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VINTAGE HUMAN CAPITAL, GROWTH, AND THE DIFFUSION OF NEW TECHNOLOGY

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ABSTRACT

We present a model of vintage human capital. The economy exhibits exogenous deterministic technological change. Technology requires skills that are specific to the vintage. A stationary competitive equilibrium is defined and shown to exist and be unique, as well as Pareto optimal. The stationary equilibrium is characterized by an endogenous distribution of skilled workers across vintages. The distribution is shown to be single peaked and, under general conditions there is a lag between the time when a technology appears and the peak of its usage, what is known as diffusion. An increase in the rate of exogenous technological change shifts the distribution of human capital to more recent vintages and increases the relative wage of the unskilled workers in each vintage.

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Section 1. Introduction

Neoclassical models of growth have proved to be useful in both theoretical and empirical analysis. The key source of growth in per capita output in such models is exogenous technological change which is often assumed to be disembodied. Casual empiricism suggests that actual technological change is embodied in very specific types of skills as well as specific types of physical capital. The Schumpeterian notion of "creative destruction" relies heavily on capital specificities. In a world characterized by such specificities we would expect that new, more productive technologies will advance more slowly that they would in a world where all capital is costlessly transferable.

In this paper, we construct a model in which we consider the polar extreme of costlessly transferable capital. Instead, decisions on investment are irreversible. The first model of this kind of 'putty clay' capital was by Solow [1960] who examined situations where the types of capital could be aggregated and economy wide output represented by a single production function.

Our interest is in developing a model in which we can analyze the interaction between capital specificities and the rate of advance of new technology. It is undeniable that dramatic advances in technology (the invention of computers and word processors comes to mind) achieve large scale use only after a prolonged period of time. Even after a long period of time we still find the abacus or the typewriter useful for some purposes. In addition, as Mansfield [1968] points out "it took 20 years or more for all of the major firms (in several industries) to install

centralized traffic control, car retarders, by-product coke ovens and continuous annealing." It is important to note that none of these inventions were patentable by their users.

Clearly, the fact that the capital goods have already been produced implies that, it would not be rational simply to destroy them. We are interested, however, in going a step further. We wish to examine circumstances under which even though in a sense newer technologies are superior, resources are used to create capital which is specific to older technologies. Continuing with our earlier example we want to understand not only why typewriters are used but why they continue to be produced.

One possible explanation for this phenomenon is that an important component of capital is in the form of human capital. The skills involved in a production process are sometimes transferable only to a limited extent to new production processes. In addition, it is often true these skills are acquired only by participation in the production process itself.

These considerations prompt us to develop a model of human capital which is acquired in the process of production and is specific to the particular technology of production. In order to focus our attention on this problem we abstract from physical capital entirely. The learning of new skills or the transfer of existing knowledge presumably occurs largely from older workers to younger workers.

We try to capture these phenomena in an overlapping generations model in which a new generation is born in each period and lives for two periods. There is a single commodity produced

in each period. At the beginning of each period, a new technology becomes available which is more productive than any of the preexisting technologies. We assume that this exogenous technical change is deterministic and that the new technology is more productive by a constant factor than the technology that became available one period earlier.

Young workers must decide which vintage of technology they should enter. Once they have entered a particular vintage, they acquire skills which are specific to that vintage and they are committed to that vintage for the rest of their lives. All workers in a given vintage acquire the same skills. The preexisting distribution of old workers implies a distribution for the marginal productivity of young workers across different vintages. The implied wage profiles over time and across vintages will, of course, in equilibrium equate the present value of wages across vintages for a given generation of workers. The existence of complementarities in production between skilled and unskilled workers then, in general, will attract unskilled workers to vintages that have skilled workers. Thus in equilibrium old technologies continue to be used.

We establish that, for such a model, a unique stationary competitive equilibrium exists and is Pareto optimal. The stationary equilibrium is characterized by an endogenous distribution of skilled workers across vintages. This distribution is single-peaked. Under fairly general conditions we show that there is a lag between the time that a technology appears and the peak level of output from the use of that technology. The wage rates for

unskilled workers increases monotonically with the age of the vintage while the wage rates for skilled workers declines monotonically with the age of the vintage.

We also examine the effect of a change in, the rate of exogenous technological change. In a stationary state, is also the growth rate of the economy. We show that an increase in this growth rate shifts the associated stationary distribution to more recent vintages. Furthermore, it reduces the time lag between the introduction of new technologies and the peak of their usage. In other words, in faster growing economies new technologies diffuse more rapidly.

We also show that an increase in the growth rate causes wage profiles over time for any given generation to become flatter. In a sense, therefore, the return form investing in human capital by working in newer vintages where current wages are lower falls for each individual. However, since the distribution of skilled workers also shifts to newer vintages in which future income is larger the effect on overall investment in human capital is ambiguous.

We also examine the effect of a change in the rate at which individuals discount future consumption. An increase in the discount factor can be interpreted as an increase in the arrival rate of new technologies. We show that an increase in the discount factor results in more rapid diffusion of new technologies. Rosenberg [1976] advanced the conjecture that if technological advance is expected to be unusually rapid or if new technologies are substantially better than old technologies that firms

might well delay the adoption of new technologies to wait for yet better ones in the future. This model suggests that this conjecture ignores the important possibility that in such environments, even though future technologies are much better, new technologies made available today are better than those discovered in the past. In our equilibrium model, at least in the steady state, we show that these tradeoffs are decisively resolved in the direction of more rapid adoption. It is of course possible that considerations of uncertainty in technological advance might reverse these results.

We examine the effect of population growth and show that more rapid growth rates in population increase the rate of adoption of new technologies in the steady state. We also show that an increase in population growth rates reduces the average wage of unskilled workers while raising the average wage of skilled workers.

Jugenfelt [1986] develops a related model in which capital specificities arise solely because of the fact that workers must be trained to produce new products. The key variable in his model is the length of training time. He shows that an increase in this variable leads to an increase in the number of old products which continue to be produced. Since there are no complementarities in his model it cannot generate the result that resources are invested to create capital which is specific to old technologies.

We present the model in Section 2. In Section 3 we prove that a stationary equilibrium exists and is unique. In

section 4 we characterize the equilibrium and prove some comparative steady state results. In section 5 we show that the stationary competitive equilibrium is Pareto optimal. Some concluding remarks are contained in section 6.

Section 2. The Model

We consider an overlapping generations model of agents who live for two periods. The set of agents born in each period is given by the interval [0,1] with uniform distribution. Our structure has the following features: i) Exogenous technical change causes new technologies to appear in every period. ii) Individuals investments specific to a vintage so that the new technologies are diffused over time by the optimal decisions of agents. In our model, these investments take the form of human capital.

A new technology appears in every period. This technology is given by the production function

$$\gamma^t f(N,Z)$$

where t denotes the period in which the technology appeared, N is the input of unskilled workers and Z is the input of experienced workers.

(A.1) The following assumptions are made on the production function.

i. f has constant returns to scale

ii. $f(N,0) = \omega_0 N$ where $\omega_0 \geq 0$

iii. $f(\cdot,Z)$ is strictly increasing and strictly concave for each $Z > 0$.

In every period there are two generations of workers who live for two periods each, the experienced (old) and the unskilled (young). Young workers can choose to work in only one vintage. Experience is acquired by working in a firm in a particular vintage as an unskilled worker when young and is specific to the vintage corresponding to the firm's technology. The amount of expertise acquired by two young agents working in a firm of the same vintage is exactly the same. This will simplify the decision problem of young agents--as will be detailed later--who will just have to choose which vintage to enter on the basis of the wage offered and the valuation that the market will give to their specific expertise in the following period. We also assume that

(A.2) Old agents have zero productivity in the unskilled tasks.

This is an assumption of convenience. It simplifies the analysis and most of our results also hold when old agents are allowed to perform the tasks of young, unskilled workers.

Agents have preferences defined over the two periods they live given by utility function

$$u(c_1,c_2) = c_1 + \beta c_2 \text{ where } 0 < \beta < 1.$$

As usual in growth models some bound on the rate of technological change must be given. The following assumption plays a key role only in the issue of the optimality of the equilibrium:

(A.3) $\beta\gamma < 1$.

As we mentioned earlier, when they are young, agents can work in only one vintage. In the following period, they will have acquired expertise in that vintage. Hence in each period there is a distribution of old agents across existing vintages. To make this precise, it will be convenient to introduce the following notation:

The letter 't' will index time and the letter 'τ' will index the vintage of the technology, with the following interpretation:

When referring in period t to technology of vintage τ, we will be referring to the technology that appeared τ periods before. For example $\tau = 2$ indicates the vintage that appeared in $t - 2$. Notice also that the same vintage in period $t + 1$ will have $\tau = 3$.

Let μ_t be the distribution of experience of old agents across vintages $\tau \in \{0,1,2,\ldots\}$. Thus $\mu_t(\tau)$ indicates the number (more precisely mass) of old agents with experience in vintage τ. These are the old people who young worked in the vintage that appeared in $t - \tau$. Since there are no experienced workers in the 'just born' vintage, $\mu_t(0) = 0$ for all t. We will often refer to μ_t as the <u>state of the economy</u>.

The existence of constant returns to scale makes irrelevant the number and distribution of property rights of firms in each vintage. For simplicity, and without loss of generality, we will assume that each old agent in a particular vintage 'runs' a firm and competitively hires young agents.

Let $w(t,\tau,\mu_t)$ indicate the minimum wage needed to attract unskilled workers to vintage τ at period t when the state of the economy is μ_t. Old agents with skills solve the following problem:

$$(1) \qquad v(t,\tau,\mu_t) = \max_{n\geq 0} \gamma^{t-\tau} f(n,1) - w(t,\tau,\mu_t)n.$$

As a consequence of (A.1) there is a unique solution to the above problem which will be denoted by $n(t,\tau,\mu_t)$.

Recall that $f(N,0) = \omega_0 N$. Consequently, in any vintage τ at any time t a young worker can always assure himself of a wage equal to $\gamma^{t-\tau}\omega_0$. We therefore have

$$(2) \qquad w(t,\tau,\mu_t) \geq \gamma^{t-\tau}\omega_0.$$

In particular, in any vintage where there are no skilled workers, (2) must hold with equality if the mass of young workers entering that vintage is positive. Hence in the newest vintage, for example, if $w(t,0,\mu_t) > \gamma^t\omega_0$ then no young workers enter there because the minimum wage required to attract them to the newest technology exceeds the feasible wage.

We will now analyze the decision problem faced by young agents born in period t. If they decide to enter vintage τ, their earnings in the following period will be $v(t+1,\tau+1,\mu_{t+1})$ since in the following period they will be skilled in vintage $\tau + 1$. Young agents will be assumed to have perfect foresight on the returns to experience in each vintage. Since they maximize discounted earnings, for them to be indifferent as to which vintage to enter the following must be satisfied:

$$w(t,1,\mu_t) + \beta v(t+1,2,\mu_{t+1}) = w(t,0,\mu_t) + \beta v(t+1,1,\mu_{t+1})$$

$$(3) \quad w(t,2,\mu_t) + \beta v(t+1,3,\mu_{t+1}) = w(t,1,\mu_t) + \beta v(t+1,2,\mu_{t+1})$$

$$w(t,\tau,\mu_t) + \beta v(t+1,\tau+1,\mu_{t+1}) = w(t,\tau-1,\mu_t) + \beta v(t+1,\tau,\mu_{t+1})$$

for all τ.

Let $N(t,\tau,\mu_t)$ denote the mass of young workers who enter vintage τ at time t. If $\mu_t(\tau) > 0$, it follows that a necessary condition for market equilibrium is that $N(t,\tau,\mu_t) = n(t,\tau,\mu_t)\mu_t(\tau)$. Of course in such a case the mass of skilled workers in vintage $\tau + 1$ at time $t + 1$, $\mu_{t+1}(\tau+1)$, is now given by the mass of young workers at vintage τ at time t, $N(t,\tau,\mu_t)$.

In order to complete the description of the environment, we will assume that at period 0 there is a set of old agents indexed by [0,1] with uniform distribution. We assume also that they have experience on a set of existing technologies and that the corresponding distribution of expertise is given by μ_0. Thus $\mu_0(\tau)$ is the mass of those workers experienced in vintage $-\tau$, i.e., the vintage with production function $\gamma^{-\tau}f(n,z)$.

We can now define an equilibrium for this economy.

<u>Definition: A competitive equilibrium</u> for this economy is:

a. a wage function $w(t,\tau,\mu_t)$

b. an employment function $n(t,\tau,\mu_t)$

c. the mass of young workers who enter each vintage, $N(t,\tau,\mu_t)$

d. a sequence of distribution functions $\{\mu_t\}$ such that:

   i. $n(t,\tau,\mu_t)$ shows the right side of (1).

ii. $w(t,\tau,\mu_t)$ makes young agents indifferent as to which vintage to enter, i.e., $w(t,\tau,\mu_t)$ satisfies equations (2) and (3).

iii. $\sum_{t=1}^{\infty} n(t,\tau,\mu_t) = 1$ where $N(t,\tau,\mu_t) > 0$ and $\mu_t(\tau) > 0$ implies $w(t,\tau,\mu_t) = \gamma^{t-\tau}\omega_0$.

iv. $\mu_{t+1}(\tau) = n(t,\tau-1,\mu_t)$ for $\tau \geq 1$, and $N(t,\tau,\mu_t) = \mu_t(\tau)n(t,\tau,\mu_t)$ if $\mu_t(\tau) > 0$.

Conditions i. and ii. state that agents make their decisions optimally. Condition iii. is the labor market clearing condition. Condition iv. states that the law of motion for μ_t is precisely the one generated by the optimal rules described.

In the rest of the paper we will concentrate our attention on the <u>stationary equilibrium</u>, i.e., a competitive equilibrium with the additional condition:

v. $\mu_t = \mu_{t+1} = \mu$ for all t.

We will establish that a stationary distribution exists and is unique. Then we will analyze the properties the economy has if it were at a stationary equilibrium.

Section 3. <u>Existence of a Stationary Equilibrium</u>

We will first establish some necessary and sufficient conditions for the existence of a stationary equilibrium. Then we will show that under the assumptions made, these conditions are satisfied by a unique set of equilibrium values. Since in the stationary equilibrium μ_t is constant, we will suppress μ_t as an argument to the functions defined above.

Proposition 1

Suppose $w^*(\cdot)$, $n^*(\cdot)$, $N^*(\cdot)$, $\mu^*(\cdot)$ is a stationary equilibrium. Then

1. $\mu^*(\tau) > 0$ implies $\mu^*(\tau') > 0$ all $\tau' \leq \tau$.
2. $N^*(\cdot)$ is independent of t and $N^*(t,0) > 0$ all t.
3. $n^*(t,\tau)$ is independent of t.
4. $w^*(t,\tau) = \gamma^t w^*(0,\tau)$ for all t.
5. $v^*(t,\tau) = \gamma^t v^*(0,\tau)$ all t.

Proof

1. Suppose $\mu^*(\tau) = 0$ and $\mu^*(\tau+1) > 0$. Then, for all t, $N^*(t,\tau) > 0$. Therefore $w^*(t,\tau) = \gamma^{t-\tau}\omega_0$. Recall that $w^*(t,\tau+1) \geq \gamma^{t-(\tau+1)}\omega_0$. Therefore $w^*(t,\tau+1) \geq \gamma w^*(t,\tau)$ all t. From (1) it then follows that for all t, $v^*(t,\tau+1) < v^*(t,\tau)$. Using equation (3) we have that

$$w^*(t,\tau) + \beta v^*(t+1,\tau+1) = w^*(t,\tau-1) + \beta v^*(t+1,\tau).$$

Therefore $w^*(t,\tau) = \gamma^{t-\tau}\omega_0 > w^*(t,\tau-1)$.
But $w^*(t,\tau-1) \geq \gamma^{t-(\tau-1)}\omega_0 \geq \gamma^{t-\tau}\omega_0$. We have established a contradiction.

2. Since $\mu^*(\tau+1) = N^*(t,\tau)$, it follows that $N^*(\cdot)$ is independent of t. Suppose $N^*(t,0) = 0$. Then $\mu^*(1) = 0$ and $\mu^*(\cdot) = 0$ for all τ. But $\sum N^*(\cdot) = 1$. Thus $N^*(t,0) > 0$ all t.

3. Recall that $N^*(t,\tau) = n^*(t,\tau)\mu^*(\tau)$ if $\mu^*(\tau) > 0$ and $N^*(\cdot)$ is independent of t. Hence, $n^*(\cdot)$ is independent of t. Furthermore, from part (1) it follows that $N^*(t,\tau) > 0$ implies $\mu^*(\tau) > 0$.

4. Note that

$$n(t,\tau) = \text{argmax } \gamma^t f(n,1) - w(t,\tau)n$$

$$= \text{argmax } f(n,1) - \frac{[w(t,\tau)]}{\gamma^t} n.$$

Since $n(\cdot)$ is independent of τ, either, $n(\cdot)$ is zero or $w(t,\tau)/\gamma^t$ is independent of t. Hence $w(t,\tau) = \gamma^t w(0,\tau)$ for all τ such that $n^*(\tau) > 0$. Furthermore, for all τ such that $n^*(\tau) > 0$

$$v(t,\tau) = \max \gamma^t f(n,1) - \gamma^t w(0,\tau)n$$

$$= \gamma^t v(0,\tau).$$

Let T be the smallest value of τ such that $n^*(\tau) = 0$. Or, $n^*(T-1) > 0$ and $n^*(T) = 0$. It follows from equations (3) that

$$w(t+1,T-1) + \beta v(t+1,\tau) = \gamma[w(t,T-1)+\beta v(t,T)].$$

Therefore $v(t+1,T) = \gamma v(t,T)$.

For all $\tau \geq T$, it is clear that $w^*(t,\tau)$ can be set equal to $\gamma^t w(0,\tau)$ without loss of generality.

As a consequence of the above proposition, we can simplify considerably the notation employed from now on. We have suppressed the μ_t arguments and we can now suppress the τ arguments from the functions used. This leaves us only with 'τ' as the only argument. We will thus define:

$$n_\tau = n(t,\tau,\mu)$$

$$w_\tau = w(t,\tau,\mu)$$

$$v_\tau = v(t,\tau,\mu)$$

$$\mu_\tau = \mu(\tau).$$

Given these facts it is now possible to rewrite equation (3). It will be convenient to write the profit v_τ as a function of the wage w_τ. Then using equation (3) and Proposition 1 we have that a necessary condition for a stationary equilibrium is that there exist sequences $\{w_\tau\}$ which satisfy:

$$w_1 + \beta\gamma v_2(w_2) = w_0 + \beta\gamma v_1(w_1)$$

$$(4) \qquad w_2 + \beta\gamma v_3(w_3) = w_1 + \beta\gamma v_2(w_2)$$

$$w_\tau + \beta\gamma v_{\tau+1}(w_{\tau+1}) = w_{\tau-1} + \beta\gamma v_\tau(w_\tau) \text{ for all } \tau$$

subject to $w_\tau \geq \gamma^{-\tau}\omega_0$ where $v_\tau(w) = \max_n \gamma^{-\tau}f(n,1) - wn$.

Suppose for the moment we could find a sequence $\{w_\tau\}_{\tau=1}^{\infty}$ that satisfied equation (4) with $w_0 = \omega_0$. This would imply a sequence $\{n_\tau\}_{\tau=1}^{\infty}$ given by $n_\tau = \text{argmax } \gamma^{-\tau}f(n,1) - w_\tau n$, the optimal input decision if there are specialized workers in vintage τ. Market clearing and Proposition 1 require that

$$(5) \qquad \sum n_\tau \mu_\tau = 1 - N_0.$$

Using the fact that $\mu_{\tau+1} = \mu_\tau n_\tau$ and $\mu_1 = N_0$, given N_0 (or μ_1) we obtain by induction the whole sequence $\{\mu_\tau\}$ by setting $\mu(\tau) = \mu_1 \prod_{\tau'=1}^{\tau-1} n_{\tau'}$. Then

$$(6) \qquad \mu_\tau n_\tau = (\mu_1 \prod_{\tau'=1}^{\tau-1} n_\tau) n_\tau = \mu_1 \prod_{\tau'=1}^{\tau} n_\tau,$$

and thus equation (5) can be rewritten as

$$(7) \qquad \mu_1 \sum_{\tau=1}^{\infty} \prod_{\tau'=1}^{\tau} n_\tau = 1 - \mu_1.$$

If $\mu_1 = 0$ the left side is equal to zero and the right side equals one. If μ_1 equals one then the right side equals zero. The left side is continuous and nondecreasing in μ_1 and the right side is strictly decreasing. Hence as long as equation (7) is well-defined, then given $\{n_\tau\}$ there exists a unique μ_1 consistent with this equation. As will be shown in Proposition 4, n_τ decreases monotonically to zero. Hence there is a T such that $n_t < 1$ and so

$$\mu_1 \sum_{\tau=1}^{\infty} \prod_{\tau'=1}^{\tau} n_\tau = \mu_1 \sum_{\tau=1}^{T} \prod_{\tau'=1}^{\tau} n_\tau + \mu_T \prod_{\tau'=1}^{T} n_\tau \sum_{\tau=T+1}^{\infty} \prod_{\tau'=T+1}^{\tau} n_\tau$$

$$\leq \mu_1 \sum_{\tau=1}^{T} \prod_{\tau'=1}^{\tau} n_\tau + \mu_T \prod_{\tau'=1}^{T} n_\tau \sum_{\tau=0}^{\infty} n_T^\tau$$

so that equation (7) is well-defined.

This suggests the following procedure for finding an equilibrium:

<u>Step 1</u>. Obtain a solution to equation (4)

<u>Step 2</u>. Find the corresponding input demands $\{n_\tau\}$

<u>Step 3</u>. Find μ_1 from equation (7).

This is summarized in the following proposition.

<u>Proposition 2</u>

$w^*(\cdot)$, $n^*(\cdot)$, $N_0^*(\cdot)$, and μ^* is a stationary equilibrium if and only if there exist $\{w_\tau\}, \{n_\tau\}$ and μ_1 such that

$\mu^*(\tau)$ is given by equation (6)

$n^*(t,\tau,\mu) = n_\tau$ for all t

$N^*(t) = \mu_1$

$w^*(t,\tau,\mu) = \gamma^t w_\tau$ for all t

$\{w_\tau\}$ satisfies equation (4) and $w_0 = \omega_0$. The optimal input decisions are given by n_τ given w_τ and μ_1 and $\{n_\tau\}$ satisfy equation (7).

Proof. That these conditions are sufficient can easily be checked. By Proposition 1, $w^*(t,\tau,\mu) = \gamma^t w^*(0,\tau,\mu)$. Hence we can set $w_\tau = w^*(t,\tau,\mu)$ and also by Proposition 1 $w_0 = \omega_0$. The construction above shows that the rest of the conditions follow from this one. □

The next proposition states the existence and uniqueness of a solution to equation (4).

Proposition 3

There is a unique sequence $\{w_\tau\}$ which shows equation (4).

Proof. See Appendix

Theorem 1. There exists a unique stationary competitive equilibrium for the economy described in section 2.

Proof. Follows immediately from Propositions 2 and 3.

Section 4. Properties of the Equilibrium

The first question to be asked is about the distribution of skilled workers across vintages. We show in Proposition 4 that the wage rate of unskilled workers, w_τ, is increasing in the age of the vintage and that the wage paid to skilled workers, v_τ, is decreasing in the age of the vintage. Since productivity is decreasing with the age of the vintage it follows that n_τ is decreasing in τ.

Proposition 4

In a steady state, wage rates for unskilled workers increase with the vintage and wages of skilled workers decrease with the vintage. Formally, $w_{\tau+1} > w_\tau$, $v_{\tau+1} < v_\tau$, $n_{\tau+1} < n_\tau$ all τ.

Proof. Suppose for τ, $w_{\tau+1} \leq w_\tau$. From equation (4) it follows that $v_{\tau+2}(w_{\tau+2}) \geq v_{\tau+1}(w_{\tau+1})$. From the definition of $v(\cdot)$ we then have that $\gamma w_{\tau+2} < w_{\tau+1}$. By induction $\gamma w_{s+1} < w_s$ all $s \geq \tau+1$. We also have that $\gamma^{-s} f_1(n_s,1) = w_s$. Therefore, $f_1(n_{s+1},1) < f_1(n_s,1)$. Since $f(\cdot,1)$ is strictly concave, n_s is a strictly increasing sequence. Furthermore, since $v_s(\cdot)$ is nondecreasing, n_s cannot be bounded from above. Hence, n_s is an unbounded, increasing sequence. This clearly conflicts with the fact that $\sum_\tau \mu_\tau = 1$ and $\mu_{\tau+1} = n_\tau \mu_\tau$. Hence, $w_{\tau+1} > w_\tau$. From equation (4) we have that $v_{\tau+1} < v_\tau$. By definition of n, $n_{\tau+1} < n_\tau$. □

We show in Proposition 5 below that if an Inada condition is satisfied then all vintages are used in a stationary equilibrium.

Proposition 5

If $f_1(0,1) = \infty$ then $\mu_\tau > 0$ for all τ. Otherwise there exists some T such that $\mu_\tau > 0$ if and only if $\tau \leq T$.

Proof. Let T be the smallest number such that $\mu_T = 0$. Recall that in a stationary equilibrium $\mu_T = \mu_1 \prod_{\tau=1}^{T-1} n_\tau$ and $\mu_1 = N_0$. We have already established that $N_0 > 0$. Consequently if $\mu_T = 0$ then $\mu_{\tau'} = 0$ for all $\tau' \geq T$ and $n_{\tau-1} = 0$. If $f_1(0,1) = \infty$ then for any finite $w_{\tau-1}$, $n_{\tau-1} > 0$.

On the other hand, suppose that $f_1(0,1) < \infty$. A necessary condition for an equilibrium is that if $n_\tau > 0$ then $w_\tau = \gamma^{-\tau} f_1(n_\tau, 1) \leq \gamma^{-\tau} f_1(0,1)$. Consequently w_τ must converge to zero. However since w_τ is an increasing sequence it must be bounded away from zero. □

Clearly, the first part of Proposition 4 depends critically on assumption A.2. If old agents could work as unskilled workers in any vintage then the wages of unskilled workers cannot exceed the wages of skilled workers. Therefore we would need to impose the condition that $v_\tau \geq w_\tau$ for all τ, τ'. In this case the number of vintages will be finite. Other than that, none of our results change. We have established in Proposition 3 that there is a unique solution to equation (4). Let $\pi_\tau = \max[v_\tau, \max_s w_s]$. We have also shown in Proposition 4 that w_s is increasing in s and v_s is decreasing in s. Furthermore, v_s decreases monotonically to zero. Hence, there is some T such that $v_\tau < w_\tau$, all $\tau > T$. If we allow old workers to work anywhere they choose, equation (4) would read $w_\tau + \beta\gamma\pi_{\tau+1} = \text{constant}$. It is straightforward to verify that all our propositions go through with minor modifications.

It is of interest to examine the shape of the distribution of skilled workers as well as the distribution of output. We establish below that employment of skilled workers will rise and then fall with the age of the vintage.

Proposition 6. (Single peakedness)

There exists T such that for all $\tau \leq T$, $\mu_\tau \geq \mu_{\tau-1}$ and for $\tau > T$, $\mu_\tau \leq \mu_{\tau-1}$. Furthermore, if $\omega_0 = 0$ and $\beta\gamma \leq f_1(1,1)/f_2(1,1)$ then $T \geq 2$.

Proof. We have established that n_τ is decreasing in τ. Let T be the smallest τ such that $n_\tau < 1$. Recall that $\mu_\tau = \mu_{\tau-1}n_{\tau-1}$. Consequently, for $\tau \leq T$, $\mu_\tau \geq \mu_{\tau-1}$ and for $\tau > T$, $\mu_\tau \leq \mu_{\tau-1}$.

We have from equation (4) and $\omega_0 = 0$ that

$$\beta\gamma v_1(w_1) = w_1 + \beta\gamma v_2(w_2).$$

By the definition of $v(\cdot)$ and the fact that $w_1 = \gamma^{-1}f_1(n_1,1)$ we have that

$$\beta\left[f(n_1,1)-f_1(n_1,1)n_1\right] \geq \frac{1}{\gamma} f_1(n_1,1).$$

Therefore

$$\frac{\beta\gamma f_2(n_1,1)}{f_1(n_1,1)} \geq 1. \tag{8}$$

The numerator of this inequality is increasing in n_1 and the denominator is decreasing. Hence, if

$$\beta\gamma \leq \frac{f_1(1,1)}{f_2(1,1)} \tag{9}$$

then $n_1 > 1$. □

We have established that under mild conditions the peak of the distribution of skilled workers will occur for some $T \geq 2$. In order to obtain sharper results about the peak of this distribution as well as results about the peak of the distribution of output, we consider a particular production function. Assume that the production function is Cobb-Douglas: $f(N,Z) = N^{\alpha}Z^{1-\alpha}$. It is plausible to assume that $\alpha \leq 1/2$. Inequality (9) which guarantees that $T \geq 2$ can then be written as $\beta\gamma(1-\alpha)/\alpha \leq 1$. In Proposition 5 below we strengthen this condition to ensure that $T \geq 3$ and the peak of the distribution of output occurs at vintage $\tau \geq 2$.

<u>Proposition 7</u>

If $\beta\gamma(1-\alpha)/\alpha + \gamma[\beta\gamma(1-\alpha)/\alpha]^{1-\alpha} \leq 1$ then $T \geq 3$ and the peak of the output distribution occurs at a vintage $\tau \geq 2$.

<u>Proof</u>. It follows from inequality (8) that $n_1 \geq \alpha/\beta\gamma(1-\alpha)$. Since $\alpha < 1$, $n_1^{\alpha-1} \leq [\alpha/\beta\gamma(1-\alpha)]^{\alpha-1}$. From equation (4) we have that $w_2 \leq w_1 + \beta\gamma v_2(w_2)$. Hence using the fact that $w_\tau = \gamma^{-\tau}f_1(n_\tau,1)$ we have that

$$n_2^{\alpha-1} \leq \gamma n_1^{\alpha-1} + \frac{\beta\gamma(1-\alpha)}{\alpha} n_2^{\alpha}.$$

It follows that

$$(10) \qquad n_2^{\alpha-1} \leq \gamma\left[\frac{\beta\gamma(1-\alpha)}{\alpha}\right]^{1-\alpha} + \frac{\beta\gamma(1-\alpha)}{\alpha} n_2^{\alpha}.$$

Suppose that $n_2 < 1$. Then the right side of inequality (10) is at most 1. Hence $n_2^{\alpha-1} \leq 1$ so $n_2 > 1$.

Note that output at vintage τ is given by $\gamma^{-\tau}f(\mu_\tau n_\tau, \mu_\tau)$. Therefore output at vintage 2 is greater than output at vintage 1 if and only if

$$(11) \qquad \gamma^{-1}f(\mu_2 n_2, \mu_2) > f(\mu_1 n_1, \mu_1).$$

Since $\mu_2 = \mu_1 n_1$, inequality (11) is satisfied iff $\gamma^{-1}n_2^{\alpha} > n_1^{\alpha-1}$. Recall however that $w_2 \geq w_1$. Hence $\alpha\gamma^{-1}n_2^{\alpha-1} \geq \alpha n_1^{\alpha-1}$. But $n_1 > n_2 > 1$. Therefore $\gamma^{-1}n_2^{\alpha} > n_1^{\alpha-1}$. □

It is of interest to examine the effect of a change in the rate of technological change on the stationary distribution. Our main result is that when $\gamma' > \gamma$ then the distribution corresponding to the higher growth rate, say μ', will be dominated in the sense of stochastic dominance by the original distribution. In other words, when the growth rate increases the distribution of skilled workers is concentrated among more recent vintages. This also implies that the rate of diffusion of new technologies is higher if the economy grows more rapidly.

<u>Proposition 8</u>

Consider two economies with $\gamma' > \gamma$ and associated stationary distributions μ' and μ respectively. Then $n(\tau,\mu') < n(\tau,\mu)$ for all τ such that $n(\tau,\mu) > 0$. Furthermore,

$$w(\tau,\mu') \geq \left(\frac{\gamma}{\gamma'}\right)^{\tau} w(\tau,\mu).$$

<u>Proof</u>. See Appendix.

We use this result to prove:

Proposition 9

Consider two economies with $\gamma' > \gamma$. Let μ', μ denote the respective stationary distributions. Then μ stochastically dominates μ', i.e.,

$$\sum_{\tau=1}^{t} \mu_\tau \le \sum_{\tau=1}^{t} \mu'_\tau \text{ for all } t.$$

Proof. From Proposition 8 we have that $n(\tau,\mu') < n(\tau,\mu)$. Hence if $\mu'(1) \le \mu(1)$ then $\mu'(\tau) < \mu(\tau)$ for all $\tau > 1$ and hence $\sum_{\tau=1}^{\infty} n_\tau \mu'_\tau < 1 - \mu_1$ and thus $\mu'_1 > \mu_1$. Let $T = \{\min t : \mu'(t) < \mu(t)\}$. Then $T \ge 2$. For $t < T$, $\sum_{\tau=1}^{t} \mu'(\tau) > \sum_{\tau=1}^{t} \mu(t)$. Since $n(t,\mu') < n(t,\mu)$ for all t and $\mu'(T) < \mu(T)$, by construction $\mu'(t) < \mu(t)$ for all $t \ge T$. But then for any $t \ge T$,

$$\sum_{\tau=1}^{t} \mu'(\tau) = 1 - \sum_{\tau>t} \mu'(\tau) > 1 - \sum_{\tau>t} \mu(\tau) = \sum_{\tau=1}^{t} \mu(\tau),$$

which establishes the result. □

Our next result shows that the earnings profile becomes flatter as the growth rate of the economy increases. From Proposition 8

$$w(\tau,\mu') \ge \left[\frac{\gamma}{\gamma'}\right]^{\tau} w(\tau,\mu).$$

But

$$v'_t(w) = \max_n \gamma'^{-t} f(n) - wn \le \left[\frac{\gamma}{\gamma'}\right]^{t} \left[\gamma^{-t} f(n) - wn\right] \le \left[\frac{\gamma}{\gamma'}\right]^{t} v_t(w).$$

Since $\gamma' > \gamma$,

$$\frac{\gamma' v'_{\tau+1}(w'_{\tau+1})}{w'_\tau} \le \frac{\gamma v_{\tau+1}(w_{\tau+1})}{w_\tau}. \tag{12}$$

This establishes that the earnings profile becomes flatter with a higher growth rate. One measure of investment in human capital is foregone earnings. If an individual joins a sufficiently old vintage we have shown that future earnings will be close to zero and current wages will be high. This individual would then be making very small investments in human capital. The present value of income is equated across vintages. Hence individuals who join new vintages will be making large investments in human capital. These can be measured by the ratio of future earnings to the wage that implies no investment. Of course the latter equals the present value of earnings. Inequality (12) implies that this measure of investment in human capital declines. However, note that aggregate investment in human capital does not necessarily fall since the distribution of skill levels shifts to vintages with higher rates of investment.

We now examine the effect of a change in the discount factor on the stationary distribution. If we had assumed that the economy was populated by a single consumer with preferences of the form $\sum_{t=0}^{\infty} \beta^t c_t$, then the obvious interpretation of an increase in the discount factor would be a decrease in the length of time between the arrival of new technologies. As we show in section 5, if $\beta\gamma < 1$, then the equilibrium in the model maximizes the infinite stream of output discounted at a rate β. Hence, it is possible to reinterpret our model as consisting of infinitely lived identical agents who maximize the discounted stream of consumption. We show in Propositions 10 and 11 below that an increase in the discount factor results in a more rapid rate of diffusion.

Proposition 10

Consider two economies with $\beta' > \beta$ with associated stationary distributions μ' and μ respectively. Let $w(\tau,\mu')$ and $w(\tau,\mu)$ denote the wage rates in vintage τ and $n(\tau,\mu')$, $n(\tau,\mu)$ denote the input decisions in vintage τ in the two economies. Then $w(\tau,\mu') > w(\tau,\mu)$ and $n(\tau,\mu') < n(\tau,\mu)$ all τ.

Proof. See Appendix. □

We then have

Proposition 11

Let μ' and μ denote the steady state distributions for two economies characterized by discount factors β' and β respectively. Then μ stochastically dominates μ'.

Proof. Parallels Proposition 9 exactly and is omitted. □

With some minor modifications, it is straightforward to incorporate population growth in our analysis. Let L_t denote the mass of young workers at time t. We will assume that $L_{t+1} = (1+\lambda)L_t$ so that the rate of population growth is λ. The only modification we need to make is our definition of competitive equilibrium is to note that $\sum_{\tau=0}^{\infty} N(t,\tau,\mu_t) = (1+\lambda)^t$. The definition of a stationary equilibrium now, in analogous fashion, is that

$$\mu_{t+1}(\tau) = (1+\lambda)\mu_t(\tau). \tag{13}$$

Thus, the relative proportions of skilled workers across vintages remains unchanged. Note that the present value conditions in equation (4) remain unchanged. We have established that

these have a unique solution. Obviously this solution depends only on β, λ, and $f(\cdot,\cdot)$. Hence, the distribution of wage rates and consequently employment decisions, n, remain unchanged. From the definition of an equilibrium,

$$\mu_{t+1}(\tau+1) = n(\tau)\mu_t(\tau)$$

$$\therefore \quad \mu_t(\tau+1) = \frac{n(\tau)}{1+\lambda}\mu_t(\tau).$$

Let $n'(\tau) = n(\tau)/1+\lambda$. The equilibrium condition at time 0 is then

$$(14) \qquad \sum_{\tau=0}^{\infty} \mu_t n'_\tau + \mu_1 = 1.$$

An increase in the rate of population growth implies a reduction in n'. From the proof in Proposition 9 we have that this results in a change in the distribution towards newer vintages. Hence there are relatively more skilled workers in new vintages in economies with rapid growth in population. Obviously there are relatively more unskilled workers in newer vintages as well. Since wage rates for unskilled workers are increasing with the age of the vintage, the average wage rate of unskilled workers falls with an increase in the growth rate of the population. Furthermore, the rate of human capital accumulation is also higher.

Section 5. Optimality of the Competitive Equilibrium

In this section, we establish that if the growth rate of the economy is not too large, the competitive equilibrium is Pareto optimal. We will need to assume

$$(A.3) \qquad \beta\gamma < 1.$$

This is a standard condition in models of economic growth. We need to ensure that the discounted consumption stream is bounded to ensure that our social welfare function is well-defined. Let $c_t = (c_{1t}, c_{2t})$ denote the consumption when young and old of a representative agent born at time t. We have assumed that $U(c_{1t}, c_{2t}) = c_{1t} + \beta c_{2t}$. The initial old care only about consumption in the first period denoted by $c_{2,-1}$. Let $c = (c_{2,-1}, (c_t)_{t=0}^{\infty})$. We will assume that a planner has preferences given by

$$(14) \qquad W(c) = c_{2,-1} + \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t}).$$

Let $y_t = c_{2t-1} + c_{1t}$. Thus y_t denotes the total output at time t. The planner is assumed to have available a total labor endowment of two units in each period. Let $N(t,\tau)$ denote labor allocated to vintage τ at time t in the unskilled task. Let $z(t,\tau)$ denote skilled labor allocated to vintage τ at time t. In keeping with our assumptions, $z(t,\tau) = N(t-1,\tau-1)$. Also $\sum_{\tau=0}^{\infty} N(t,\tau) \leq 1$ all t. The problem faced by the planner is then

$$(15) \qquad \max \sum_{t=0}^{\infty} \beta^t y_t$$

$$(16) \quad \text{s.t. } 0 \leq y_t \leq \sum_{\tau=0}^{\infty} \gamma^{t-\tau} f(N(t,\tau), N(t-1,\tau-1))$$

$$(17) \qquad \sum_{\tau=0}^{\infty} N(t,\tau) \leq 1 \text{ all } t$$

$$(18) \qquad N(t-1,0) = 0 \text{ all } t$$

$$(19) \qquad N(t,\tau) \geq 0 \text{ all } \tau, \text{ all } t$$

given $N(-1,\tau)$.

Let λ_t denote the Lagrange multiplier on constraint (17). The necessary first order conditions are then given by

$$(20) \qquad \beta^t \gamma^{t-\tau} f_1\big(N(t,\tau),N(t-1,\tau-1)\big) + \beta^{t+1}\gamma^{t+1-(\tau+1)} f_2\big(N(t+1,\tau+1),N(t,\tau)\big) = \lambda_t.$$

Let $\hat{\lambda}_t = (\beta\gamma)^{-\tau}\lambda_t$. Then equation (20) can be rewritten to read

$$(21) \qquad \gamma^{-\tau} f_1\big(N(t,\tau),N(t-1,\tau-1)\big) + \beta\gamma^{-\tau} f_2\big(N(t+1,\tau+1),N(t,\tau)\big) = \hat{\lambda}_t.$$

Let $w(t,\tau) = \gamma^{-\tau} f_1\big(N(t,\tau),N(t-1,\tau-1)\big)$ and $v(t,\tau) = \gamma^{-\tau} f_2\big(N(t,\tau),N(t-1,\tau-1)\big)$.

Equation (21) is then obviously identical to equation (4). The fact that $f(\cdot,\cdot)$ is homogenous of degree one implies that $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$ are homogenous of degree zero. Let $n(t,\tau) = N(t,\tau)/N(t-1,\tau-1)$. Euler's Theorem implies that

$$f_2\big(n(t,\tau),1\big) = f\big(n(t,\tau),1\big) - n(t,\tau) f_1\big(n(t,\tau),1\big).$$

Hence,

$$v(t,\tau) = f\big(n(t,\tau),1\big) - w(t,\tau)n(t,\tau).$$

It is clear that the competitive equilibrium solves the same problem as does the planner. Note that consumers are indifferent in the competitive equilibrium about the timing of their consumptions. Hence, the competitive equilibrium is Pareto optimal.

Section 6. Concluding Remarks

We have presented a model of investment in technology specific human capital. The central result is that such specificities lead to a lag between the time that a new technology becomes available and the peak of its usage. In that sense, this model is consistent with the slow diffusion of new technologies. It is certainly true that slow diffusion can be a consequence of the fact that consumers must learn how to use new products. Our focus, however, is on the fact that producers must acquire the skills necessary to produce the new product cheaply. In our model the marginal product of investment in such human capital is high when older workers already possess the required level of skill.

Our main result is that an increase in the rate of change of technology implies an increase in the rate of diffusion. We also show that the wage profiles over time are flatter in older technologies than in newer ones. In that sense, the value of investing in a newer technology is higher and our model is one of human capital accumulation. The equilibrium we describe is Pareto optimal.

An obvious extension of our model would be to allow for uncertainty in the rate of technological innovation. We conjecture that in such a case, a technological innovation which is substantially better than average will attract a large number of young workers and lead to larger than average investment in the newest technology. Since this capital is specific to the technology, in subsequent periods relatively few young workers will be attracted to even newer technologies. These technologies will then be adopted and diffused at a slower rate than average.

The assumption of exogenous technical change obviously does not do justice to the reality of the process of innovation which requires the use of resources. In addition, it would be of interest to examine a model where technological innovation as well as adoption are jointly and endogenously determined. One possible modification of our model would be to let the productivity of the newest vintage relative to the previous one, γ, be determined by the number of workers who enter the newest industry. In such a case workers in the newest vintages can be thought of as engaging in innovative activity.

An alternative possibility is to think of spillover effects as implying that if workers are skilled in relatively new technologies then the productivity of the newest technology is higher. In such a case, the rate of growth of the economy and the rate of adoption of new technologies are both determined endogenously. The resulting equilibrium is not necessarily Pareto optimal and the model can be used to study the effect of policy interventions on the growth rate of the economy.

The existence of this externality may well cause the equilibrium not to be Pareto optimal. The effects of various policies to remedy this externality could then be examined. In any case, we conjecture that an exogenous improvement in the technology of innovation will lead, as in this paper, to an increase in the rate of diffusion. The earning profiles will also likely get flatter with such an improvement.

Appendix

Proposition 3

There exists a unique sequence $\{w_t\}$ that solves equation (4).

Proof. To establish this result we will first show that if we truncate the system at any T and impose $v_T = 0$, then there is a unique solution to the truncated problem. Then we will show that the sequence of solutions to the truncated problem converges and that the limit is a solution to the original problem. Finally we establish that there is no other solution to equation (4).

Step 1: Truncated Problem

Consider the problem

$$
\begin{aligned}
w_1 + \beta\gamma v_2(w_2) &= w_0 + \beta\gamma v_1(w_1) \\
w_2 + \beta\gamma v_3(w_3) &= w_1 + \beta\gamma v_2(w_2) \\
&\vdots \\
w_T &= w_{T-1} + \beta\gamma v_T(w_T)
\end{aligned}
\tag{4'}
$$

subject to $w_\tau \geq 0$.

Alternatively, this is equivalent to

$$
\begin{aligned}
w_T &= w_0 + \beta\gamma v_1(w_1) \\
w_T &= w_1 + \beta\gamma v_2(w_2) \\
&\vdots \\
w_T &= w_{T-1} + \beta\gamma v_T(w_T)
\end{aligned}
$$

subject to $w_\tau \geq 0$.

Note that we have not imposed the constraint that $w_\tau \geq \gamma^{-\tau}\omega_0$. We will show that there is a unique solution to 4'. Furthermore this solution has w_τ an increasing function of τ. Clearly our solution solves the original problem.

We will now show that the set of w_T such that the w_τ induced by backward induction in (4') are all nonnegative is nonempty. We will proceed by induction. Notice that $w_{T-1} = w_T - \beta\gamma v_T(w_T)$ and the right side is increasing in w_T and goes to infinity as $w_T \to \infty$. Hence there exists some w_T that makes $w_{T-1} \geq 0$. Suppose that for a given w_T and for all $\tau \geq \tau'$ the induced w_τ is nonnegative. If w_T increases, then w_{T-1} increases and inductively w_τ increases for all $\tau \geq \tau'$. Thus, as $w_T \to \infty$, $w_{\tau'-1}$ given by

$$w_{\tau'-1} = w_T - \beta\gamma v_{\tau'}(w_{\tau'})$$

also goes to ∞ and thus there exists some w_T such that $w_{\tau'-1}$ is nonnegative. Hence there exists some w_T such that the induced sequence $\{w_t\}_{\tau=0}^{\infty}$ is nonnegative.

Suppose w_T is such that all w_τ are nonnegative. Then $w_{T-1} = w_T - \beta\gamma v_T(w_T) \leq w_T$. By induction we will show that w_τ is increasing in τ. Assuming $w_\tau \leq w_{\tau+1}$ we have that $w_{\tau-1} = w_T - \beta\gamma v_\tau(w_\tau) \leq w_T - \beta\gamma v_\tau(w_{\tau+1}) \leq w_T - \beta\gamma v_{\tau+1}(w_{\tau+1}) = w_\tau$. Hence w_τ in increasing in τ.

Suppose that in the above case, $w_0(w_T) > 0$. Start decreasing w_T. For any $0 < \epsilon < w_0(w_T)$ at some point some $w_\tau = \epsilon$. But in that case the corresponding ω_0 will be no greater than ϵ. Hence there exists some $\underline{w}$ such that if $w_T = \underline{w}$ then $w_\tau(\underline{w})$

≥ 0 and $\omega_0(w_T) \leq w_0$. As $w_\tau \to \infty$ we already have shown that $w_0(w_\tau) \to \infty$.

We just need to establish that the mapping $w_0(w_T)$ is continuous. We will proceed by induction. For $\tau = T - 1$. $w_{T-1} = w_T - \beta\gamma v_T(w_T)$. Since v_T is continuous, $w_{T-1}(w_T)$ is a continuous function. Suppose $w_{\tau+1}$ is a continuous function of w_T. Then $w_\tau = w_T - \beta\gamma v_{\tau+1}(w_{\tau+1})$ which is a continuous function of $w_{\tau+1}$ and by composition of w_T.

The above implies that for any $\omega_0 > 0$ there exists a solution to problem equation (4'). Furthermore, since $\omega_0(w_T)$ is strictly increasing, there is a unique solution to this problem.

Step 2: Convergence of the Truncated Solutions

We proceed to show that these solutions converge. More precisely, letting $\bar{w}_t$ be the unique solution to the truncated problem we establish that $\bar{w}_t \to \bar{w} < \infty$.

We will first establish that $\bar{w}_t \leq \bar{w}_{t+1}$. Suppose, to the contrary, that $\bar{w}_t > \bar{w}_{t+1}$. We will first show that this implies that $w_1(\bar{w}_t) < w_1(\bar{w}_{t+1})$. If this were not the case, then

$$\bar{w}_t = \omega_0 + \beta\gamma v_1(w_1(\bar{w}_t)) \leq \omega_0 + \beta\gamma v_1(w_1(\bar{w}_{t+1})) = \bar{w}_{t+1}$$

so that $w_1(\bar{w}_t) < w_1(\bar{w}_{t+1})$.

We now show that the contradiction hypothesis implies that $w_\tau(\bar{w}_t) < w_\tau(\bar{w}_{t+1})$ for all $\tau \leq t$. Suppose $w_\tau(\bar{w}_t) < w_\tau(\bar{w}_{t+1})$ and that $w_{\tau+1}(\bar{w}_t) \geq w_{\tau+1}(\bar{w}_{t+1})$. Then repeating the argument used above for w_1 we obtain a contradiction. Hence for all $\tau \leq t$, $w_\tau(\bar{w}_t) < w_\tau(\bar{w}_{t+1})$ and in particular, $\bar{w}_t < w_t(\bar{w}_{t+1}) < w_t(\bar{w}_{t+1}) + \beta\gamma v_{t+1}(\bar{w}_{t+1}) = \bar{w}_{t+1}$. This proves that $\bar{w}_{t+1} \geq \bar{w}_t$, as desired.

We now turn to the other side of the inequality. Suppose, to the contrary, that $\bar{w}_t < w_t(\bar{w}_{t+1})$. Then

$$w_{t-1}(\bar{w}_{t+1}) = \bar{w}_{t+1} = \beta\gamma v_t(w_t(\bar{w}_{t+1})) > \bar{w}_t - \beta\gamma v_t(\bar{w}_t) = w_{t-1}(\bar{w}_t).$$

By the same argument if $w_\tau(\bar{w}_{t+1}) > w_\tau(\bar{w}_t)$ the same will be true for all $\tau' \le \tau$. Hence by induction $\omega_0(\bar{w}_{t+1}) > \omega_0(\bar{w}_t) > \omega_0$.

We have thus established that $w_t(\bar{w}_{t+1}) < \bar{w}_t < \bar{w}_{t+1}$. As a consequence the following inequality holds

$$|\bar{w}_{t+1}-\bar{w}_t| \le |\bar{w}_{t+1}-w_t(\bar{w}_{t+1})| = \beta\gamma v_{t+1}(\bar{w}_{t+1}).$$

But as can easily be checked, $v_{t+1}(\bar{w}_{t+1}) \le \gamma^{-1}v_t(\bar{w}_{t+1})$ so that

$$|\bar{w}_t-\bar{w}_1| \le \gamma^{-t}[\beta\gamma v_1(\bar{w}_t)] \le \gamma^{-t}\beta\gamma v_1(\omega_0).$$

This implies that $\{\bar{w}_t\}$ is a Cauchy sequence so it converges.

Denoting the limit of $\{\bar{w}_t\}$ by $\bar{w}$, we now proceed to show that the solution to equation (4) induced by $\bar{w}$ from equation (4') is well-defined. We will proceed again by induction

$$\bar{w} = \omega_0 + \beta\gamma v_1(w_1) \text{ gives } w_1$$

$$\bar{w} = w_1 + \beta\gamma v_2(w_2) \text{ gives } w_2.$$

In order for this to be well-defined, we need that $\bar{w} > w_1$. Suppose, to the contrary, that $\bar{w} \le w_1$. Since $\bar{w}_t \le \bar{w}$ for all t, $w_1(\bar{w}_t) \ge w_1(\bar{w})$ and so the above would imply that $\bar{w}_t \le w_1(\bar{w}_t)$, a contradiction. Suppose $w_\tau(\bar{w})$ is well-defined. Then $\bar{w} = w_\tau(\bar{w}) + \beta\gamma v_{\tau+1}(w_{\tau+1})$ and we thus require that $\bar{w} > w_\tau(\bar{w})$. Since $\bar{w}_\tau$ was assumed well-defined it is easy to see that $w_\tau(\bar{w}_t) \ge w_\tau(\bar{w})$ and so the above would imply that $\bar{w}_t \le w_\tau(\bar{w}_t) + \beta\gamma v_{\tau+1}[w_{\tau+1}(\bar{w}_t)]$, and

hence $w_{\tau+1}(\bar{w}_t)$ would not be be well-defined. This proves that the sequence $w_\tau(\bar{w})$ is a well-defined solution to equation (4).

Step 3: This is the Only Solution

Suppose $\bar{w}'$ is another equilibrium. We will denote by w'_τ the wage induced for vintage τ by $\bar{w}'$. We will first show that $\bar{w}' \geq \bar{w}$. Suppose to the contrary that $\bar{w}' < \bar{w}$. Then there exists some t such that $w'_t \leq w'_t + \beta\gamma v_{t+1}(w'_{t+1}) = \bar{w}' < \bar{w}_t$, where $\bar{w}_t$ corresponds to the truncated solution at t. By the inductive argument used in step 2, $w'_1 < w_1(\bar{w}_t)$. But this implies that

$$\bar{w}' = \omega_0 + \beta\gamma v_1(w'_1) \leq \omega_0 + \beta\gamma v_1\big(w_1(\bar{w}_t)\big) = \bar{w}_t,$$

which yields a contradiction. This establishes that $\bar{w}' \geq \bar{w}$.

We now show that $\bar{w}' \leq \bar{w}$, which will complete the proof. Suppose to the contrary that $\bar{w}' > \bar{w}$. Then there is some t such that $\bar{w}' \geq w'_t > \bar{w} \geq \bar{w}_t$. This implies that $w'_1 > w_1(\bar{w}_t)$ and hence that $\bar{w}' = \omega_0 + \beta\gamma v_1(w'_1) \leq \omega_0 + \beta\gamma v_1\big(w_1(\bar{w}_t)\big) = \bar{w}$, which yields a contradiction. □

We will now prove some results which are used in Proposition 6. Let $\gamma' \geq \gamma$ and let $\bar{w}_t$ and $\bar{w}'_t$ correspond to the wage for the problem truncated at t for γ and γ', respectively. For simplicity we will denote by w'_τ and w_τ the wages for period τ corresponding to the problem truncated at t for γ' and γ, respectively.

Lemma 1

$$\bar{w}'_t \leq \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t.$$

<u>Proof</u>. Suppose to the contrary that $\bar{w}'_t < [\gamma/\gamma']^t \bar{w}_t$. If $w'_{t-1} \geq [\gamma/\gamma']^{t-1} w_{t-1}$ then

$$\bar{w}'_t = w'_{t-1} + \beta\gamma v'_t(w'_{t-1}) > \left[\frac{\gamma}{\gamma'}\right]^{t-1}\left[w_{t-1}+\beta\gamma v_t(w_t)\right]$$

$$= \left[\frac{\gamma}{\gamma'}\right]^{t-1} \bar{w}_t \geq \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t$$

which is a contradiction. We will now show by backward induction that the contradiction hypothesis implies $w'_1 < \gamma/\gamma'$, w_1. Suppose that $w'_\tau < [\gamma/\gamma']^\tau w_\tau$. Then if $w'_{\tau-1} \geq [\gamma/\gamma']^{\tau-1} w_{\tau-1}$,

$$\bar{w}'_t = w'_{\tau-1} + \beta\gamma v'_\tau(w'_\tau) > \left[\frac{\gamma}{\gamma'}\right]^{\tau-1}\left[w_{\tau-1}+\beta\gamma v_\tau(w_\tau)\right]$$

$$= \left[\frac{\gamma}{\gamma'}\right]^{\tau-1}\bar{w}_t \geq \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t$$

a contradiction to the above hypothesis.

Hence if $\bar{w}_t < [\gamma/\gamma']^t \bar{w}_{t+1}$, then $w'_1 < \gamma/\gamma' \, w_1$. But then $\bar{w}'_t = \omega_0 + \beta\gamma v'_1(w'_1) > \omega_0 + \beta\gamma v_1(w_1) = \bar{w}_t$. This establishes that $\bar{w}'_t \geq [\gamma/\gamma']^t \bar{w}_t$. □

<u>Lemma 2</u>

$$w'_1 \leq \frac{\gamma}{\gamma'} w_1.$$

<u>Proof</u>. Suppose that $w'_1 < \gamma/\gamma' \, w_1$. Then

$$(*) \qquad \bar{w}'_t = \omega_0 + \beta\gamma' v_1(w'_1) > \omega_0 + \beta\gamma' \frac{\gamma}{\gamma'} v_1\left(\frac{\gamma}{\gamma'} w_1\right) = \bar{w}_t.$$

We now show that the above implies that $w'_\tau < [\gamma/\gamma']^\tau w_\tau$ for all $\tau < t$. Suppose that $w'_\tau < [\gamma/\gamma']^\tau w_\tau$ for all $\tau < T$. Then if $w'_T \geq [\gamma/\gamma']^\tau w_T$,

$$\bar{w}_t' = w_{T-1}' + \beta\gamma' v_T'(w_T') < \left[\frac{\gamma}{\gamma'}\right]^{T-1}\left[w_{T-1}+\beta\gamma v_T(w_T)\right]$$

$$= \left[\frac{\gamma}{\gamma'}\right]^{T-1} \bar{w}_t \le w_t,$$

contradicting (*). Hence $w_1' < \gamma/\gamma'$, w_1 implies that $w_\tau' < [\gamma/\gamma']^\tau w_\tau$ for all $\tau < t$. But then

$$\bar{w}_t' = w_{t-1}' + \beta\gamma' v_t'(\bar{w}_t') < w_{t-1}' + \beta\gamma' v_t'\left(\left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t\right)$$

$$< \left[\frac{\gamma}{\gamma'}\right]^{t-1}\left[w_{t-1}+\beta\gamma(\bar{w}_t)\right] \le \bar{w}_t$$

contradicting (*). This establishes that $w_1' \ge \gamma/\gamma'$, w_1. □

<u>Proposition 8</u>

If $\gamma' \ge \gamma$ then $n(\tau,\mu') \le n(\tau,\mu)$ where μ' and μ are the invariant distributions corresponding to γ' and γ, respectively.

<u>Proof</u>. We will show that for any truncated sequence

$$w(\tau,\mu') \ge \left[\frac{\gamma}{\gamma'}\right]^\tau w(\tau,\mu).$$

Since $n_\tau'\left[(\gamma/\gamma')^\tau w\right] = n_\tau(w)$, this suffices to prove the result.

By Lemma 1 $\bar{w}_t' \ge [\gamma/\gamma']^t \bar{w}_t$. Suppose that $w_\tau \ge [\gamma/\gamma']^\tau w_\tau$ for all $\tau \ge T$. We will show that

$$w_{T-1}' \ge \left[\frac{\gamma}{\gamma'}\right]^{T-1} w_{T-1}.$$

We will show that if this is not true then by induction (oh no, ..., not again!) $w_1' < \gamma/\gamma'\ w_1$ therefore contradicting Lemma 2. Hence suppose the contrary, that $w_{T-1}' < [\gamma/\gamma']^{T-1} w_{T-1}$. Then

$$\text{(A.2)} \qquad \bar{w}_t' < \left[\frac{\gamma}{\gamma'}\right]^{T-1}\left[w_{T-1}+\beta\gamma v_{T-1}(w_{T-1})\right] = \left[\frac{\gamma}{\gamma'}\right]^{T-1} \bar{w}_t.$$

We will now show that as a consequence of the above assumption $w'_{T-2} < [\gamma/\gamma']^{T-2} w_{T-2}$. Suppose not. Then

$$\bar{w}'_t \geq [\tfrac{\gamma}{\gamma'}]^{T-2}[w_{T-2}+\beta\gamma v_{T-1}(w_{T-1})] \geq [\tfrac{\gamma}{\gamma'}]^{T-1}\ \bar{w}_t,$$

contradicting equation (A.2). Also, if $w'_{T-3} \geq [\gamma/\gamma']^{T-3} w_{T-3}$ then $\bar{w}'_t \geq [\gamma/\gamma']^{T-3}[w_{T-3}+\beta\gamma v_{T-2}(w_{T-2})] \geq [\gamma/\gamma']^{T-1}\ \bar{w}_t$ contradicting equation (A.2). Applying this same argument inductively we obtain that $w'_1 < \gamma/\gamma'\ w_1$, a contradiction to Lemma 2.

Hence, as desired, $w'_{T-1} \geq [\gamma/\gamma']^{T-1} w_{T-1}$. Since T was chosen arbitrarily, this establishes that $w'_\tau \geq [\gamma/\gamma']^{\tau} w_\tau$, so the proof is complete. □

Proposition 10

Consider two economies with discount factors β' and β, with $\beta' > \beta$. Then $w'_\tau > w_\tau$ all τ and $n'_\tau < n_\tau$ all τ.

Proof. For convenience, let w'_τ, w_τ represent the solutions to the truncated problem 4'. Suppose $w'_1 \leq w_1$. We will show that this leads to a contradiction. From equation (4') we have that

$$w_0 + \beta\gamma v_1(w_1) = w_1 + \beta\gamma v_2(w_2)$$

$$w_0 + \beta'\gamma v_1(w'_1) = w'_1 + \beta'\gamma v_2(w'_2).$$

Subtracting from the first equation from the second and noting that $w'_1 \leq w_1$ we have after some rearranging that

$$\beta'[v_1(w'_1)-v_2(w'_2)] \leq \beta[v_1(w_1)-v_2(w_2)].$$

Recall that $\beta' > \beta$ and $v(\cdot)$ in decreasing function. Hence $w_2' < w_2$. By induction it is easy to show that $w_\tau' < w_\tau$ all τ. In particular, $w_T' < w_T$. But

$$w_0 + \beta\gamma v_1(w_1) = w_T > w_T' = w_0 + \beta'\gamma v_1(w_1').$$

Since $\beta' > \beta$ and $v_1(\cdot)$ is a decreasing function we have established a contradiction. Therefore $w_1' > w_1$.

Consider the following induction hypothesis. Suppose $w_\tau' \geq w_\tau$ and $w_{\tau+1}' \leq w_{\tau+1}$. We will show that $w_{\tau+2}' < w_{\tau+2}$ and arrive at a similar contradiction. From equation (4) we have that

$$w_\tau + \beta\gamma v_{\tau+1}(w_{\tau+1}) = w_{\tau+1} + \beta\gamma v_{\tau+2}(w_{\tau+2})$$

and

$$w_\tau' + \beta'\gamma v_{\tau+1}(w_{\tau+1}') = w_{\tau+1}' + \beta'\gamma v_{\tau+2}(w_{\tau+2}').$$

Subtracting the first equation from the second and using the induction hypothesis we have

$$\beta'\gamma\left[v_{\tau+1}(w_{\tau+1}')-v_{\tau+2}(w_{\tau+2}')\right] \leq \beta\gamma\left[v_{\tau+1}(w_{\tau+1})-v_{\tau+2}(w_{\tau+2})\right].$$

But $\beta' > \beta$. Hence $w_{\tau+2}' < w_{\tau+2}$. Again this implies that $w_T' < w_T$. But

$$w_T' = w_\tau' + \beta'\gamma v_{\tau+1}(w_{\tau+1}') < w_T = w_\tau + \beta\gamma v_{\tau+1}(w_{\tau+1}).$$

Since $w_0' = w_0$, it follows that $w_1' > w_1$. Similarly, $w_\tau' > w_\tau$ for all τ. Hence $w_T' > w_T$ which is a contradiction. Hence, for every truncated problem $w_\tau' > w_\tau$ for all τ. It follows that the same inequality holds in the limit. □

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