t+1)$, and $r(t)$. First, note that since utility is increasing in $c_t^h(t)$ and in $c_t^h(t+1)$, utility maximizing choices will satisfy (3) at equality. This implies that we can use (3) at equality and express h's utility as a function of a single variable. Thus, for example, we can express utility in terms of $c_t^h(t)$ as

$$u_t^h[c_t^h(t),c_t^h(t+1)] = u_t^h(c_t^h(t),w_t^h(t+1)+r(t)[w_t^h(t)-c_t^h(t)]).$$

We will express utility in terms of a new variable called saving and denoted s_t^h for h in generation t. We define s_t^h as $w_t^h(t) - c_t^h(t)$, the amount by which the endowment when young exceeds consumption when young. Using (3) at equality, utility as a function of saving is

(4) $$u[c_t^h(t), c_t^h(t+1)] = u[w_t^h(t) - s_t^h, w_t^h(t+1) + r(t)s_t^h].$$

Our general assumptions about utility imply that there is a unique value of s_t^h which maximizes the right-hand side (RHS) of (4) and that this maximum is given by the value of s_t^h at which the first derivative of the RHS of (4) is zero. Using the chain rule, the first derivative of the RHS of (4) is $-u_{t1}^h(\ ,\ ) + r(t)u_{t2}^h(\ ,\ )$, where, as above, u_{tj}^h is the partial derivative of $u_t^h(\ ,\ )$ w.r.t. its jth argument. Upon equating this derivative to zero, we have

(5) $$u_{t1}^h(\ ,\ )/u_{t2}^h(\ ,\ ) = r(t).$$

This says that the value of saving which maximizes utility is that which equates the MRS to r(t), which is the price of time t good in units of time t+1 good.

Here are two examples for which equation (5) can be easily solved for saving, s_t^h. These examples will be used repeatedly in exercises.

<u>Example 1</u>. $u_t^h(c_t^h(t), c_t^h(t+1)) = c_t^h(t)[c_t^h(t+1)]^\beta$, $\beta > 0$. Then $u_{t1}^h = [c_t^h(t+1)]^\beta$, $u_{t2}^h = \beta c_t^h(t)[c_t^h(t+1)]^{\beta-1}$, and $u_{t1}^h(\ ,\ )/u_{t2}^h(\ ,\ ) = c_t^h(t+1)/\beta c_t^h(t)$.

Using the definition of s_t^h and (3) at equality, we can rewrite this as

$$u_{t1}^h(\ ,\ )/u_{t2}^h(\ ,\ ) = [w_t^h(t+1) + r(t)s_t^h]/\beta[w_t^h(t) - s_t^h]$$

Thus, for this example, equation (5) takes the form

$$[w_t^h(t+1) + r(t)s_t^h]/\beta[w_t^h(t) - s_t^h] = r(t).$$

Verify that this can be solved for s_t^h and that the solution is

$$s_t^h = \beta w_t^h(t)/(1+\beta) - w_t^h(t+1)/(1+\beta)r(t)$$

<u>Example 2</u>. $u_t^h(c_t^h(t),c_t^h(t+1)) = [c_t^h(t)]^{1/2}+\beta[c_t^h(t+1)]^{1/2}$, $\beta > 0$.

Verify that for this example, equation (5) can be solved for s_t^h to obtain

$$s_t^h = \beta^2 r(t)w_t^h(t)/[1+\beta^2 r(t)] - w_t^h(t+1)/r(t)[1+\beta^2 r(t)].$$

In general, we will denote the value of s_t^h that maximizes (4) by the function $s_t^h[r(t),w_t^h(t),w_t^h(t+1)]$, which we call a saving function. Sometimes we will denote the function by $s_t^h(\quad)$ or even by s_t^h. For example, if h has the example 1 utility function we write $s_t^h[r(t),w_t^h(t), w_t^h(t+1)] = \beta w_t^h(t)/(1+\beta) - w_t^h(t+1)/(1+\beta)r(t)$. (For this case, compute the partial derivatives of s_t^h w.r.t. $r(t)$, $w_t^h(t)$, and $w_t^h(t+1)$. Interpret what you find in terms of income and substitution effects on the demand for $c_t^h(t)$ and $c_t^h(t+1)$.

In general, our assumptions about utility functions and the assumption that $w_t^h(t) > 0$ and $w_t^h(t+1) > 0$ imply that $s_t^h(r(t),w_t^h(t),w_t^h(t+1))$ has the following properties:

(a) It is continuous.

(b) There is a unique value of $r(t)$, say r^*, such that $s_t^h = 0$ and for $r(t) > r^*$, $s_t^h > 0$; while for $r(t) < r^*$, $s_t^h < 0$.

(c) $s_t^h(r(t)) < w_t^h(t)$ for all $r(t)$.

The saving function for h in generation t gives us the relationship between utility maximizing choices and prices. In order to complete the description of a competitive equilibrium, we must describe market clearing conditions.

In the present model, we assume that the total amount of time t good available is the sum of what individuals start with. Then, the condition that total consumption of time t good be equal to the goods available is

$$(6) \qquad \sum_h c^h_{t-1}(t) + \sum_h c^h_t(t) = \sum_h w^h_{t-1}(t) + \sum_h w^h_t(t) \text{ for all } t \geq 1,$$

where summation over h means summation over the members in the relevant generation.

Next, we use some implications of the individual budget constraints. At t = 1, the trivial budget constraint of h in generation 0 is $c^h_0(1) \leq w^h_0(1)$, which implies that choices satisfy this at equality or that $\sum_h c^h_0(1) = \sum_h w^h_0(1)$. By (6), this implies $\sum_h c^h_1(1) = \sum_h w^h_1(1)$. This equality and (3) at equality summed over h imply $\sum_h c^h_1(2) = \sum_h w^h_1(2)$. Then, using (6) again, we get $\sum_h c^h_2(2) = \sum_h w^h_2(2)$. By the obvious induction argument then, we get for all $t \geq 1$,

$$(7) \qquad \sum_h c^h_t(t) = \sum_h w^h_t(t),$$

which we can write as

$$(8) \qquad \sum_h \left(w^h_t(t) - c^h_t(t)\right) = 0, \text{ for } t \geq 1.$$

Since $w^h_t(t) - c^h_t(t)$ is saving of h in generation t, (8) says that total savings for the members of generation t must be zero in equilibrium.

We have already expressed the dependence between the choice of saving and r(t) for h in generation t by the function $s^h_t(\quad)$. Let us define the function $S_t[r(t)]$ by

$$(9) \qquad S_t[r(t)] = \sum_h s^h_t(\quad).$$

We call S_t the aggregate saving function on the part of generation t. (Note that we here use a notation that leaves implicit the dependence of aggregate saving on endowments.) The requirement that utility maximizing choices be market clearing is, therefore, the requirement that

(10) $$S_t[r(t)] = 0; \qquad t = 1, 2, \ldots .$$

Here are some exercises.

1. Find the $S_t[r(t)]$ function for the following cases:

(i) $N(t) = 100$; each h in generation t has the example 1 utility function with $\beta = 1$ and $w_t^h(t) = 2$, $w_t^h(t+1) = 1$.

(ii) $N(t) = 100$; each h has the example 1 utility function with $\beta = 1$, and

$$\left(w_t^h(t), w_t^h(t+1)\right) = \begin{matrix} (2,1);\ h = 1, 2, \ldots, 50 \\ (1,1);\ h = 51, 52, \ldots, 100 \end{matrix}$$

2. Use some or all of properties (a)-(c) of s_t^h listed above to prove the following about $S_t[r(t)]$, the aggregate saving function for generation t:

(a) $S_t[r(t)]$ is continuous.

(b1) There exists some value of $r(t)$, say $\underline{r}(t)$, such that if $r(t) \leq \underline{r}(t)$, then $S[r(t)] < 0$.

(b2) There exists some value of $r(t)$, say $\bar{r}(t)$, such that if $r(t) \geq \bar{r}(t)$, then $S[r(t)] > 0$.

(c) $S_t[r(t)] < \sum_h w_t^h(t)$.

3. Use properties (a), (b1), (b2) of the $S_t[r(t)]$ function to prove the following: For each t, there exists at least one value of $r(t)$ that satisfies equation (10).

4. Describe completely the competitive equilibria for the following economies:

(i) $N(t) = 100$ for all $t \geq 0$. Each member of generation t, $t \geq 1$ has the example 1 utility function with $\beta = 1$ and $\left(w_t^h(t), w_{t-1}^h(t)\right) = (2,1)$ for all h and $t \geq 1$.

(ii) Same as (i) except that $\left(w_t^h(t), w_{t-1}^h(t)\right) = (1,2)$ for all h and $t \geq 1$.

(iii) Same as (i) except that each member of generation t, $t \geq 1$, has the example 2 utility function with $\beta = 1$.

(iv) Same as (i) except that for all $t \geq 1$

$$\left(w_t^h(t), w_{t-1}^h(t)\right) = \begin{cases} (2,1); & h = 1, 2, \ldots, 50 \\ (1,1); & h = 51, 52, \ldots, 100 \end{cases}$$

(v) Same as (i) except that for all $t \geq 1$

$$\left(w_t^h(t), w_{t-1}^h(t)\right) = \begin{cases} (2,1); & h = 1, 2, \ldots, 60 \\ (1,1); & h = 61, 62, \ldots, 100 \end{cases}$$

(vi) Same as (i) except that for all h

$$\left(w_t^h(t), w_{t-1}^h(t)\right) = \begin{cases} (2,1) \text{ for } t = 1, 3, 5, \ldots \\ (1,1) \text{ for } t = 2, 4, 6, \ldots \end{cases}$$

5. (Pareto optimality) Under the assumption that the total amount of time t good available is the sum of individual endowments; namely, that $Y(t) = \sum_h w_t^h(t) + \sum_h w_{t-1}^h(t)$ for all $t \geq 1$. Prove that the competitive equilibrium of each of the following economies of the last exercise is <u>not</u> Pareto optimal: (i), (iii)-(vi). (You may find it helpful to review the material on feasible, Pareto superior and Pareto optimal allocations.)

At this point, we want to introduce a simple kind of taxation into our model--endowment taxation. This kind of taxation is often called "lump-sum" taxation because the amount to be paid is unaffected by choices that the individual makes. In our model, such taxes can either be payable when young (a tax on $w_t^h(t)$ for h in generation t) or payable when old (a tax on $w_t^h(t+1)$ for h in generation t). Whether payable when young or when old, we assume (unless we explicitly say otherwise) that people when young know the tax they must pay when old and take account of it when they decide how much to save. For a given tax scheme and given endowments, to find saving in the presence of the tax simply use net-of-tax or after-tax endowments in the saving function. By the way, taxes can be either positive (the individual makes a payment) or negative (the individual receives a payment or transfer).

6. (Taxes on young, transfers to old, or "social security" schemes) Here, we consider only balanced budget schemes, those for which the taxes collected from the young at any date t equal the transfers at that date to the old. Assume throughout that people when young know and take into account the transfers they will receive when old.

(i) Consider the exercise 4(i) economy and consider a scheme in which each person when young is taxed one unit and each person when old receives one unit as a transfer. Describe the competitive equilibrium and show that it is Pareto superior to that which obtains in the absence of the scheme.

(ii) Consider the same social security scheme within the context of the exercise 4(iv) economy. Compare the equilibria with and without the scheme.

(iii) Consider the exercise 4(i) economy but with $N(0) = 100$ and $N(t) = 2N(t-1)$ for all $t \geq 1$. If each person when young is taxed one unit,

what is the maximum that can be given to every person when old? Compare the competitive equilibria with and without this scheme.

(iv) Consider the exercise 4(i) economy but with $N(0) = 100$, $N(t) = 2N(t-1)$ for $t = 1, 2, \ldots, 10$ and $N(t) = N(10)$ for $t \geq 10$. Describe some alternative social security schemes for this economy. Relate the "difficulties" that arise to those that currently plague the U.S. social security system.

7. (Consumption vs. income taxes) To get started, we write versions of equations (1) and (2) that include a flat rate consumption tax, z_c, and a flat rate income tax, z_y:

$$c_t^h(t) \leq (1-z_y)w_t^h(t) - \ell^h(t) - z_c c_t^h(t) \tag{1'}$$

$$c_t^h(t+1) \leq (1-z_y)\left[w_t^h(t+1)+r(t)\ell^h(t)\right] - z_c c_t^h(t+1). \tag{2'}$$

Note that under the income tax, z_y, interest received is taxable and interest paid is deductible. Show, without making any particular assumptions about preferences or endowments, that all nonnegative pairs (z_c, z_y) satisfying $(1-z_y)/(1+z_c) = g$ for a given g satisfying $0 < g < 1$ are equivalent. As part of your answer, define equivalent.

We now want to introduce government borrowing into the model. It is easy to show that if government debt consists of sure (safe or certain) titles to the consumption good in the future, then one period debts or loans of the government must bear the same rate of return as private loans. We will assume such safety of government debt and the implied equality of rates of return. If we do, then we do not have to derive a special demand for government bonds or debt; individuals are indifferent as to whether their saving takes the form of debt issued by other members of their generation or debt issued by the

government. The effect of the presence of government debt is captured by: (a) the requirement that private saving at date t be equal to the time t value of outstanding government debt at t, both measured in units of the time t good, and (b) the requirement that government and private debt earn the same return. Requirement (a) says that we now equate the function $S_t[r(t)]$ to the time t value of outstanding government debt, not to zero.

8. (Endowment taxes vs. government borrowing) Consider the economy of exercise 4(i). Assume there is a government which will raise 25 units of time 1 good and transfer these units to members of generation 0. Compare the following ways of financing this scheme.

(a) The government collects 1/4 of a unit of time 1 good from each member of generation 1.

(b) The government at t = 1 sells securities which are titles to time 2 good (one-period bonds). At t = 1 it also announces that the members of generation 1 will be taxed equally at t = 2 (when they are old) to pay off bonds. (Hint: Show that the equilibrium under scheme (a) is also an equilibrium under this scheme.)

(c) The government at t = 1 sells securities which are titles to time 2 good (one-period bonds). It sells enough to get 25 units of time 1 good. It will tax the members of generation 2 equally at t = 2 to pay off the securities it sells at t = 1.

The result that the equilibrium under 8(a) is also an equilibrium under 8(b) is a particular instance of a general result that often goes by the name Ricardian equivalence, named after David Ricardo, an early 19th century English economist. The general result which we will prove can be stated as follows.

Proposition. Given an equilibrium under some pattern of lump-sum taxation and government borrowing, alternative (intertemporal) patterns of lump-sum taxation that keep the present value (at the given equilibrium rates of return) of each individual's total tax liability equal to that in the given equilibrium are equivalent to the following sense. Corresponding to each such pattern is a pattern of government borrowing and lending such that the given equilibrium consumption allocation, including consumption of the government, and the given equilibrium (rates of return) are an equilibrium under the alternative taxation pattern.

Proof. We can write the budget set of h in generation t, for $t \geq 1$, as:

(i) $c_t^h(t) = w_t^h(t) - \tau_t^h(t) - \ell^h(t)$

(ii) $c_t^h(t+1) = w_t^h(t+1) - \tau_t^h(t+1) + r(t)\ell^h(t)$

where the τ's are lump-sum taxes (the subscript refers to the generation).

Combining these, by eliminating $\ell^h(t)$, we have

(iii) $c_t^h(t) + c_t^h(t+1)/r(t) = w_t^h(t) + w_t^h(t+1)/r(t) - \left[\tau_t^h(t)+\tau_t^h(t+1)/r(t)\right].$

Let us denote the quantities and rates of returns in the given equilibrium by the respective symbols with a "-" over them. Thus, for all h and $t \geq 1$, $\bar{c}_t^h(t)$, $\bar{c}_t^h(t+1)$, $\bar{\ell}^h(t)$, $\bar{\tau}_t^h(t)$, $\bar{\tau}_t^h(t+1)$, and $\bar{r}(t)$ satisfy (i)-(iii) and are such that

(a) $\left(\bar{c}_t^h(t),\bar{c}_t^h(t+1)\right)$ maximizes h's utility subject to (iii) at $r(t) = \bar{r}(t)$ and the "-" taxes.

(b) $\sum\bar{\ell}^h(t) = \bar{B}(t)/\bar{r}(t)$, where the summation is over the members of generation t and where $\bar{B}(t)$ is the number of one period bonds at t, each bond being a title to one unit of time t+1 good.

Now consider any other tax pattern that for each h satisfies

(iv) $\tau_t^h(t) + \tau_t^h(t+1)/\bar{r}(t) = \bar{\tau}_t^h(t) + \bar{\tau}_t^h(t+1)/\bar{r}(t)$,

the condition that the present value of taxes be the same.

It follows from (iii) that when faced with $r(t) = \bar{r}(t)$, h has the same set of affordable consumption bundles as under the "-" taxation pattern. It follows that $(\bar{c}_t^h(t),\bar{c}_t^h(t+1))$ is again a utility maximizing consumption bundle. However, to achieve $(\bar{c}_t^h(t),\bar{c}_t^h(t+1))$ under $r(t) = \bar{r}(t)$ and the alternative taxation pattern, h must lend a different amount; in particular, (i) implies that the new level of lending, $\ell^h(t)$, is related to $\bar{\ell}_t^h$ by

(v) $\ell^h(t) = \bar{\ell}^h(t) - [\tau_t^h(t)-\bar{\tau}_t^h(t)]$.

Summing (v) over h in generation t and using (b), we have

(vi) $\sum\ell^h(t) = \bar{B}(t)/\bar{r}(t) - \sum[\tau_t^h(t)-\bar{\tau}_t^h(t)]$.

Now we must show that this level of total lending by the members of generation t clears the market in loans. Since the market clearing condition is that $\sum\ell^h(t)$ equals government borrowing, we must show that the RHS of (vi) is the alternative level of government borrowing at t implied by the alternative taxes and the condition that government consumption is unchanged.

The alternative level of government borrowing at each date $t \geq 1$ must satisfy

(vii) $B(t)/\bar{r}(t) = \bar{G}_t - [\tau_t(t)+\tau_{t-1}(t)] + B(t-1)$

where $\bar{G}_t$ is government consumption, $\tau_t(t)$ is total time t taxes on generation t, and $\tau_{t-1}(t)$ is total time t taxes on generation t-1. We are given that

(viii) $\bar{B}(t)/\bar{r}(t) = \bar{G}_t - [\bar{\tau}_t(t)+\bar{\tau}_{t-1}(t)] + \bar{B}(t-1)$.

Solving (viii) for $\bar{G}_t$ and substituting the result into (vii), we get for all $t \geq 1$

$$\text{(ix)} \quad B(t)/\bar{r}(t) = \bar{B}(t)/\bar{r}(t) - [\tau_t(t)-\bar{\tau}_t(t)] - [\tau_{t-1}(t)-\bar{\tau}_{t-1}(t)] + [B(t-1)-\bar{B}(t-1)].$$

For $t = 1$, $B(t-1) = \bar{B}(t-1)$ because $\bar{B}(0)$ is an inherited amount of debt which is not affected by policies adopted at $t = 1$. Also, as a consequence of the present value condition on taxes for the people who are old at $t = 1$, $\tau_0(1) = \bar{\tau}_0(1)$. (Since it is too late at $t = 1$ to adjust the taxes members of generation 0 paid when they were young, we do not vary the taxes they must pay when old.) Thus (ix) implies

$$\text{(x)} \quad B(t)/\bar{r}(t) = \bar{B}(t)(\bar{r}/t) - [\tau_t(t)-\bar{\tau}_t(t)]$$

for $t = 1$.

We now establish by induction that (x) holds for all $t > 1$.

For $t \geq 2$, (iv) can be lagged once and summed over h to get $\tau_{t-1}(t) - \bar{\tau}_{t-1}(t) = -[\tau_{t-1}(t-1)-\bar{\tau}_{t-1}(t-1)]\bar{r}(t)$. Upon substituting this into (ix), we get for all $t \geq 2$

$$\text{(xi)} \quad B(t)/\bar{r}(t) = \bar{B}(t)/\bar{r}(t) - [\tau_t(t)-\bar{\tau}_t(t)] + \{[\bar{\tau}_{t-1}(t-1)-\bar{\tau}_{t-1}(t-1)]\bar{r}(t-1)+B(t-1)-\bar{B}(t+1)\}.$$

Notice that if (x) holds for t-1, then the term in curly brackets, { }, in (xi) is zero. This, in turn, by (xi) implies that (x) holds for t. This induction step, together with the previously established result that (x) holds for $t = 1$, implies that (x) holds for all $t \geq 1$. Since the RHS of (x) is equal to the RHS of (vi), we have established market clearing for loans at every date. This completes the proof.

Notice that the proposition is general in that nothing specific was assumed about preferences, endowments, or taxes. The crucial hypothesis is the one concerning the present value of taxes on each individual, condition (iv) in the proof. You should be able to demonstrate that schemes (a) and (c) of problem 8 do not satisfy that condition. We now consider still another financing scheme that when compared to scheme (a) does not satisfy that condition.

9. (Borrowing forever) As in 8(c), the government sells enough securities at $t = 1$ to get 25 units of time 1 good which it transfers to members of generation 0. However, instead of taxing the members of generation 2 at $t = 2$, it sells enough new securities (new one-period bonds) to pay off the securities it issued at $t = 1$. At every subsequent date, it keeps doing this; it sells enough new securities to pay off the securities that come due. As completely as you can, describe the competitive equilibrium under this scheme for the exercise 4(i) economy. Is the equilibrium under this scheme Pareto superior to the competitive equilibrium in the absence of any policy? Is it Pareto optimal?

10. Analyze the financing scheme of problem 9 for two other economies: the economy of exercise 4(ii) and the economy of exercise 4(iv).

III. Long-term assets, expectations, and perfect foresight equilibrium

In exercises 8-10 of section II, we considered the consequences of government borrowing that takes the form of one-period bonds or loans. We now want to study the consequences of a government sale of longer-term bonds--bonds which pay off in two periods or in three periods or in k periods. For example, given the economy of exercise 4(i) of section II and given that the government at $t = 1$ sells enough bonds which are titles to time 3 good to raise 25 units of time 1 good and will tax the members of generation 3 equally at $t = 3$ to pay off the bonds, we want to describe what happens at $t = 1$, 2, and 3 and thereafter.

Without loss of generality, we will suppose that each bond is a promise to pay 1 unit of the consumption good at the date when it comes due. (In the language typically used, these are zero "coupon" bonds or "pure discount" bonds.) As we did above, we assume that promises are believed and, in fact, honored.

Now, if at any time t, a young person buys a bond which does not come due at t+1, then such a person must plan to sell it at t+1, presumably to some member of generation t+1. We may suppose that at t there is a price of such bonds, denoted $p_k(t)$ for bonds that pay off in k periods, at t+k. However at t, a potential purchaser of such bonds must guess the price at the next date. (After all, the purchasers at t+1, the members of generation t+1, are not yet present in the economy.) We begin our analysis of this situation by supposing that member h of generation t has a "guess" or a "forecast" about the price of such bonds at t+1. We denote the guess by $p_{k-1}^{h,e}(t+1)$ and assume it is not negative. (Here the superscript "h,e" stands for h's expectation. The subscript is k-1 because at t+1, the bonds pay off in k-1 periods.) Moreover, we treat this guess or forecast as one that h holds with certainty.

We begin by describing a version of the budget set for member h of generation t implied by the presence of a market in such bonds and h's forecast.

$$(11) \qquad c_t^h(t) = w_t^h(t) - \ell^h(t) - p_k(t)b_k^h(t)$$

$$(12) \qquad c_t^h(t+1) = w_t^h(t+1) + r(t)\ell^h(t) + p_{k-1}^{h,e}(t+1)b_k^h(t).$$

In addition to the consumption bundle, h chooses $\ell^h(t)$, one-period loans granted by h, and $b_k^h(t)$, the number of k-period bonds h purchases. The "givens" to individual h are the endowment, the w's; the time t price per bond, $p_k(t)$; the time t gross interest rate on one-period loans, r(t); and the price at which h expects to be able to sell the bonds at t+1, $p_{k-1}^{h,e}(t+1)$. For the moment, we do not ask how h arrived at this expectation.

Since $\ell^h(t)$ is not constrained as to sign, the above two constraints are equivalent to the single constraint obtained by, say, solving (11) for $\ell^h(t)$ and substituting the solution into (12). The result can be written

$$(13) \qquad r(t)c_t^h(t) + c_t^h(t+1) = r(t)w_t^h(t) + w_t^h(t+1) - b_k^h(t)[r(t)p_k(t)-p_{k-1}^{h,e}(t+1)].$$

As an exercise, show that (13) is equivalent to (11) and (12) in the same way as you showed earlier that (3) is equivalent to (1) and (2).

Given only that utility is increasing in both components of consumption, we can draw some conclusions from (13) about h's demand for bonds, h's choice of $b_k^h(t)$. If the term in square brackets that multiplies $b_k^h(t)$ is positive, then h chooses $b_k^h(t) = 0$ (we do not permit a negative demand for government bonds); if that term is negative, then h chooses to have an infinite number of bonds; if it is zero, then h is indifferent about the magnitude of $b_k^h(t)$. These remarks lead us to the following proposition:

Proposition. If there is unanimity--namely, $p^{h,e}_{k-1}(t+1) = p^{e}_{k-1}(t+1)$ for all h in generation t--and if some k period bonds exist at t, then in any equilibrium (in which, in particular, desired bond purchases equal the bonds supplied)

$$(14) \qquad r(t)p_k(t) = p^{e}_{k-1}(t+1).$$

Proof. As an exercise, present a proof.

The conclusion of this proposition says that the price of bonds, $p_k(t)$, the rate of return on loans, $r(t)$, and the expected price of bonds, $p^{e}_{k-1}(t+1)$, satisfy a present value formula, or, equivalently, that the one-period expected rate of return on bonds, $p^{e}_{k-1}(t+1)/p_k(t)$, equals the rate of return on one-period loans, $r(t)$. From now on, we assume unanimity, namely $p^{h,e}_{k-1}(t+1) = p^{e}_{k-1}(t+1)$ for all h in generation t.

Our first task is to define what we call a time t temporary equilibrium. Given $p^{e}_{k-1}(t+1)$, a time t temporary equilibrium is a bond price $p_k(t)$, an $r(t)$, bond purchases, loans granted and received at t, and a pattern of consumption of time t good which clears the markets for bonds, loans, and the consumption of good at time t. We want to describe how we go about finding a temporary equilibrium.

First, note that for prices that satisfy (14), (13) becomes (3). In other words, for prices that satisfy (14), which are the only kind that are consistent with equilibrium, the set of affordable consumption bundles depends only on $r(t)$ (and endowments) in the same way as described by (3). Moreover, if for a given $r(t)$, h wants to dissave--i.e., chooses $c^{h}_{t}(t) > w^{h}_{t}(t)$, then h borrows. If instead h wants to save--i.e., chooses $c^{h}_{t}(t) < w^{h}_{t}(t)$, then h either lends or buys bonds or some of both. Since in equilibrium, the rates of return on loans and bonds are equal, h cares only about total saving, not

its composition between loans granted and bonds purchased. Thus, maximization of utility subject to (13) gives us the same saving function for h as we found in section II; namely, s_t^h. And by summing these functions over all the members of generation t, we get the same aggregate saving function on the part of generation t; namely, $S_t[r(t)]$.

One equilibrium condition is that saving by the members of generation t, $S_t[r(t)]$, equals the time t value of government bonds supplied by both the members of generation t-1 and the government. If we assume that only bonds which pay off in k periods are outstanding at t, the number of which we denote by $B_k(t)$, the equilibrium condition is

$$(15) \qquad S_t[r(t)] = p_k(t)B_k(t).$$

Equation (15) contains two unknowns, r(t) and $p_k(t)$. We need at least one other condition if we are to hope to determine values for $p_k(t)$ and r(t). The other condition is (14). We define a temporary equilibrium in terms of both conditions.

A <u>temporary equilibrium at t</u>, then, is a pair $(r(t),p_k(t))$ that satisfies (14) and (15). Note that the "givens" are an S_t <u>function</u>, which, in turn, depends on the preferences and endowments of members of generation t; the number of bonds, $B_k(t)$; and the anticipated price of bonds at t+1, $p_{k-1}^e(t+1)$. Different "givens" will, in general, imply different solutions.

One way to "solve" (14) and (15) is to solve (14) for $p_k(t)$ and to substitute the result into (15). This gives us

$$(16) \qquad r(t)S_t[r(t)] = [p_k^e(t+1)]B_k(t).$$

Equation (16) contains only one unknown r(t). If we can solve it for r(t), then we can use that solution and (14) to solve for $p_k(t)$. Do the general

assumptions made imply that there exists at least one value of r(t) that satisfies (16)?

As an exercise, describe completely the time t temporary equilibrium for each of the following economies.

(i) $N(t) = 100$, $u_t^h(c_t^h(t), c_t^h(t+1)) = c_t^h(t)c_t^h(t+1)$, $(w_t^h(t), w_t^h(t+1)) = (2,1)$ for all h, $B_k(t) = 100$, and $p_{k-1}^e(t+1) = 1$.

(ii) Same as (i) except that $p_{k-1}^e(t+1) = 1/2$.

Notice that the temporary equilibrium depends on, among other things, the price anticipation, $p_{k-1}^e(t+1)$. Thus, if we do not go beyond this way of modeling, we must say that what happens depends on, among other things, what people believe about prices in the future. Economists have generally been reluctant to leave matters there, mainly because of a suspicion that beliefs about prices in the future can be explained.

In the context currently under discussion, economists have for a long time used as a hypothesis about beliefs something called perfect foresight. The hypothesis of perfect foresight is the hypothesis that forecasts or expectations are correct or that they equal the corresponding realizations.

To introduce this concept most simply, let us first suppose that $k = 2$; namely, that all the bonds outstanding at t pay off at t+2. Let us also suppose that at t+1 only bonds that pay off at t+2 will be outstanding. In this context, perfect foresight is the hypothesis $p_{k-1}^e(t+1) = p_{k-1}(t+1)$, or with $k = 2$, $p_1^e(t+1) = p_1(t+1)$. We first show how to apply the hypothesis. Then we will say why it is an obvious and attractive hypothesis in this context. Finally, we will note some of its implications.

In the circumstances being assumed here, we can use the perfect foresight hypothesis as follows. First, solve for $p_1(t+1)$. Second, use that

solution and perfect foresight to find $r(t)$ from (16), and then $p_2(t)$ from (14). Since we have already done the second step, here we will describe only the first step.

At t+1, the bonds that pay off at $t = 2$ are one-period bonds. Suppose the total number of such government bonds is $B_1(t+1)$. Then $p_1(t+1)$ must satisfy

$$(17) \qquad S_{t+1}[r(t+1)] = p_1(t+1)B_1(t+1).$$

Again, we have one equation in two unknowns, $r(t+1)$ and $p_1(t+1)$. However, a second equation is implied by the condition that the rate of return on these bonds at t+1 be $r(t+1)$, or that

$$(18) \qquad 1/p_1(t+1) = r(t+1).$$

Here are two exercises which ask you to apply the perfect foresight hypothesis.

1. Consider the economy of exercise 4(i) of section II. Suppose the government at $t = 1$ sells 25 bonds, each of which is a title to one unit of time 3 good--i.e., each bond when issued is a two-period bond. The proceeds from the bond sale are distributed in equal amounts to the members of generation 0; while the members of generation 3, the young at $t = 3$, are taxed equally to pay off the bonds. Describe the competitive equilibrium under the perfect foresight hypothesis.
2. Same as Exercise 1 except that instead of selling 25 bonds at $t = 1$, the government sells enough two-period bonds to raise 25 units of the time 1 good. Again, describe the competitive equilibrium under the perfect foresight hypothesis.

Perfect foresight is an attractive hypothesis because people have an incentive to form correct forecasts. (We will devote some class discussion to explaining this incentive.) Given this incentive, if you describe people as holding views or making forecasts that are not correct, you are probably giving a faulty description of their behavior. Perfect foresight has long been used in contexts like the one described above because it is easy to apply in that context. In the next section, we will be using perfect foresight in contexts in which it is not so easy to apply.

The following exercises deal with two closely related consequences of perfect foresight.

3. (Long-term and short-term rates of return) Consider a pure discount bond which at time t is a promise to one unit of time t+k good. Suppose its price at t is $p_k(t)$. The gross internal rate of return on this bond is defined to be $[1/p_k(t)]^{(1/k)}$.

(a) Under the perfect foresight hypothesis, prove the following:

$$[1/p_k(t)]^{1/k} = [r(t)r(t+1) \ldots r(t+k-1)]^{(1/k)}$$

where r(t+j) is the equilibrium one-period return at time t+j, and where the right side is a product of k terms raised to the power 1/k.

(b) Show, using an example, that the result is not true if perfect foresight does not hold.

4. (Irrelevance of the maturity composition of government debt)

(a) Same as Problem 2 except that instead of selling two-period bonds at t = 1, the government sells one-period bonds at t = 1 and at t = 2 sells enough one-period bonds to pay off those issued at t = 1. Describe the

equilibrium and compare it with that of problem 2. Comment on the role of the perfect foresight hypothesis.

(b) Formulate a general proposition concerning irrelevance of the maturity composition of government debt.

(c) Present a proof of your part (b) proposition.

IV. The pricing of an infinitely long-lived asset under perfect foresight

Here we suppose that there exists in the economy some amount of "land," A, where A is measured in acres, say. At each date t, this land throws off a "crop" consisting of D(t) (for dividend) units of the time t good. The crop is proportional to the acreage in the sense that xA acres of land throw off a time t crop equal to xD(t). In our market schemes, we will assume that A is owned initially by the people who are old at t = 1. In a competitive equilibrium, this land is sold each period; at time t, the (old) members of generation t-1 sell it to the (young) members of generation t. Without loss of generality, we assume that land is sold ex-dividend, after the old at t get the time t crop. We let p(t) be the (ex-dividend) price per acre of land in units of time t good. Let d(t) denote the crop per acre or D(t)/A. We are interested in, among other things, how p(t) gets determined for $t \geq 1$.

The presence of land does not lead us to change what we have been assuming about preferences or utility. People still "care" only about how much they consume. What land does is provide an alternative way of converting endowment of time t good into consumption of time t+1 good.

At any date t, all the land is offered for sale by the old at any positive price. In other words, the supply of land on the part of the old is perfectly inelastic at the quantity A. Thus, the price is determined by the demand for land on the part of the young at t.

We begin by describing one version of the budget set for member h of generation t, some young person:

$$c_t^h(t) = w_t^h(t) - \ell^h(t) - p(t)a^h(t) \tag{19}$$

$$c_t^h(t+1) = w_t^h(t+1) + r(t)\ell^h(t) + a^h(t)d(t+1) + a^h(t)p^{h,e}(t+1). \tag{20}$$

In addition to the consumption bundle, h chooses $\ell^h(t)$, one-period loans granted by h, and $a^h(t)$, acres of land purchased by h. The "givens" to the individual are the endowment, the w's; the time t land price, p(t); the time t gross interest rate, r(t); the per acre crop at t+1, d(t+1); and the price that h expects to be able to sell the land for at t+1, $p^{h,e}(t+1)$. For the moment, we do not ask how h arrived at this belief.

Since $\ell^h(t)$ is not constrained as to sign, the above two constraints are equivalent to the single constraint obtained by, say, solving (19) for $\ell^h(t)$ and substituting the solution into (20). The result can be written

$$(21) \qquad r(t)c_t^h(t) + c_t^h(t+1) = r(t)w_t^h(t) + w_t^h(t+1) - a^h(t)\left[r(t)p(t)-d(t+1)-p^{h,e}(t+1)\right].$$

As an exercise, show that (21) is equivalent to (19) and (20) in the same way as you showed earlier that (3) is equivalent to (1) and (2).

Given only that utility is increasing in both components of consumption, we can draw some conclusions from (21) about h's demand for land, h's choice of $a^h(t)$. If the term in square brackets that multiplies $a^h(t)$ is positive, then h chooses $a^h(t) = 0$ (we do not permit a negative demand for land); if that term is negative, then h chooses to have an infinite amount of land; if it is zero, then h is indifferent about the magnitude of $a^h(t)$. These remarks lead us to the following proposition.

<u>Proposition</u>. If there is unanimity--namely $p^{h,e}(t+1) = p^e(t+1)$ for all h in generation t--then in any equilibrium (in which desired land purchases equal the land supplied)

$$(22) \qquad r(t)p(t) = d(t+1) + p^e(t+1).$$

Proof. As an exercise, present a proof.

The conclusion of this proposition says that the price of land, $p(t)$, the rate of return on loans, $r(t)$, and the expected payoff from the land, $d(t+1) + p^e(t+1)$, satisfy a present value formula, or, equivalently, that the rate of return on land, $[d(t+1)+p^e(t+1)]/p(t)$, equals the rate of return on loans, $r(t)$.

From now on, we assume unanimity; namely, $p^{h,e}(t+1) = p^e(t+1)$ for all h in generation t.

Again, we first define a time t temporary equilibrium. Given $p^e(t+1)$, a time t temporary equilibrium is a land price $p(t)$, an $r(t)$, land purchases, loans granted and received at t, and a pattern of consumption of time t good which clears the markets for land, loans, and the consumption good at time t.

We want to describe how we go about finding a temporary equilibrium.

First, note that for prices that satisfy (22), (21) becomes (3). In other words, for prices that satisfy (22), which are the only kind that are consistent with equilibrium, the set of affordable consumption bundles depends only on $r(t)$ (and endowments) in the same way as described by (3). Moreover, if for a given $r(t)$, h wants to dissave--i.e., chooses $c_t^h(t) > w_t^h(t)$, then h borrows. If, instead, h wants to save--i.e., chooses $c_t^h(t) < w_t^h(t)$, then h either lends or buys land or some of both. Since in equilibrium, the rates of return on loans and land are equal, h cares only about total saving, not its composition between loans granted and land purchases. Thus, maximization of utility subject to (21) gives us the same saving function for h as we found in section II; namely, s_t^h. And by summing these functions over all the members of generation t, we get the same aggregate saving function on the part of generation t; namely, $S_t[r(t)]$.

Assuming that there are no government bonds outstanding, the equilibrium condition now is

(23) $S_t[r(t)] = p(t)A.$

We can "derive" (23) in the same way as we derived (10). In place of (6) we get the new market clearing condition

(24) $\sum_h c^h_{t-1}(t) + \sum_h c^h_t(t) = \sum_h w^h_{t-1}(t) + \sum_h w^h_t(t) + D(t).$

But total consumption of the old at t now satisfies

(25) $\sum_h c^h_{t-1}(t) = \sum_h w^h_{t-1}(t) + D(t) + p(t)A.$

Upon substituting from (25) into (24), we get (23).

Note, by the way, that we now assume that Y(t), the total amount of time t good available in this economy, is equal to the sum of individual endowments plus the total crop, the right-side of (24). It is important to remember this if you are asked questions about feasible or optimal consumption allocations.

You can interpret equation (23) as saying that net saving on the part of generation t is equal to net dissaving on the part of generation t-1, all of which is in the form of sales of land.

Equation (23) contains two unknowns, r(t) and p(t). We need at least one other condition if we are to hope to determine values for p(t) and r(t). The other condition is (22).

A <u>temporary equilibrium at t</u>, as above, is a pair $[r(t),p(t)]$ that satisfies (22) and (23). Note that the "givens" are an S_t <u>function</u>, which, in turn, depends on the preferences and endowments of members of generation t; an amount of land, A; the per acre dividend at t+1, d(t+1); and the anticipated

price of land at t+1, $p^e(t+1)$. Different "givens" will, in general, imply different solutions.

One way to "solve" (22) and (23) is to solve (22) for p(t) and substitute the result into (23). This gives us

$$(26) \qquad r(t)S_t[r(t)] = [d(t+1)+p^e(t+1)]A.$$

Equation (26) contains only one unknown r(t). If we can solve (26) for r(t), then we can use that solution and (22) to solve for p(t).

We will use the term "sequence of temporary equilibria" to refer to a sequence comprised of a time t temporary equilibrium for each $t \geq 1$.

As an exercise, describe completely the sequence of temporary equilibria for each of the following economies:

(i) $N(t) = 100$, $u_t^h(c_t^h(t), c_t^h(t+1)) = c_t^h(t)c_t^h(t+1)$ and $(w_t^h(t), w_t^h(t)) = (2,1)$ for all h and $t \geq 1$. $A = 100$, $d(t) = 1$, and $p^e(t+1) = 1$ for all $t \geq 1$.

(ii) Same as (i) except that $p^e(t+1) = 1/2$ for all $t \geq 1$.

(iii) Same as (i) except that $d(t) = 1/2$ if t is odd and $d(t) = 3/2$ if t is even.

(iv) Same as (i) except that for all $t \geq 1$

$$(w_t^h(t), w_{t-1}^h(t)) = \begin{cases} (2,1) \text{ for } h = 1, 2, \ldots, 50 \\ (1,1) \text{ for } h = 51, 52, \ldots, 100 \end{cases}$$

(v) Same as (i) except that for all h

$$(w_t^h(t), w_{t-1}^h(t)) = \begin{cases} (2,1) \text{ for } t = 1, 3, 5, \ldots \\ (1,1) \text{ for } t = 2, 4, 6, \ldots \end{cases}$$

Notice that the sequence of temporary equilibria depends on, among other things, the sequence of price anticipations. Thus, if we do not go beyond this way of modeling, we must say that what happens depends on, among other things, what people believe about prices in the future.

One approach to explaining beliefs is to impose hypotheses about the relationship between beliefs about prices in the future and other variables. One version of this approach is to hypothesize that beliefs about prices in the future are formed by extrapolation from current and previous prices. One particular version is called static expectations; it says that people believe the price in the future will be the same as the current price. In our notation, the static expectations hypothesis is expressed as $p^e(t+1) = p(t)$ for all t.

Here is an exercise. In each of the economies of the last exercise, replace the sequence of price anticipations by the static expectations hypothesis and find the implied sequence of temporary equilibria.

We have now found sequences of temporary equilibria for several economies. In some, the hypothesized beliefs turn out to be correct, in others not. Notice, in particular, that a hypothesis like static expectations is consistent with beliefs being correct in some economies and not in others.

Why, though, should we be concerned about whether hypothesized beliefs turn out to be correct? We are concerned about this because people have an incentive to try to be correct, an incentive which is no weaker than the incentive to maximize profit or utility. That being so, a modeling approach which ignores that incentive is not likely to give good predictions.

Perfect foresight or rational expectations competitive equilibrium

Instead of describing how people form correct beliefs, we will insure consistency with the incentive to be correct by again imposing the

hypothesis that beliefs are correct. Usually this is called making use of a perfect foresight or rational expectations equilibrium concept.

Here is a definition applicable to this land economy. A perfect foresight competitive equilibrium consists of sequences for $p(t)$ and $r(t)$ and the other endogenous (dependent) variables such that the time t values are a time t temporary equilibrium for $p^e(t+1) = p(t+1)$. More generally, a perfect foresight competitive equilibrium is a sequence of values for the endogenous variables with the property that the tth term of the sequence is a time t temporary equilibrium given that beliefs about prices in the future are given by the subsequent term(s) in the sequence.

From now on the term "equilibrium" will be used to refer to a competitive, perfect foresight equilibrium.

It is more difficult to apply perfect foresight in this model than it is to apply it in the setups we studied in section III. There, our examples were carefully constructed so that we could always find the perfect foresight forecasts applicable to some future date and then work backwards. Here, that is not possible.

Let's begin by working in a somewhat formal way with equations (22) and (23). The definition of an equilibrium (perfect foresight) tells us to set $p^e(t+1) = p(t+1)$. Suppose we do that in (22) and write the result as

$$r(t)p(t) = d(t+1) + p(t+1). \tag{27}$$

Next, suppose we solve (27) for $r(t)$ and substitute the result into (23) to get

$$S_t[(d(t+1)+p(t+1))/p(t)] = p(t)A. \tag{28}$$

If we can find a nonnegative $p(t)$ sequence that satisfies (28) for all $t \geq 1$, we will have found an equilibrium. This is so because given such a $p(t)$ sequence, we can easily find a corresponding $r(t)$ sequence (from (27)). Then these together can easily be shown to satisfy the definition of an equilibrium.

Here are three economies we will use in several exercises.

<u>Economy 1</u>. $N(t) = 100$, $u_t^h(c_t^h(t), c_t^h(t+1)) = c_t^h(t)c_t^h(t+1)$, and $(w_t^h(t), w_{t-1}^h(t)) = (2,1)$ for all h and $t \geq 1$. Also, $A = 100$ and $d(t) = 1$ for all $t \geq 1$.

<u>Economy 2</u>. Same as Economy 1 except that for all $t \geq 1$

$$(w_t^h(t), w_{t-1}^h(t)) = \begin{cases} (2,1) \text{ for } h = 1, 2, \ldots, 50 \\ (1,1) \text{ for } h = 51, 52, \ldots, 100 \end{cases}$$

<u>Economy 3</u>. Same as Economy 1 except that $d(t) = 1/2$ if t is odd and $d(t) = 3/2$ if t is even.

As an exercise, write the explicit forms of equation (28) for $t = 1$, 2, 3, and 4 for each of the above economies. Next write in a succinct way the explicit forms of equation (28) for all $t \geq 1$ for each of those economies.

Notice that (28) for $t = 1$ can be written $S_1[(d(2)+p(2))/p(1)] = p(1)A$. Given that we know the form of the function S_1 (as you do in the last exercise) and that we know $d(2)$ and A, this equation contains two unknowns $p(1)$ and $p(2)$. Hence, it is not likely that we can solve for a unique pair that satisfies it. We can get another equation in $p(2)$ from (28) for $t = 2$--namely, $S_2[(d(3)+p(3))/p(2)] = p(2)A$--but this contains another unknown, $p(3)$. Obviously, if we proceed to $t = 3$, to $t = 4$, and so on, we will always be one equation short. (Notice, by the way, that if we somehow knew $p(T)$ for some $T > 1$, then we could use these equations to find $p(T-1)$, $p(T-2)$, ... $p(1)$.)

Sometimes when we have fewer equations than unknowns, there are any number (an infinite number) of solutions. For example, there are an infinite number of pairs (x,y) that satisfy the single equation $y = 10 + 2x$, since for any x, there is a value of y for which the equation holds. To consider whether this is necessarily our situation, we proceed as follows. We could pick arbitrarily a positive value of p(1) and use equation (28) for $t = 1$ to get a value of p(2) (or perhaps several). Then, we could use that value of p(2) in equation (28) for $t = 2$ to get a value of p(3) and so on. If this procedure always generates a nonnegative p(t) sequence, then our situation would be analogous to having one linear equation in two unknowns.

As an exercise, try this procedure for the explicit forms of equation (28) you found in the last exercise. What you should find is that the procedure eventually implies negative values of p(t) for almost all values of p(1). This shows that we cannot start with any value of p(1) and find a corresponding sequence that is an equilibrium because, remember, an equilibrium consists of nonnegative prices.

Now try to prove that each of economies 1 and 2 has a unique equilibrium and that that equilibrium has the property that p(t) is a constant for all $t \geq 1$. Also, determine in which economy the price of land is higher. Finally, for each economy, write an equation that the constant price of land satisfies.

Before we turn to an analysis of economy 3 and then to a final set of exercises on this model, some general comments on uniqueness of equilibrium are in order. The above examples show that even though we seem to be "one equation short," it can happen that there is a unique equilibrium. This does not mean that uniqueness of equilibrium is the typical situation in our model. It is well known that multiple equilibria can arise in the standard

model, one with a finite number of individuals and commodities. That being so, we should be very surprised if multiple equilibria could not arise in our model which has an infinite number of individuals and commodities. One lesson to be learnt from our examples, though, is the following. The mere fact that our equilibrium conditions are a difference equation system without "enough" initial conditions is not by itself sufficient to imply the existence of multiple equilibria.

Now we will study economy 3. Economies 1 and 2 are easy to analyze because for those economies an equilibrium p(t) sequence is one that satisfies an equation of the form $p(t+1) = f[p(t)]$ for all $t \geq 1$, where f is some function that does not depend on t. We cannot quite do this for economy 3. For economy 3, we can, however, find two functions, call them f_1 and f_2, such that an equilibrium p(t) sequence is one that satisfies:

$$(29) \qquad p(t+1) = \begin{cases} f_1(p(t)) & \text{if } t \text{ is odd} \\ f_2(p(t)) & \text{if } t \text{ is even} \end{cases}$$

(As an exercise, write explicit expressions for f_1 and f_2 for economy 3. Also, sketch those functions in the $(p(t),p(t+1))$ plane and try to analyze solutions graphically.)

Notice that equation (29) says that any equilibrium satisfies $p(2) = f_1(p(1))$, $p(3) = f_2(p(2))$, $p(4) = f_1(p(3))$, $p(5) = f_2(p(4))$, and so on. It follows by substitution that any equilibrium p(1), p(3), and p(5) must satisfy $p(3) = f_2[f_1(p(1))]$, $p(5) = f_2[f_1(p(3))]$ and, more generally, that for any equilibrium:

$$(30) \qquad p(t+2) = f_2[f_1(p(t))] \equiv g_1(p(t)) \text{ if } t \text{ is odd.}$$

As an exercise, write an explicit expression for the composite function, g_1, for economy 3.

Equation (30) is useful because it says that the odd dated terms of an equilibrium p(t) sequence satisfy $p(t+2) = g_1(p(t))$ where g_1 is a function that does not depend on time. As an exercise, treat $p(t+2) = g_1(p(t))$ as a first-order difference equation and analyze its solutions. In particular, prove that there is one and only one nonnegative sequence which satisfies $p(t+2) = g_1(p(t))$ and that this is a constant sequence. Denote this constant $\bar{p}_1$ and show that $\bar{p}_1 > 0$. Next let $\bar{p}_2 = f_1(\bar{p}_1)$ and prove the following: the sequence $p(t) = \bar{p}_1$ if t is odd and $p(t) = \bar{p}_2$ if t is even satisfies (29) and is the only sequence that satisfies (29). Also determine whether $\bar{p}_2 > \bar{p}_1$ or vice versa. For this last question, it may be helpful to note that by substitution, the even dated terms of an equilibrium for economy 3 must satisfy the composite function $p(t+2) = f_1[f_2(p(t))]$.

Now, here are some exercises that require applying the model of this section.

1. (Land prices and the size of the crop) Consider economy 1 but with $d(t) = d > 0$ for all t. Show that the only equilibrium land price sequence is a constant sequence and that the value of the constant, p, say, depends on d. Does $p \to 0$ as $d \to 0$? Explain.

   Next answer this same question for an economy exactly like economy 1 but with $(w_t^h(t), w_{t-1}^h(t)) = (1,2)$.

2. (Time-series vs. cross-section observations) Suppose you have time-series data for economy 3, data which consist of observations on the pairs $(d(t), p(t))$ for different dates. Plot these pairs on a chart with d(t) on the horizontal axis and p(t) on the vertical axis. Now consider the

following questions. What are the equilibrium prices of land in two other economies that are identical to economy 3 except that in one d(t) is always 1/2, while in the other d(t) is always 3/2? Is there any simple way to use the time-series observations for economy 3 to answer these questions? Why?

3. (An "estimation" problem) You are an economist living in economy 3. You are asked what the equilibrium land price would be in economies that are identical except that they have different time patterns of crops--for example, crops that do not vary over time. Here is what you know. You have data on A and on $\{d(t),p(t)\}$ for your economy. You also know that the aggregate saving function for every generation has the <u>form</u> $S[r(t)] = a_0 - a_1/r(t)$. However, you do not start out knowing the pair of parameters (a_0,a_1). Outline in detail a procedure for answering the question.
4. (International "capital" flows and controls on "capital" flows) Consider a world economy that consists of two countries, country V and country W. Country V has the characteristics of economy 1 while country W has the same characteristics except that it contains twice as much land--200 acres, each acre of which has a crop of one unit of the consumption good at each date. Assume that neither people nor land can physically move between countries, but that the consumption good can be transported without cost between countries.

Here are elements of some alternative government policy rules: (a) laissez-faire (LF) at t--At time t, any member of generation t can buy land located anywhere. Also, there can be borrowing and lending between residents of different countries. (b) portfolio autarky (PA) at t--A member of generation t of a particular country can buy land located in that country only and can lend to or borrow from residents of that country only.

(i) Describe and compare equilibria under PA for all $t \geq 1$ and under LF for all $t \geq 1$. Describe who is better off under which policy. Prove that the equilibrium under PA is not Pareto optimal.

(ii) Describe the equilibria under PA for $t = 1$, LF for $t \geq 2$. Assume that the entire policy (path or rule) is known at $t = 1$. (Hint: First solve for what happens for $t \geq 2$ and then use the $t = 2$ outcome to solve for what happens at $t = 1$.)

(iii) Does what happens at $t = 1$ depend on whether LF or PA will be in effect for $t \geq 2$? What does this suggest about why we analyze policy <u>rules</u> or <u>paths</u> rather than different <u>actions</u> at $t = 1$?

5. (Consumption vs. income taxes) For economies with land, formulate and prove an equivalence result between consumption and income taxes along the lines of the proposition in exercise 7 of section II.

6. (A tax on land rents) Read a little about Henry George, a 19th century American who advocated a single tax, a tax on land rents. Describe how you would use the model of this section to analyze the effects of the kind of tax system George favored.

V. Intertemporal production

We begin with a very simple production technology which you can think of as a storage technology. We assume that for all $t \geq 1$, k units of time t good can be converted into xk units of time t+1 good, where k is any nonnegative number and x is some given nonnegative number (which does not depend on time or k).

Feasible allocations

We let K(t) denote the total amount of time t good stored at time t or used as input for the storage technology. A path of total consumption C(1), C(2), ..., C(t) is feasible if given K(0), there exists a nonnegative K(t) sequence for $t \geq 1$ that satisfies:

$$C(t) + K(t) \leq Y(t) + xK(t-1); \text{ all } t \geq 1. \tag{31}$$

Note that we say "given K(0)" because at t = 1, our starting date, the amount of time 0 good placed into storage at time 0 is already determined. (The symbols other than K(t) and x have the same meaning they had in section I.)

The right side of (31) represents the total amount of time t good available to the economy. The left side represents total usage of that good, consumption, C(t), plus investment, K(t). Thus, (31) says that total use of time t good cannot exceed the amount available.

Here are two exercises.

1. Suppose N(t) = 1 and Y(t) = 1 for all $t \geq 0$ and x = 2 and K(0) = 0. Also, suppose that $u_t^h(c_t^h(t), c_t^h(t+1)) = c_t^h(t)c_t^h(t+1)$ for all h and $t \geq 1$. Show that the following allocation is feasible but not Pareto optimal:

$$(c_t^h(t), c_{t-1}^h(t)) = (1/2, 1/2) \text{ for all h and } t \geq 0.$$

2. Prove the following. Any feasible allocation that is such that

$$u^h_{t1}\big(c^h_t(t),c^h_t(t+1)\big)/u^h_{t2}\big(c^h_t(t),c^h_t(t+1)\big) < x$$

for some h and some $t \geq 1$ is not Pareto optimal.

Competitive equilibrium

We begin by writing a revised budget set, revised versions of (19) and (20). These are:

$$c^h_t(t) = w^h_t(t) - \ell^h(t) - p(t)a^h(t) - k^h(t) \tag{32}$$

$$c^h_t(t+1) = w^h_t(t+1) + r(t)\ell^h(t) + a^h(t)[d(t+1)+p^e(t+1)] + xk^h(t) \tag{33}$$

where $k^h(t)$, the only new symbol, stands for the amount of time t good that member h in generation t places into storage. Here are several exercises.

3. Prove the following. In any competitive equilibrium, perfect foresight or not, $r(t) \geq x$ for all t. (Use an argument similar to that used to get equation (22) in section IV.)
4. Describe a no-land economy with an equilibrium that satisfies $r(t) > x$ for all $t \geq 1$.
5. Describe a no-land economy with an equilibrium that satisfies $r(t) = x$ for all $t \geq 1$.
6. Describe an economy with land and with an equilibrium (perfect foresight, here and below) with $r(t) > x$ for all $t \geq 1$.
7. Describe an economy with land and with an equilibrium with $r(t) = x$ for all $t \geq 1$.
8. (Taxes on land rents again) Study the effects of substituting a tax on land rents for a flat rate tax on endowments.

9. (Consumption vs. income taxes again) Consider a no-land model in which the equilibrium is such that storage occurs at every date and in which $x > 1$. Show that consumption and income taxes are not in general equivalent in the sense of the proposition listed in exercise 7 of section II. Assume here that the income tax is levied on the endowment, on gross interest as in (2') and on the gross return from storage; namely, $xk^h(t-1)$ at t. Also, show that in general an equilibrium with $z_y > 0$ is not Pareto optimal. Discuss whether the same is true for $z_c > 0$.