A Rational Expectations Equilibrium
Model of the Cyclical Behavior of
Inventories and Employment

Martin S. Eichenbaum

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University of Minnesota
and
Federal Reserve Bank of Minneapolis

ABSTRACT

A critical roadblock to modelling inventories of finished goods has been the claim that production and inventory decisions of a perfectly competitive firm are determined independently of each other. A basic goal of this study is to specify fundamental preferences of economic agents, technologies, constraints and market structures that are, in a rough way, capable of generating patterns of serial correlation and cross correlation between inventories and employment of factors of production that are consistent with those observed in the data. The claim is made that the time series for inventories, output and employment can be interpreted as emerging from a well specified dynamic, stochastic competitive equilibrium in which economic agents are assumed to form rational expectations about variables not included in their information sets. Inventories and employment will not be related in a direct way if and only if the price elasticity of demand for output is equal to infinity.

I. Introduction

Despite the general concensus among economists that inventory fluctuations play an important role in the dynamics of output and the employment of factors of production, there seems to be little agreement as to what that role is. A critical roadblock to modelling inventories of finished goods has been the claim that the production and inventory of finished goods decisions of a perfectly competitive firm are determined independently of each other (Blinder [3], Blinder [4], and Blinder and Fischer [5]). In particular, the first-order necessary conditions for the maximization of the expected present value of the profits of a perfectly competitive firm which produces storable goods are alleged to decompose into two blocks: one for inventories, the other for production decisions. This decomposition theorem can be interpreted as an exclusion restriction on the feedback part of the closed loop system for employment of factors of production and inventories of finished goods. If $L(t + J)$ represents the firm's employment of labor at time $(t + J)$ and $I(t + J)$ represents the firm's inventories of finished goods at time $(t + J)$, the feedback part of the closed loop system for $L(t + J)$ and $I(t + J)$ will be of the form:

$$\begin{bmatrix} L(t + 1) \\ I(t + 1) \end{bmatrix} = \sum_{J=0}^{\infty} A_J \begin{bmatrix} L(t - J) \\ I(t - J) \end{bmatrix}$$

where

$$A_J = \begin{bmatrix} a_J & 0 \\ 0 & b_J \end{bmatrix} \quad \text{for all nonnegative } J.$$

The result of this "decomposition theorem" has been an inability to relate changes in the employment of factors of production and inventories of

finished goods at the microeconomic level in a perfectly competitive environment.

Despite the claims that the decomposition problem arises at the level of the perfectly competitive firm, there are no claims that output is independent of inventory stocks at the industry level, only that those effects are indirect. Unfortunately, the assertion that, in a competitive environment, the decomposition theorem holds at the firm level but fails at the industry level has not been investigated. Instead, the apparent inability of perfectly competitive models to generate results consistent with the data has prompted some to view inventories as a phenomenon which is incapable of being addressed within the confines of competitive equilibrium theory (Arrow [1], Blinder and Fischer [5], and Honkapohja and Ito [12]).

Blinder [4], Blinder and Fischer [5] and Hay [11] have examined the problem from the perspective of a monopolist, while Ito and Honkapohja [12], Bryant [6] and Mills [25] have concentrated on finished goods inventories as a buffer stock in the face of fluctuating demand and quantity restrictions. Others such as Lovell [15] and Hay [11] have viewed the time series on inventories of finished goods as having been generated by the lagged adjustment of actual inventories to some optimal "equilibrium" level of inventories which is typically postulated to be a linear function of sales. In essence all of these efforts revolve around the notion that the observed data are somehow fundamentally inconsistent with notions of competitive equilibrium.

This study attempts to deal with the above issues in the spirit of

recent attempts to interpret economic time series as resulting from the interaction of economic agents who face and solve nontrivial dynamic stochastic optimization problems. A basic goal of the analysis is to specify fundamental preferences of economic agents, technologies, constraints, and market structures that are, in a rough way, capable of generating patterns of serial correlation and cross correlation consistent with those observed in the data. In particular, we will claim that it is possible to interpret the time series for inventories, output, sales and employment of factors of production as the laws of motion emerging from a well specified dynamic, stochastic competitive equilibrium in which economic agents are assumed to form rational expectations about variables not included in their information sets. Because the critical issues in the literature seem to revolve around the behavior of firms, we adopt an approach in which the critical dynamics emerge from the supply side rather than the demand side. In particular, the cyclical behavior of employment and inventories is modelled as being generated by speculation on future commodity prices, rental rates of factors of production and the costs of holding inventories. Moreover, these dynamics arise from the technology and constraints that economic agents face rather than from ad hoc adjustments to equilibrium.

The role of equilibrium is extremely important in this paper. The position is taken that the decomposition problem is not, as some have claimed, an aggregation problem, but rather an equilibrium problem. Viewed in this way, the paper may be looked at as an example pointing out the importance of generalizing single agent decision theory to market

contexts and concepts of equilibrium.[1] As it turns out, the decomposition theorem alluded to holds at the industry level if and only if the rewards of any one agent do not depend on the decisions of other agents. In terms of our problem, this is equivalent with saying that the industry demand curve is infinitely elastic.

The paper also attempts to provide a tractable empirical model of the cyclical behavior of employment and inventories of finished goods which is consistent with recent developments in dynamic economic theory. Lucas [17], Lucas and Sargent [21] and Sargent [31], among others, have pointed out the importance of the observation that the behavioral rules of rational economic agents cannot be expected to remain invariant to changes in the constraints that they face, such as the laws of motion describing the evolution over time of the prices of goods that they buy and sell. Changes in perceptions of these laws of motion will, in general, generate changes in agents' decision rules. This view, if taken seriously, necessitates rethinking what classes of objects should be regarded as "structural" or invariant to changes in the environment which affect agents' constraints, such as changes in government policy rules. What is clear is that decision rules such as supply and demand functions do not belong to this class of structural objects. As such, estimation should not be aimed at identifying the coefficients of decision rules. Instead, it should be aimed at identifying the fundamental parameters of agents' constraints, preferences and technology.

To be successful, such an estimation strategy must be based on a theoretical model which provides the econometrician with a mapping from

the fundamental parameters of the problem to decision rules of agents and the resulting laws of motion of market-determined variables. If this can be done in a way such that the free parameters of preferences, technologies and constraints are identifiable econometrically, it offers the modeller, in principle, the ability to predict how market-determined variables respond to changes in the environment that alter agents' constraints.

A related but somewhat different issue is the observation, made by Sargent [31] and Hansen and Sargent [9], that regardless of whether a given variable directly impinges on agents' objective functions, to the extent that the variable Granger causes or helps predict variables which directly affect objective functions, they belong in agents' optimal decision rules. While much work has been devoted to deriving the implications of the noninvariance of decision rules, especially in macroeconomics (Sargent and Wallace [ ], McCallum [ ]), relatively less attention has been paid to this latter issue. This paper hopes to point out that ignoring the Granger causality issue may lead to serious misinterpretations of the data as well as leading to misleading theoretical results.

The models employed in this paper involve setups in which the environment and decision rules of agents can be modelled as time invariant linear stochastic difference equations. There are a number of advantages to such an approach. One of the central messages of existing research on dynamic economic theory is that the decision rules of optimizing agents are not invariant to changes in the constraints that they face. In our context, the dynamic demand schedules of competitive firms for factors of production and dynamic supply functions for goods to the market are

systematic functions of the stochastic processes facing agents.

An advantage of specifying agents' objective functions as being quadratic and their constraints as being linear is that the relevant dynamic stochastic optimization theory is tractable analytically. Furthermore, it permits one to exploit the property of certainty equivalence which enables us to separate the maximum problem which agents face into two parts, an optimization problem and a forecasting problem. The resulting decision rules will depend separately on the parameters of taste and technology and on the parameters describing the environment in which agents exist. Furthermore, when the environment is described in terms of stationary Markov processes, the equilibrium laws of motion of the system, while being highly nonlinear in the fundamental parameters to be estimated, take the form of a system of time invariant linear stochastic difference equations. This last feature is extremely convenient from the point of view of estimation.

The remainder of this paper is organized as follows. Section II contains a detailed critique of the decomposition theorem. Section II.1

develops a model in which there exists a well defined inventory policy for perfectly competitive firms. The rational expectations competitive equilibrium of this industry is calculated in Section II.2. The rational expectations equilibrium for the monopoly case is calculated in Section II.3. Section II.4 contrasts the laws of motion describing the time paths of inventories, employment and prices emerging from a dynamic Nash equilibrium with those that emerge from the competitive and monopoly equilibria. Section III discusses estimation strategies which are aimed at the fundamental parameters of agents' objective functions and constraints which include the various stochastic processes that economic agents care about but have no control over.[2]

II. An Equilibrium Model of Inventories and Employment

In Section II.1 we deal with perfectly competitive firms who maximize the expected present value of profits. Firms are allowed to hold inventories of finished goods as well as to produce output. The firms' problem is solved by using the discrete time calculus of variations. Optimal decision rules are then calculated. The first-order necessary conditions, which take the form of a set of Euler equations and associated transversality conditions, decompose into two separate blocks, one for inventories, the other for production decisions. However, when an explicit dynamic rational expectations equilibrium is calculated in Section II.2, the equilibrium laws of motion for the employment of labor and inventories of finished goods are simultaneously determined at the industry level. In Section II.3, the n-plant monopoly case is considered.

In Section II.4 the Nash equilibrium is examined.

II.1 The Problem of the Representative Firm

This section develops a linear-quadratic model of the behavior of perfectly competitive firms who produce storable output.

$S(t)$ = sales of the representative firm at time t

n = number of firms in the industry, assumed constant over time

$(1 - \sigma)$ = the rate of depreciation per unit of time of inventories of finished goods

$L(t)$ = the amount of labor employed by the representative firm during time period t

$Q(t)$ = output of the representative firm at time t

$I(t)$ = stock of inventories of finished goods of the representative firm at the end of period t

$P(t)$ = price of a finished good sold at period t

$\omega(t)$ = rental rate of labor at time t

$Z(t)$ = a (P x 1) vector whose first element is ω_t and whose second element is $P(t)$; the remaining elements of $Z(t)$ are variables that help to predict future ω_t's and/or future P_t's

$\rho(t)$ = a random shock to costs

$\psi(t)$ = a random shock to costs

Both $\rho(t)$ and $\psi(t)$ are observed by the firm but are unobserved by the econometrician.

B = a discount factor, $0 < B < 1$

a, d, e, f and g are scalar constants, $0 < a < 1$, $e > 0$, $g > 0$, $d \geq 0$, $f \geq 0$

We also define the following polynomials in the lag operator L:

$\zeta(L) = I - \sum_{J=1}^{q} \zeta_J L^J$, where the ζ_J are (p x p) matrices and I is the (p x p) identity matrix

$\delta_{\psi}(L) = 1 - \sum_{J=1}^{r_\psi} \delta_{\psi_J} L^J$, where the δ_{ψ_J} are scalars

$\delta_{\rho_J}(L) = 1 - \sum_{J=1}^{r_\rho} \delta_{\rho_J} L^J$, where the δ_{ρ_J} are scalars.

The representative firm is assumed to maximize the expected present value of profits. Output of the firm is governed by

$$Q(t) = aL(t)$$

and sales of the firm are

$$S(t) = Q(t) - [I(t) - \sigma I(t - 1)].$$

Hence, the total revenue of the firm at time (t) is

$$TP(t) = P(t)[aL(t) - I(t) + \sigma I(t - 1)].$$

In order to reproduce the decomposition result at the firm level, we must assume that the costs of production and changing the level of inventories of finished goods are additively separable.

$$TC(t) = TC_1(t)[L(t), L(t - 1), \omega(t), \rho(t)] + TC_2(t) \cdot [I(t), I(t - 1), \psi(t)]$$

In particular,

$$TC_1(t) = \frac{e}{2}[L(t) - L(t - 1)]^2 + \frac{d}{2}[L(t) + \rho(t)]^2 + \omega(t)L(t)$$

$$TC_2(t) = \frac{g}{2}[I(t) - \sigma I(t - 1)]^2 + \frac{f}{2}[I(t) + \psi(t)]^2$$

where:

$$\delta_\psi(L)\psi(t) = U_t^\psi \qquad (1)$$

$$\delta_\rho(L)\rho(t) = U_t^\rho \qquad (2)$$

and

$$\zeta(L)Z(t) = U^z(t) \qquad (3)$$

where

$$U_t^\psi = \psi(t) - E[\psi(t)/\Omega_{t-1}]$$

$$\psi_t^\rho = \rho(t) - E[\rho(t)/\Omega_{t-1}]$$

$$U_t^z = Z(t) - E[Z(t)/\Omega_{t-1}]$$

E is the linear least squares projection operator and Ω_t is the information set at time t. We assume that Ω_t includes at least [I(t - 1), I(t - 2),..., L(t - 1), L(t - 2),..., $\psi(t)$, $\psi(t - 1)$, $\psi(t - 2)$, ..., $\rho(t)$, $\rho(t - 1)$,..., $\omega(t)$, $\omega(t - 1)$, $\omega(t - 2)$, ..., P(t), P(t - 1), ...].

The terms $\frac{e}{2}[L(t) - L(t - 1)]^2$ and $\frac{g}{2}[I(t) - \sigma I(t - 1)]^2$ represent costs of adjustment which are internal to the firm. These costs are postulated to rise at an increasing rate as a function of the absolute value of the change in labor inputs and the absolute value of the change in inventories of finished goods.

The term $\frac{d}{2}[L(t) + \rho(t)]^2$ is included for two reasons. The first reason is to represent the notion that there are costs internal to the firm

associated with large size. The second and more important reason is that it will give us a model of the error term which we may contrast wiht the one that emerges from the setup of Hansen and Sargent [9].[3]

The term $\frac{f}{2}[I(t) + \psi(t)]^2$ is included for the latter reason as well as to capture the nature of the various ad hoc inventory holding cost functions that appear in the literature. Setting d = g = 0 will not significantly alter the qualitative conclusions that emerge from the model.

The term $\sigma bI(t - 1)$ represents benefits accruing to the firm due to beginning of the period inventory stocks. The presence of such a term emphasizes that the inventories in this model are held by firms which have reasons to hold them in addition to speculative motives. As is emphasized in Aiyagari, Eckstein and Eichenbaum [ ], the solution of the model in which inventories are held only by speculators may be considerably different, although the rough correlations between the time series remain the same. The returns or reductions to costs due to beginning of period inventories are represented in the above form so as to retain a decomposable environment.

We assume

$$E[U_t^{\psi}|\Omega_{t-1}] = E[U_t^{\rho}|\Omega_{t-1}] = 0 ,$$

and

$$E[U_t^Z|\Omega_{t-1}] = 0 \quad \text{and} \quad E[U_t^Z(U_t^Z)^T] = \Sigma$$

where Σ is a positive definite (p x p) matrix,

$$E[(U_t^{\psi})^2] = \mathrm{Var}(U^{\psi})$$

$$E[(U_t^{\rho})^2] = \mathrm{Var}(U^{\rho}).$$

Hence,

U_t^{ψ} is a white noise which is fundamental for $\psi(t)$

U_t^ρ is a white noise which is fundamental for $\rho(t)$

and U_t^z is a (p x 1) vector white noise which is fundamental for $Z(t)$[4]

We further assume that $\rho(t)$, $\psi(t)$ and $Z(t)$ are jointly covariance stationary stochastic processes. Sufficient conditions for this are that the roots of $\delta_\omega(Z) = 0$, $\delta_\rho(Z) = 0$ and $\det \zeta(Z) = 0$ lie outside the unit circle.[5] Note that given our specification of Ω_t, the firm knows $\{\rho_t, \rho_{t-1}, \ldots, \psi_t, \psi_{t-1}, \ldots, Z_t, Z_{t-1}, \ldots\}$ but does not know with certainty $\{\rho_{t+J}\}_{J=1}^{\infty}$, $\{Z_{t+J}\}_{J=1}^{\infty}$.

The firm's problem is to choose a linear contingency plan for L(t) and I(t) to maximize its expected present value

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t \Pi(t) = \lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{P(t)[aL(t) - I(t) + \sigma I(t-1)]$$
$$- \omega(t)L(t) - \frac{d}{2}[L(t) + \rho(t)]^2 - \frac{e}{2}[L(t) - L(t-1)]^2$$
$$+ b\sigma I(t-1) - \frac{f}{2}[I(t) + \psi(t)]^2 - \frac{g}{2}[I(t) - \sigma I(t-1)]^2 \qquad (4)$$

subject to L_{-1} and I_{-1} given and to (1), (2) and (3). The symbol $E_t[Z(t + J)]$ represents the mathematical expectation of $Z(t + J)$ conditioned on information available at time t, i.e., $E_t(Z(t + J)) = E[Z(t + J)/\Omega_t]$.

Notice that the above problem decomposes into two separate problems:

(I) Choose a linear contingency plan for L(t) to maximize

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{P(t)aL(t) - \omega(t)L(t) - \frac{d}{2}[L(t) + \rho(t)]^2$$
$$- \frac{e}{2}[L(t) - L(t-1)]^2\} \qquad (5)$$

subject to L_{-1} given and to (2) and (3).

(II) Choose a linear contingency plan for $I(t)$ to maximize

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{-P(t)I(t) + [P(t) + b]\sigma I(t-1) - \frac{f}{2}[I(t) + \psi(t)]^2 - \frac{g}{2}[I(t) - \sigma I(t-1)]^2\} \tag{6}$$

subject to I_{-1} given and to (1) and (3).

It can be shown that (I) and (II) are well defined problems with well defined solutions. Hence, the non-existence theorem fails at the firm level. However, if by decomposition it is meant that the feedback part of $I(t)$ does not depend on $L(t - J)$, $J \geq 0$ and the feedback part of $L(t)$ does not depend on $I(t - J)$, $J \geq 0$, then the decomposition theorem does not fail at the firm level. In particular, the solutions to (I) and (II) will not enable us to justify a direct effect of inventory stocks or inventory investment on production decisions. Notice however that the presence of $P(t)$ in both objective functions and possible correlations between $P(t)$ and $\omega(t)$ imply cross-equation restrictions on the decision rules for $L(t)$ and $I(t)$. Therefore, $L(t)$ and $I(t)$ will, in general, be correlated.

We now proceed to solve problems (I) and (II). As noted in Section I, problems such as (I) and (II) which have quadratic return functions and linear constraints exhibit certainty equivalence or the separation principle.[6] In particular, we may separate forecasting from optimization considerations. Consider problem (I) where $\{\omega(t)\}_{t=0}^{\infty}$, $\{P(t)\}_{t=0}^{\infty}$ and $\{\rho(t)\}_{t=0}^{\infty}$ are known and are of exponential order less than $1/\sqrt{B}$. We solve the problem by use

of the discrete time calculus of variations.

The first-order necessary conditions are obtained by differentiating (5) with respect to $t = 0, 1, \ldots, T-1$ and setting each derivative equal to zero. The result is the system of Euler equations

$$BL(t+1) + \phi L(t) + L(t-1) = \frac{1}{e}[\omega(t) + d\rho(t) - aP(t)]$$

$$t = 0, 1, \ldots, T-1 \qquad (7)$$

$$\phi_1 = -[(1 + B) + \frac{d}{e}]$$

Differentiating with respect to $L(T)$ we obtain the transversality condition

$$\lim_{T\to\infty} E_0 B^N [P(T)\dot{a} - \omega(T) - dL(T) - d\rho(T) - eL(T) - L(T-1)] = 0 \quad (8)$$

which, given our assumptions, is clearly satisfied.

To solve the Euler equations for $t = 0, 1, 2, \ldots,$ subject to the initial conditions and the transversality conditions, we factor the characteristic polynomial of (7)

$$[1 + \frac{\phi_1}{B} Z + \frac{1}{B} Z^2] = [1 - \lambda_1 Z][1 - \lambda_2 Z]$$

Equating powers of Z gives

$$-\frac{\phi_1}{B} = \lambda_1 + \lambda_2, \ \lambda_1\lambda_2 = \frac{1}{B} \ \text{ or } \lambda_2 = \frac{1}{\lambda_1 B}$$

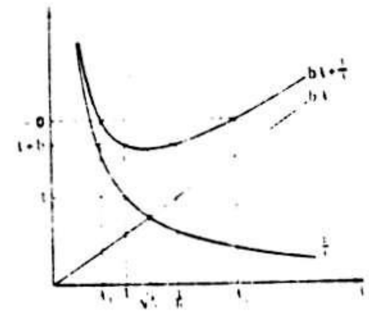
Hence

$$-\frac{\phi_1}{B} + \lambda_1 + \frac{1}{\lambda_1 B} \ \text{ or } (1 + B) + \frac{d}{e} = \lambda_1 B + \frac{1}{\lambda_1}$$

Now the function $\lambda_1 B + \frac{1}{\lambda_1}$ attains a minimum at $\lambda_1 = 1/\sqrt{B}$ and is equal to

$2\sqrt{B}$ there. For $0 < B < 1$, $1 + B \geq 2\sqrt{B}$ with equality at $B = 1$. Further, $-\phi_1 \geq 1 + B$ with equality if $d = 0$. Hence, with $d > 0$, the solutions for λ_1 and λ_2, as depicted in Figure 1, are real and distinct.

FIGURE 1



Without loss of generality, let λ_1 be the smaller root. Then the above implies that $0 < \lambda_1 < 1 < 1/\sqrt{B} < \lambda_2$. Hence, we rewrite (7) as

$$(1 - \lambda_1 L)(1 - \lambda_2 L)L(t + 1) = \frac{1}{Be}[\omega(t) + d\rho(t) - aP(t)]$$

In order to satisfy the transversality condition, we must solve the unstable root forwards and the stable root backwards. As such we have

$$L(t + 1) = \lambda_1 L_1(t) + \frac{1}{eB(1 - \lambda_2 L)^{-1}}[\omega(t) + d\rho(t) - aP(t)]$$

or

$$L(t + 1) = \lambda_1 L_1(t) - \frac{\lambda_1}{e}\sum_{J=0}^{\infty}(\lambda_1 B)^J[\omega(t + J + 1) + d\rho(t + J + 1) - aP(t + J + 1)]$$

Recalling that $\{P(t)\}_{t=1}^{\infty}$, $\{\omega(t)\}_{t=1}^{\infty}$ and $\{\rho(t)\}_{t=1}^{\infty}$ are in fact unknown at $t = 0$, we replace the terms

$$\sum_{J=0}^{\infty} (\lambda_1 B)^J [\omega(t + J + 1) + d\rho(t + J + 1) - aP(t + J + 1)]$$

by

$$\sum_{J=0}^{\infty} (\lambda_1 B)^J [E_{t+1}\omega(t + J + 1) + dE_{t+1}\rho(t+J+1) - aE_{t+1}P(t+J+1)]$$

Hence,

$$L(t + 1) = \lambda_1 L(t) - \frac{\lambda_1}{e} \sum_{J=0}^{\infty} (\lambda_1 B)^J [E_{t+1}\omega(t + J + 1)$$

$$+ dE_{t+1}\rho(t + J + 1) - aE_{t+1}P(t + J + 1) \qquad (9)$$

Now, (9) is not yet a decision rule because terms like $E_t P(t + k)$, $E_t\rho(t + k)$ and $E_t\omega(t + k)$, $k > 0$, must be expressed as functions of variables included in the information sets of agents at time t, Ω_t. Recall that firms are maximizing (5) subject to (2) and (3), the objective stochastic processes. In the context of this problem this is the content of the rational expectations assumption. That is to say that the parameters of the perceived stochastic processes that agents use in making decisions are the same as the objective parameters of those processes. Hansen and Sargent [9] derive formulas for terms such as

$$\sum_{J=0}^{\infty} \delta^J E_t Z(t+J)$$

where Z(t) is a (Px1) vector. In particular, if

$$\zeta(L)Z(t) = U^Z(t)$$

then the moving average representation for Z(t) is

$$Z(t) = \zeta^{-1}(L)U^Z(t)$$

and

$$\sum_{J=0}^{\infty} \delta^J E_t Z(t + J) = [\frac{I - L^{-1}\delta\zeta^{-1}(\delta)\zeta(L)}{1 - \delta L^{-1}}] Z(t)$$

$$= \zeta^{-1}(\delta)[I + \sum_{J=1}^{q-1} (\sum_{J=k+1}^{q} \delta^{k-J}\zeta_k)L^J]Z(t) \tag{10}$$

We use the above prediction formula to obtain the following closed form for the firm's decision rule that expresses the restrictions imposed across the decision rule and the parameters of the stochastic process of $\omega(t)$, $P(t)$ and $\rho(t)$.[8] Defining ℓ_i, $i = 1, 2$, as the (1xP) row vector with one in the ith place and zero elsewhere, we obtain

$$L(t) = \lambda_1 L(t - 1) - \frac{\lambda_1}{e} \{(\ell_1 - a\ell_2)\}[\frac{I - L^{-1}\lambda_1 B\zeta^{-1}(\lambda_1 B)\zeta(L)}{1 - \lambda_1 BL^{-1}}]Z(t)$$

$$- \frac{\lambda_1 d}{e} [\frac{1 - L^{-1}\lambda_1 B\delta_\rho^{-1}(\lambda_1 B)\delta_\rho(L)}{1 - \lambda_1 BL^{-1}}] \rho(t) \tag{11}$$

Notice that although the agent cares only about $\{P(t + J)\}_{j=0}^{\infty}$, $\{\omega(t + J)\}_{J=0}^{\infty}$ and $\{\rho(t + J)\}_{J=0}^{\infty}$, in the sense that only those stochastic processes directly affect their objective functions, equation (11) implies that the current and past values of all variables that help predict $\omega(t)$, $P(t)$ and $\rho(t)$ belong in the decision rule. Alternatively, all processes that agents see and Granger cause $\omega(t)$, $P(t)$ and/or $\rho(t)$ belong in the agent's decision rule for $L(t)$.[9]

We now consider problem II. The relevant Euler equations are

$$B\sigma g I(t + 1) - [g(1 + B\sigma^2) + f]I(t) + g\sigma I(t - 1) = P(t) - B\sigma P(t + 1) + f\psi(t) - Bb\sigma$$

or

$$BI(t + 1) + \phi_2 I(t) + I(t - 1) = \frac{1}{\sigma g} [P(t) - B\sigma P(t + 1) + f\psi(t) - Bb\sigma] \tag{12}$$

where

$$\phi_2 = - \frac{(1 + B\sigma^2)}{\sigma} + \frac{f}{\sigma g}$$

The associated transversality condition is

$$\lim_{T \to \infty} E_0 B^T \{- P(t) - fI(T) - f\psi(T) - gI(T) + g\sigma I(T - 1)\} = 0 \tag{13}$$

As before we must factor the characteristic polynomial

$$[1 + \frac{\phi_2}{B} Z + \frac{Z^2}{B}] = (1 - \lambda_3)(1 - \lambda_4)$$

Equating powers of Z implies that

$$- \frac{\phi_2}{B} = \lambda_3 + \lambda_4, \ \lambda_3 \lambda_4 = \frac{1}{B} \text{ or } \lambda_4 = \frac{1}{\lambda_3 B}$$

Given our assumptions, we see as before that $0 < \lambda_3 < 1 < 1/B < \lambda_4$.

Hence,

$$I(t + 1) - \lambda_3 I(t) = \frac{1}{B\sigma g} \frac{1}{1 - \lambda_4 L} [P(t) - B\sigma P(t + 1) + f\psi(t) - Bb\sigma]$$

or

$$I(t + 1) = \lambda_3 I(t) - \frac{\lambda_3}{\sigma g} \sum_{J=0}^{\infty} (\lambda_3 B)^J [P(t + J + 1) - B\sigma P(t + J + 2) + f\psi(t + J + 1)] + \frac{\lambda_3 Bb}{g(1 - \lambda_3 B)}$$

or

$$I(t) = \lambda_3 I(t - 1) - \frac{\lambda_3 f}{\sigma g} \sum_{J=0}^{\infty} (\lambda_3 B)^J E_t \psi(t + J) - \frac{\lambda_3}{\sigma g} \sum_{J=0}^{\infty} (\lambda_3 B)^J [E_t P(t + J) - B\sigma E_t P(t + J + 1)] + \frac{\lambda_3 Bb}{g(1 - \lambda_3 B)} \tag{14'}$$

As before we impose the hypothesis of rational expectations to evaluate terms like $E_t\psi(t + k)$, $E_tP(t + k)$, $k > 0$.

Using equation (10),

$$\frac{\lambda_3 f}{\sigma g} \sum_{J=0} (\lambda_3 B)^J E_t \psi(t + k) = \frac{\lambda_3 f}{\sigma g} \left[\frac{1 - L^{-1}\lambda_3 B\delta_\psi^{-1}(\lambda_3 B)\delta_\psi(L)}{1 - \lambda_3 BL^{-1}}\right] \psi(t)$$

Now

$$\sum_{J=0}^{\infty} (\lambda_3 B)^J E_t P(t + J + 1) = \sum_{J=0}^{\infty} (\lambda_3 B)^{J-1} E_t P(t + J) - \frac{1}{\lambda_3 B} P(t)$$

Hence,

$$\frac{-\lambda_3}{\sigma g} \sum_{J=0}^{\infty} (\lambda_3 B)^J E_t P(t + J) + \frac{\lambda_3 B\sigma}{g} \sum_{J=0}^{\infty} (\lambda_3 B)^J E_t P(t + J + 1)$$

$$= \frac{-\lambda_3}{g} \sum_{J=0}^{\infty} (\lambda_3 B)^J E_t P(t + J) + \frac{\lambda_3 B\sigma}{g} \sum_{J=0}^{\infty} (\lambda_3 B)^{J-1} E_t P(t + J) - \frac{\lambda_3 B\sigma}{g(\lambda_3 B)} P(t)$$

$$= \frac{1}{g} (\sigma - \lambda_3) \sum_{J=0}^{\infty} (\lambda_3 B)^J E_t P(t + J) - \frac{\sigma}{g} P(t)$$

Substituting into (14') and using (10) we obtain the decision rule

$$I(t) = \lambda_3 I(t - 1) + \frac{\lambda_3 Bb}{g(1 - \lambda_3 B)} - \frac{\sigma}{g} P(t) + \frac{1}{g} (\sigma - \lambda_3)\ell_2$$

$$\cdot \left[\frac{I - L^{-1}\lambda_3 B\zeta^{-1}(\lambda_3 B)\zeta(L)}{1 - \lambda_3 BL^{-1}}\right] Z(t)$$

$$- \frac{\lambda_3 f}{\sigma g} \left[\frac{1 - L^{-1}\lambda_3 B\delta_\psi^{-1}(\lambda_3 B)\delta_\psi(L)}{1 - \lambda_3 BL^{-1}}\right] \psi(t) \qquad (15)$$

In sum, the system of equations to be estimated assuming that the $Z(t)$ process was econometrically exogenous is:

$$L(t) = \lambda_1 L(t-1) - \frac{\lambda_1}{e}\{(\ell_1 - a\ell_2)\}\left[\frac{I - L^{-1}\lambda_1 B\zeta^{-1}(\lambda_1 B)\zeta(L)}{1 - \lambda_3 BL^{-1}}\right] Z(t)$$

$$- \frac{\lambda_1 d}{e}\left[\frac{1 - L^{-1}\lambda_1 B\delta_\rho^{-1}(\lambda_1 B)\delta_\rho(L)}{1 - \lambda_1 BL^{-1}}\right]\rho(t) \qquad (11)$$

$$I(t) = \lambda_3 I(t-1) + \frac{\lambda_3 Bb}{g(1-\lambda_3 B)} - \frac{\sigma}{g}P(t) + \frac{1}{g}(\sigma - \lambda_3)\ell_2$$

$$\cdot \left[\frac{I - L^{-1}\lambda_3 B\zeta^{-1}(\lambda_3 B)\zeta(L)}{1 - \lambda_3 BL^{-1}}\right] Z(t)$$

$$- \frac{\lambda_3 f}{\sigma g}\left[\frac{1 - L^{-1}\lambda_3 B\delta^{-1}(\lambda_3 B)\delta_\psi(L)}{1 - \lambda_3 BL^{-1}}\right]\psi(t) \qquad (15)$$

$$\delta_\psi(L)\psi(t) = U^\psi(t) \qquad (1)$$

$$\delta_\rho(L)\rho(t) = U^\rho(t) \qquad (2)$$

$$\zeta(L)Z(t) = U^Z(t) \qquad (3)$$

Notice that equations (11) and (15) are exact decision rules. However, the assumption that private agents observe the random processes $\rho(t)$ and $\psi(t)$ but that the econometrician does not justifies the existence of a disturbance term. With this interpretation the disturbance term in (11) becomes

$$\varepsilon_1(t) = \frac{-\lambda_1 d}{e}\left[\frac{1 - L^{-1}\lambda_1 B\delta_\rho^{-1}(\lambda_1 B)\delta_\rho(L)}{1 - \lambda_1 BL^{-1}}\right]\rho(t) \qquad (16)$$

and the disturbance term in (15) becomes

$$\varepsilon_2(t) = \frac{-\lambda_3 f}{g} \left[\frac{1 - L^{-1}\lambda_3 B\delta_\psi^{-1}(\lambda_3 B)\delta_\psi(L)}{1 - \lambda_3 BL^{-1}}\right] \psi(t) \tag{17}$$

and the econometric model becomes

$$L(t) = \lambda_1 L(t-1) - \frac{\lambda_1}{e}\{(\ell_1 - a\ell_2)\} \left[\frac{I - L^{-1}\lambda_1 B\zeta^{-1}(\lambda_1 B)\zeta(L)}{1 - \lambda_1 BL^{-1}}\right] Z(t) + \varepsilon_1(t) \tag{11'}$$

$$I(t) = \lambda_3 I(t-1) + \frac{\lambda_3 Bb}{g(1 - \lambda_3 B)} - \frac{\sigma}{g} P(t) + \frac{1}{g}(\sigma - \lambda_3)\ell_2 \cdot \left[\frac{I - L^{-1}\lambda_3 B\zeta^{-1}(\lambda_3 B)\zeta(L)}{1 - \lambda_3 BL^{-1}}\right] Z(t) + \varepsilon_2(t) \tag{15'}$$

and

$$\zeta(L)Z(t) = U^Z(t) \tag{3}$$

It is important to notice the existence of multiple cross-equation restrictions in the above model. In particular, there are restrictions across the parameters in agents' decision rules and the parameters of the stochastic processes that are, from the point of view of firms, uncontrollable. As will be discussed in Section III these restrictions will, given the paucity of exclusion restrictions generated by the model, be a significant source of identification.

Before proceeding, we comment on the qualitative nature of the system. Recall equations (9) and a version of (14"):

$$L(t) = \lambda_1 L(t-1) - \frac{\lambda_1}{e} \sum_{J=0}^{\infty} (\lambda_1 B)^J [E_t\omega(t+J) + dE_t\rho(t+J) - aE_t P(t+J)] \tag{9}$$

$$I(t) = \lambda_3 I(t-1) - \frac{\lambda_3 f}{\sigma g} \sum_{J=0}^{\infty} (\lambda_3 B)^J E_t \psi(t+J) + \frac{\lambda_3 Bb}{g(1-\lambda_3 B)}$$

$$+ \frac{1}{g}(\sigma - \lambda_3) \sum_{J=1}^{\infty} (\lambda_3 B)^J E_t P(t+J) - \frac{\lambda_3}{g} P(t) \quad (14'')$$

For any arbitrary set of expectations, the above imply that

$$\frac{\partial L(t)}{\partial E_t \omega(t+J)} < 0 \ \forall J \geq 1, \quad \frac{\partial L(t)}{\partial \omega(t)} < 0 \ , \quad (18)$$

and

$$\frac{\partial I(t)}{\partial P(t)} < 0 \ .$$

If, in addition, $\lambda_3 < \sigma$, which must be the case if as in most of the literature we assume $\sigma = 1$, then

$$\frac{\partial I(t+1)}{\partial E_t P(t+J)} > 0 \ \forall J \geq 1 \ . \quad (19)$$

Equation (18) simply says that employment will be an increasing function of the expected value of the marginal product of labor and a decreasing function of the expected real wage rate. In addition, the planned inventories at time (t) will be a decreasing function of the price level at time t and an increasing function of expected speculative gains.

To the extent that one is willing to identify the business cycle with P(t), the above results are a hint that inventories of finished goods behave counter-cyclically, whereas the employment of factors of production will behave pro-cyclically. We now present a simple example to investigate this conjecture.

If we specify the stochastic processes involved as

$$\sum_{J=0}^{\infty} \Theta_p^J P(t - J) = U^P(t) \qquad |\Theta_p| < 1$$

$$\omega(t)(1 - \bar{a}L)(1 - \bar{b}L) = U^{\omega}(t) \qquad |\bar{a}| < 1,\ |\bar{b}| < 1$$

$$\psi(t) = \Theta^{\omega}\psi(t - 1) + U^{\psi}(t) \qquad |\Theta^{\psi}| < 1$$

$$\rho(t) = \Theta^{\rho}\psi(t - 1) + U^{\rho}(t) \qquad |\Theta^{\rho}| < 1$$

we obtain the following decision rules

$$\begin{aligned} L(t) = \lambda_1 L(t - 1) &- \frac{\lambda_1\omega(t)}{e[1 - (\bar{a} + \bar{b})\lambda_1 B + \bar{a}\bar{b}\lambda_1^2 B^2]} \\ &+ \frac{\lambda_1^2 \bar{a}\bar{b}B}{e[1 - (\bar{a} + \bar{b})\lambda_1 B + \bar{a}\bar{b}\lambda_1^2 B^2]}\,\omega(t - 1) \\ &- \frac{a\lambda_1}{e}P(t) + \frac{a\lambda_1 B}{e}\sum_{J=0}^{\infty}\Theta_p^{J+1}P(t - J) \\ &- \frac{\lambda_1 d}{e}\,\frac{\rho(t)}{(1 - \Theta^{\rho}\lambda_1 B)} \end{aligned}$$

$$\begin{aligned} I(t) = \lambda_3 I(t - 1) &+ \frac{\lambda_3 Bb}{g(1 - \lambda_3 B)} - \frac{\lambda_3}{g}P(t) \\ &- \frac{1}{g}(\sigma - \lambda_3)\Theta^P\lambda_3 B\sum_{J=1}^{\infty}\Theta_p^J P(t - J) \\ &- \frac{\lambda_3 f}{\sigma g(1 - \Theta^{\psi}\lambda_3 B)}\,\psi(t) \end{aligned}$$

In this example, our conjecture is true. In particular,

$$\frac{\partial I(t)}{\partial P(t)} < 0,\ \frac{\partial L(t)}{\partial P(t)} > 0\ .$$

However, the above example is not meant to suggest that the signs of the

partial derivatives are necessarily invariant to the specification of the relevant stochastic processes.

II.2 Competitive Equilibrium

We now consider the competitive equilibrium of the industry in question. It is demonstrated that in a dynamic rational expectations competitive equilibrium, the decomposition theorem fails at the industry level.

Let the industry demand curve for final consumption of the good be

$$P(t) = A_0 - A_1\bar{S}(t) + U_s(t), \quad A_0, A_1 > 0 \tag{20}$$

where "-" denotes an industry-wide variable, $U_s(t)$ is a stochastic shock to demand which obeys the Markov law

$$\alpha(L)U_s(t) = V^s(t), \; \alpha(L) = \sum_{J=0}^{r_s} \alpha_J L^J,$$

$V^s(t)$ is a fundamental white noise for $U_s(t)$ and the zeroes of det $\alpha(Z)$ lie outside the unit circle. At time t, total industry sales are

$$\bar{S}(t) = n[Q(t) - I(t) + \sigma I(t - 1)]$$

or

$$\bar{S}(t) = n[aL(t) - I(t) + \sigma I(t - 1)]$$

Hence

$$P(t) = A_0 - A_1[a\bar{L}(t) - \bar{I}(t) + \sigma\bar{I}(t - 1)] + U_s(t) \tag{20'}$$

The representative firm's problem is to choose linear contingency plans for setting L(t) and I(t) as functions of information available at time t, to maximize

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{[A_0 - A_1[a\bar{L}(t) - \bar{I}(t) + \sigma\bar{I}(t-1)]aL(t) - \omega(t)L(t)$$
$$- \frac{d}{2}[L(t) + \rho(t)]^2 - \frac{e}{2}[L(t) - L(t-1)]^2\}$$

$$+ \lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{[A_0 - A_1[a\bar{L}(t) - \bar{I}(t) + \sigma\bar{I}(t-1)](-I(t) + \sigma I(t-1))$$
$$+ b\sigma I(t-1) - \frac{f}{2}[I(t) + \psi(t)]^2 - \frac{g}{2}[I(t) - \sigma I(t-1)]^2\} \qquad (21)$$

subject to I_{-1}, L_{-1} and

$$\alpha(L)U_s(t) = V^s(t) \qquad (22)$$

$$\delta_\psi(L)\psi(t) = U_t^\psi \qquad (23)$$

$$\delta_\rho(L)\rho(t) = U_t^\rho \qquad (24)$$

and

$$\zeta(L)M(t) = V^M(t) \qquad (25)$$

where $\omega(t)$ is the first element of the (p x 1) vector random process M(t) that obeys the (q x 1) order vector autoregression (24), $V^M(t)$ is a (q x 1) vector white noise that is fundamental for M(t). $E[V^M(t)[V^M(t)]^t] = \varepsilon_M$ which is known by the representative firm, and the roots of det $\zeta(Z) = 0$ are all greater than $\sqrt{B}$.

Now, problem (21) is not a well posed one until we attribute to the firm views about how the industry-wide stocks of inventories, $\bar{I}(t)$, and

industry-wide employment of labor $\bar{L}(t)$, evolve. Once that is done the firm will have views as to the laws of motion of all those random variables which it cannot control but which influence the expected present value of profits. It is clear that $\bar{I}(t)$ and $\bar{L}(t)$ enter that category because, as is clear from equation (20'), they influence P(t).

At this point we define

$$\bar{M}(t) = [M(t), M(t - 1) , \ldots, M(t - q + 1)]$$

$$\bar{U}_s(t) = [U_s(t), U_s(t - 1, \ldots, U_s(t - r_s + 1)]$$

$$\bar{\psi}(t) = [\psi(t), \psi(t - 1), \ldots, \psi(t - r_\psi + 1)]$$

$$\bar{\rho}(t) = [\rho(t), \rho(t - 1), \ldots, \rho(t - r_\rho + 1)]$$

We assume that firms view $\bar{L}(t)$ and $\bar{I}(t)$ as evolving according to

$$\bar{L}(t) = F[\bar{M}(t), \bar{U}_s(t), \bar{\psi}(t), \bar{\rho}(t), \bar{L}(t - 1), \bar{I}(t - 1), 1] \qquad (26)$$

and

$$\bar{I}(t) = G[\bar{M}(t), \bar{U}_s(t), \bar{\psi}(t), \bar{\rho}(t), \bar{L}(t - 1), \bar{I}(t - 1), 1] \qquad (27)$$

where F and G are linear functions whose parameters are known by firms in the industry.

Hence the firm maximizes (21) subject to the laws of motion (22), (23), (24), (25), (26) and (27), where the optimizing actions are over linear contingency plans of the form

$$L(t) = f[\bar{M}(t), \bar{U}_s(t), \bar{\psi}(t), \bar{\rho}(t), \bar{L}(t - 1), \bar{I}(t - 1)$$
$$L(t - 1), I(t - 1), 1]. \qquad (28)$$

$$I(t) = g[\bar{M}(t), \bar{U}_s(t), \bar{\psi}(t), \bar{\rho}(t), \bar{L}(t-1), \bar{I}(t-1), L(t-1), I(t-1), 1] \tag{29}$$

We now define a rational expectations equilibrium.[10]

Definition: A rational expectations equilibrium is four linear functions (26), (27), (28) and (29) such that

(i) given the aggregate laws of motion (26) and (27), the contingency plans (28) and (29) solve the firm's problem, and

(ii) the contingency plans of the representative firm (28) and (29) imply the aggregate laws of motion (26) and (27) so that

$$F(\cdot) = nf(\cdot)$$

and

$$G(\cdot) = ng(\cdot)$$

If we write

$$\bar{L}(t) = F_0 + F_L\bar{L}(t-1) + F_I\bar{I}(t-1) + F_M(L)M(t) + F_{u_s}U_s(t) + F_\psi(L)\psi(t) + F_\rho(L)\rho(t) \tag{26}$$

$$\bar{I}(t) = G_0 + G_L\bar{L}(t-1) + G_I\bar{I}(t-1) + G_M(L)M(t) + G_{u_s}(L)U_s(t) + G_\psi(L)\psi(t) + G_\rho(L)\rho(t) \tag{27}$$

and

$$L(t) = f_0 + C_L L(t-1) + C_I I(t-1) + f_L(L)\bar{L}(t-1) + f_1(L)\bar{I}(t-1) + f_M(L)M(t) + f_{U_s}(L)U_s(t) + f_\psi(L)\psi(t) + f_\rho(L)\rho(t) \tag{28}$$

and

$$I(t) = g_0 + d_L L(t-1) + d_I I(t-1) + g_L(L)\bar{L}(t-1) + g_I(L)\bar{I}(t-1)$$

$$+ g_M(L)M(t) + g_{u_s}(L)U_s(t) + g_\psi(L)\psi(t)$$

$$+ g_\rho(L)\rho(t) \tag{29}$$

and multiply (28) and (29) by n, we find that the actual laws of motion for $\bar{L}(t)$ and $\bar{I}(t)$ are

$$\bar{L}(t) = nf_0 + [C_L + nf_L(L)]\bar{L}(t-1) + [C_I + nf_I(L)]\bar{I}(t-1)$$

$$+ nf_M(L)M(t) + nf_{U_s}(L)U_s(t) + nf_\psi(L)\psi(t)$$

$$+ nf_\rho(L)\rho(t)$$

and

$$\bar{I}(t) = ng_0 + [d_L + ng_L(L)]\bar{L}(t-1) + [d_I + ng_I(L)]\bar{I}(t-1)$$

$$+ ng_M(L)M(t) + ng_{U_s}(L)U_s(t) + ng_\psi(L)\psi(t)$$

$$+ ng_\rho(L)\rho(t)$$

Because the rational equilibrium requires that the actual laws of motion be identically equal to the perceived laws of motion (26) and (27), we have that

$$nf_0 = F_0$$

$$ng_0 = G_0$$

$$nf_M(L) = F_M(L)$$

$$ng_M(L) = G_M(L)$$

$$nf_{U_s}(L) = F_{U_s}(L)$$

$$ng_{U_s}(L) = G_{U_s}(L)$$

$$nf_{\psi}(L) = F_{\psi}(L)$$

$$ng_{\psi}(L) = G_{\psi}(L)$$

$$nf_{\rho}(L) = F_{\rho}(L)$$

$$ng_{\rho}(L) = G_{\rho}(L)$$

$$C_L + nf_L(L) = F_L(L)$$

$$C_I + nf_I(L) = F_I(L)$$

$$d_L + ng_L(L) = G_L(L)$$

$$d_I + ng_I(L) = G_I(L)$$

We note in passing that knowledge of the aggregate laws of motion of the system enables us to solve recursively for the decision rules of the representative firm.

As in Section II.1, we begin by solving the problem by initially assuming that there is no uncertainty in the system.

By substituting equation (20) into the Euler equations derived in Section II, (7) and (12), one obtains the matrix Euler equation

$$G_1 Y(t + 1) + G_0 Y(t) + G_{-1} Y(t - 1) = J(t)$$

where

$$Y(t) = \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$G_0 = \left[\begin{array}{c|c} -e(1+B)+d+a^2nA_1 & anA_1 \\ \hline nA_1a & -(1+B\sigma^2)(g+nA_1)-f \end{array}\right]$$

$$G_1 = \left[\begin{array}{c|c} eB & 0 \\ \hline -BanA_1\sigma & B\sigma(g+nA_1) \end{array}\right]$$

$$G_{-1} = \left[\begin{array}{c|c} e & -anA_1\sigma \\ \hline 0 & \sigma(g+nA_1) \end{array}\right]$$

and

$$J(t) = \left[\begin{array}{c} -aA_0 - aU_s(t) + \omega(t) + d\rho(t) \\ \hline (1-B\sigma)A_0 + U_s(t) - B\sigma U_s(t+1) + f\psi(t) - Bb\sigma \end{array}\right]$$

Now it turns out that for our problem it is not necessary to calculate the functions f and g that determine the decision rules of the firm. Instead, we calculate the equilibrium laws of motion for the industry directly by utilizing the Lucas, Prescott method [20] of calculating a rational expectations equilibrium. In order to do this we pose the integrability question: for what optimum problem are the pair of Euler equations (30) and (31) the first-order necessary conditions?

Now the area under the demand curve for the consumption good is

$$\int_0^{\bar{S}(t)} [A_0 - A_1X(t) + U_s(t)]dX = [A_0 + U_s(t)]\bar{S}(t) - \frac{A_1}{2}\bar{S}^2(t)$$

Recalling that $\bar{S}(t) = n[aL(t) - I(t) + \sigma I(t-1)]$, we have that

$$\int_0^{\bar{S}(t)} [A_0 - A_1X(t) + U_s(t)]dX = n[A_0 + U_s(t)][aL(t) - I(t) - \sigma I(t-1)] - \frac{A_1}{2} n^2[aL(t) - I(t) + \sigma I(t-1)]^2 \qquad (32)$$

Now consider the following <u>social planning problem</u> which consists of maximizing the expected discounted area under the demand curve for final consumption of the good minus the total social costs of production and maintaining and changing inventory levels. The maximization is over contingency plans setting $\bar{L}(t) = nL(t)$ and $\bar{I}(t) = nI(t)$ as linear functions of the social planner's information set $\bar{\Omega}_t$ to be specified shortly.

Maximize

$$\begin{aligned}\lim_{N\to\infty} E_0 \sum_{t=0}^{\infty} B^t\{&n[A_0 + U_s(t)][aL(t) - I(t) + \sigma I(t-1)] \\ &- \frac{A_1n^2}{2}[aL(t) - I(t) + \sigma I(t-1)]^2 - \omega(t)nL(t) \\ &- \frac{dn}{2}[L(t) + \rho(t)]^2 - \frac{en}{2}[L(t) - L(t-1)]^2 + nb\sigma I(t-1) \\ &- \frac{fn}{2}[I(t) + \psi(t)]^2 + nb\sigma I(t-1) - \frac{gn}{2}[I(t) - \sigma I(t-1)]^2\end{aligned}$$

subject to I_{-1} and L_{-1} given and

$$\alpha(L)U_s(t) = V^s(t) \qquad (22)$$

$$\delta_\psi(L)\ (t) = U^\psi(t) \qquad (23)$$

$$\delta_\rho(L)\ (t) = U^\rho(t) \qquad (24)$$

$$\zeta(L)M(t) = V^M(t) \qquad (25)$$

The information set $\bar{\Omega}_t$ consists of at least $\{L(t - 1), I(t - 1), \bar{\omega}(t), \bar{U}_s(t), \bar{\rho}(t), \bar{\psi}(t)\}$. The parameters of the stochastic processes (22), (23), (24), and (25) are known with certainty by the social planner.

The Euler equations for the certainty case are

$$neBL(t + 1) - n[e(1 + B) + d + a^2nA_1]L(t) + neL(t - 1)$$
$$= -naA_0 - anU_s(t) + n\omega(t) + nd\rho(t) \qquad (33)$$

and

$$nB\sigma(g + nA_1)I(t + 1) - n[g(1 + B\sigma^2) + f + nA_1(1 + B\sigma^2)]I(t)$$
$$+ n\sigma(g + nA_1)I(t - 1) - B\sigma n^2A_1aL(t + 1) + n^2A_1aL(t)$$
$$= n(1 - B\sigma)A_0 + nU_s(t) - nB\sigma U_s(t + 1) + nf\psi(t) - nBb\sigma \qquad (34)$$

If one multiplies the Euler equations of the competitive firm (30) and (31) by n and uses the fact that $\bar{L}(t + 1) = nL(t + 1)$ and $\bar{I}(t + 1) = nI(t + 1)$, it becomes apparent that the two sets of Euler equations {(30), (31)} and {(33), (34)} are identical.[11] Hence, solving (33) and (34) yields the rational expectations competitive equilibrium laws of motion for $\bar{L}(t)$ and $\bar{I}(t)$, i.e., solving (33) and (34) yields an explicit solution for the parameters of

$$\bar{L}(t) = F[\bar{M}(t), \bar{U}_s(t), \bar{\psi}(t), \bar{\rho}(t), \bar{L}(t - 1), \bar{I}(t - 1), 1] \qquad (26)$$

and

$$\bar{I}(t) = G[\bar{M}(t), \bar{U}_s(t), \bar{\psi}(t), \bar{\rho}(t), \bar{L}(t - 1), \bar{I}(t - 1), 1] \qquad (27)^{12}$$

Upon consideration of equations (33) and (34), it is clear that in equilibrium, the decomposition theorem fails at the industry level. The equilibrium laws of motion for $\bar{L}(t)$ and $\bar{I}(t)$ will in general depend on each other in a direct way and not only through the exogenous stochastic processes. To the extent that the aggregate stock of inventories and the employment of labor Granger cause stochastic processes such as $\{P(t)\}$ which directly influence agents' objective functions, optimizing agents will use knowledge of $\bar{I}(t)$ and $\bar{L}(t)$ to help predict future values of such Granger-caused processes. This implies that, in a rational expectations equilibrium, those variables will be included in firms' decision rules and in the aggregate laws of motion for the system.

The above issues are related to a more general point brought up by both Theil [36] and Hay [11] who note that the form of the laws of motion of a system or a regression equation are sensitive, in a non-trivial way, to the level of aggregation. To the extent that the existence of aggregation problems presupposes the existence of more than one decision maker in the economic environment, the laws of motion for a multi-agent system must refelct the fact that the rewards of any one agent depend on the decisions of all of the agents in the system. In our context, the exact type of equilibrium is of considerable importance in determining both the qualitative and quantitative characteristics of the system. Because of this, we feel that the aggregation issues in our setup are more fundamentally equilibrium issues. While one might believe that individual

firms regard prices as an exogenous stochastic process, the usefulness of a modelling stragegy that views prices as being exogenous to the actions of all of the firms in the industry is more suspect. In moving from the firm level we are forced to view the equilibrium price sequence as being endogenous to the model with the result that the decomposition theorem fails.

At a different level, the results illustrate a general point made by Sargent [31], Lucas and Sargent [21] and Sims [34]. As opposed to many standard Keynesian macroeconometric models where identification of structural parameters is achieved by means of a priori exclusion restrictions, rational expectations and dynamic economic theory tend to work against the usual identification conditions of the exclusion variety. However, such models do supply cross-equation restrictions between the parameters of the laws of motion of the variables that are controllable by agents and the parameters of the exogenous stochastic processes.

We write the system of Euler equations (33) and (34) as

$$[G_1 L^{-1} + G_0 + BG_1^T L]Y(t) = E_t J(t) \tag{35}$$

where G_0, G_1, G_{-1}, $Y(t)$ and $J(t)$ are defined as before.

The Euler equations (35) can be solved subject to the initial conditions Y_{-1} and the terminal conditions formed by taking limits as $t \to \infty$ in (35) by the following procedure developed in Hansen and Sargent [10], who show that it is possible to factor

$$G(Z) = [G_1 Z^{-1} + G_0 + BG_1^T Z]$$

so that

$$G(Z) = C(BZ^{-1})^T C(Z) \qquad (36)$$

where C(Z) is a first order (2x2) matrix polynomial in nonnegative powers of Z, $C(Z) = C_0 + C_1 Z$.

Furthermore, the roots of det $C(Z) = 0$ are larger than $\sqrt{B}$ in modulus. The factorization (36) is unique up to post multiplication of C(L) by an orthogonal matrix.[13]

Using the factorization we write our Euler equations as

$$C(BL^{-1})^T C(L)Y(t) = E_t J(t)$$

The solution of this equation that satisfies the transversality conditions is

$$C(L)Y(t) = [C(BL^{-1})^T]^{-1} E_t J(t)$$

or

$$C_0 Y(t) + C_1 Y(t-1) = [C_0^T + C_1^T BL^{-1}]^{-1} E_t J(t)$$

or

$$Y(t) + C_0^{-1} C_1 Y(t-1) = [C_0^T C_0 + C_1^T C_0 BL^{-1}]^{-1} E_t J(t) \qquad (37)$$

which is the nonrealizable solution for the equilibrium laws of motion for the rational expectations competitive equilibrium.

The realizable solution is given by expressing the right-hand side of (37) as functions of current (time t) and past values of the variables of the model. Efficient methods for calculating the feedback polynomal of (37) as well as analytical expressions for the feedforward part exist.

Calculating the feedback part of (37) essentially involves formulating the social planning problem as an infinite time optimal linear regulator problem and solving the algebraic Riccati matrix equations by numerical methods. This formulation is discussed in Appendix A. Algorithms for computing the feedforward part of (37) are discussed in detail in Hansen and Sargent [10].

The resulting laws of motion for the competitive equilibrium will be of the form

$$Y(t) = -C_0^{-1}C_1Y(t-1) + \bar{J}(t) \tag{38}$$

where $\bar{J}(t)$ involves max $\{r_{u_s} - 1, r_\psi - 1, r_\rho - 1, q - 1\}$ lagged values of $J(t)$.

The solution to the problem will then be of the form

$$\begin{aligned}\bar{L}(t) = F_0 + F_L\bar{L}(t-1) + F_I\bar{I}(t-1) + F_M(L)M(t) + F_{u_s}(L)U_s(t) \\ + F_\psi(L)\psi(t) + F_\rho(L)\rho(t)\end{aligned} \tag{39}$$

and

$$\begin{aligned}\bar{I}(t) = G_0 + G_L\bar{L}(t-1) + G_I\bar{I}(t-1) + G_M(L)M(t) + G_{u_s}(L)U_s(t) \\ + G_\psi(L)\psi(t) + G_\rho(L)\rho(t)\end{aligned} \tag{40}$$

where

$$F_M(L) = \sum_{J=0}^{q-1} F_{M_J} L^J \quad \text{and } M_J \text{ is a } p \times p \text{ matrix } \forall J$$

$$G_M(L) = \sum_{J=0}^{q-1} G_{M_J} L^J \quad \text{and } G_J \text{ is a } p \times p \text{ matrix } \forall J$$

$$F_{u_s}(L) = \sum_{J=0}^{r_{u_s}-1} F_{u_{s_J}} L^J \qquad F_{u_{s_J}} \text{ is a scalar } \forall J$$

$$G_{u_s}(L) = \sum_{J=0}^{r_{u_s}-1} G_{u_{s_J}} L^J \qquad G_{u_{s_J}} \text{ is a scalar } \forall J$$

$$F_{\psi}(L) = \sum_{J=0}^{r_{\psi}-1} F_{\psi_J} L^J \qquad F_{\psi_J} \text{ is a scalar } \forall J$$

$$G_{\psi}(L) = \sum_{J=0}^{G_{\psi}-1} G_{\psi_J} L^J \qquad G_{\psi_J} \text{ is a scalar } \forall J$$

$$F_{\rho}(L) = \sum_{J=0}^{r_{\rho}-1} F_{\rho_J} L^J \qquad F_{\rho_J} \text{ is a scalar } \forall J$$

$$G_{\rho}(L) = \sum_{J=0}^{r_{\rho}-1} G_{\rho_J} L^J \qquad G_{\rho_J} \text{ is a scalar } \forall J$$

F_0, F_L, F_I, G_0, G_L and G_I are scalars

As in Section II.1, equations (39) and (40) are exact relationships. On the hypothesis that the econometrician does not observe current and past values of $\psi(t)$ or $\rho(t)$, the distributed lags in $\psi(t)$ and $\rho(t)$ become error terms from the econometrician's point of view.

Proposition

For the linear quadratic model under consideration in which production and inventory costs are additively separable, the rational expectations equilibrium laws of motion for labor and inventories decompose, $F_I = G_L = 0$, if and only if the elasticity of consumer demand for industry output

$$\frac{\bar{S}(t)}{P(t)} \frac{\partial \bar{S}(t)}{\partial P(t)} = -\infty.$$

Proof

$$P(t) = A_0 - A_1 \bar{S}(t) + U_s(t)$$

Hence, for finite P(t)

$$\frac{P(t)}{\bar{S}(t)} \frac{\partial \bar{S}(t)}{\partial P(t)} = -\infty \text{ if and only if } A_1 = 0.$$

If $A_1 = 0$, the social planning problem (33) can be written as the sum of two separate problems:

$$\text{(I)} \quad \operatorname*{Max} \lim_{N\to\infty} \sum_{t=0}^{N} \{B^t[n(A_0 + U_s(t))][aL(t)] - \omega(t)nL(t) - \frac{dn}{2}[L(t) + \rho(t)]^2 - \frac{en}{2}[L(t+1) - L(t)]^2\}$$

subject to L_{-1} given and

$$\zeta(L)M(t) = V^M(t) \qquad (25)$$

$$\alpha(L)U_s(t) = V^s(t) \qquad (22)$$

$$\delta_\rho(L)\rho(t) = U^\rho(t) \qquad (24)$$

and

$$\text{(II)} \quad \operatorname*{Max} \lim_{N\to\infty} \sum_{t=0}^{N} B^t\{n[A_0 + U_s(t)][-I(t) + \sigma I(t-1)] - \frac{fn}{2}[I(t) + \psi(t)]^2 + nb\sigma I(t-1) - \frac{gn}{2}[I(t+1) - \sigma I(t)]^2\}$$

subject to I_{-1} given and

$$\alpha(L)U_s(t) = V^s(t). \qquad (22)$$

$$\delta_\psi(L)\ (t) = U^\psi(t) \qquad (23)$$

and

$$\zeta(L)M(t) = V^M(t) \qquad (25)$$

To prove the converse, we note that a sufficient condition for non-decomposition is that the matrices G_0, G_1 and G_{-1} be non-diagonal, which is true if and only if $A_1 \neq 0$. Hence, a necessary condition for decomposition is that $A_1 = 0$.

Essentially, the above proposition says that if the equilibrium price seqeunce is not Granger caused by the aggregate level of inventories, it conveys no useful information to agents who are making production decisions. As such there is no reason for those agents to predict future inventory stocks. Consequently inventories will not influence production decisions. Similarly, production decisions will not influence inventory decisions.

In the absence of such restrictions, the econometric model becomes

$$\bar{L}(t) = F_0 + F_L\bar{L}(t-1) + F_I\bar{I}(t-1) + F_M(L)M(t) + F_{u_s}(L)U_s(t) + \Sigma_L(t) \tag{41}$$

$$\bar{I}(t) = G_0 + G_L\bar{L}(t-1) + G_I\bar{I}(t-1) + G_M(L)M(t) + G_{u_s}(L)U_s(t) + \Sigma_I(t) \tag{42}$$

$$\alpha(L)U_s(t) = V^s(t) \tag{22}$$

and

$$\zeta(L)M(t) = V^M(t) \tag{25}$$

As in Section II.1, estimation is subject to cross-equation restrictions. In particular, the model imposes restrictions between the parameters of (41) and (42) and the free parameters of (22) and (25).

II.3. The n-Plant Monopolist

The result obtained in Section II.2, namely that in a rational expectations competitive equilibrium the feedback part of the closed-loop system for inventories and employment decomposes if and only if the demand for industry output is infinitely elastic, irrespective of the decomposability of the decision rules of individual economic agents in the system, points out the importance of generalizing single-agent decision theory to market contexts and concepts of equilibrium. As noted, many authors have attempted to bypass the difficulties of modelling a perfectly competitive industry by considering monopolistic industries. In this section we conisder an n-plant monopolist in order to evaluate that strategy and examine the sensitivity of equilibrium solutions to industry structures in the linear quadratic framework.

As in Section II.2, demand is given by

$$P(t) = A_0 - A_1\bar{S}(t) + U_s(t),\ A_0,\ A_1 > 0 \tag{20}$$

$$\bar{S}(t) = n[aL(t) - I(t) + \sigma I(t-1)]$$

or

$$P(t) = A_0 - A_1 \sum_{i=1}^{n} [aL_i(t) - I^i(t) + \sigma I^i(t-1)] + U_s(t)$$

The problem of the n-plant monopolist is to choose a linear contingency plan for setting $L^i(t)$ and $I^i(t)$, $i = 1, 2, \ldots, n$, as functions of the information available at time t, to maximize

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{[A_0 + U_s(t)] \sum_{i=1}^{n} [aL^i(t) - I^i(t) + \sigma I^i(t-1)]$$

$$- A_1[\sum_{i=1}^{n} [aL^i(t) - I^i(t) + \sigma I^i(t-1)]]^2 - \omega(t) \sum_{n=1}^{n} L^i(t)$$

$$- \frac{d}{2} \sum_{i=1}^{n} [L^i(t) + \rho(t)]^2 - \frac{e}{2} \sum_{i=1}^{n} [L^i(t) - L^i(t-1)]^2$$

$$- \frac{f}{2} \sum_{i=1}^{n} [I^i(t) + \psi(t)]^2 + b\sigma \sum_{i=1}^{n} I^i(t-1)$$

$$- \frac{g}{2} \sum_{i=1}^{n} [I^i(t) - \sigma I^i(t-1)]^2\} \qquad (43')$$

subject to I^i_{-1} and L^i_{-1} given, $i = 1, 2, \ldots, n$ and

$$\alpha(L)U_s(t) = V^s(t) \qquad (22)$$

$$\delta^i_\psi(L)\psi^i(t) = U^\psi_i(t) \qquad (23)$$

$$\delta^i_\rho(L)\rho^i(t) = U^\rho_i(t) \qquad (24)$$

and

$$\zeta(L)M(t) = V^M(t) \qquad (25)$$

The monopolist's information set Ω_t consists of at least $\{L^1(t-1), \ldots L^n(t-1), I^1(t-1), \ldots I^n(t-1), \bar{\psi}^1(t), \ldots \bar{\psi}^n(t), \bar{\rho}^1(t), \ldots \bar{\rho}^n(t), \bar{U}_s(t), \bar{M}(t)\}$. The parameters of the stochastic processes (22) - (25) are known with certainty by the monopolist.

In order to facilitate comparison of the solution with that of the competitive equilibrium case where all firms were assumed to be identical, we assume that the monopolists' n plants are identical.

We therefore write (43') as

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{n[A_0 + U_s(t)][aL(t) - I(t) + \sigma I(t-1)]$$

$$- A_1 n^2[aL(t) - I(t) + \sigma I(t-1)]^2 - \frac{dn}{2}[L(t) + \rho(t)]^2$$

$$- \omega(t)nL(t) - \frac{en}{2}[L(t) - L(t-1)]^2$$

$$- \frac{nf}{2}[I(t) + \psi(t)]^2 + nb\sigma I(t-1)$$

$$- \frac{ng}{2}[I(t) - \sigma I(t-1)]^2 \qquad (43)$$

and

$$L_{-1}^J = L_{-1}^i,\ I_{-1}^J = I_{-1}^i \quad \forall i,\ J = 1, \ldots, n$$

and

$$\rho^i(t) = \rho^J(t),\ \psi^i(t) = \psi^J(t) \quad \forall t \text{ and } \forall i,\ J = 1, \ldots, n$$

If one compares (33), the social planner's problem, and (43), the monopolist's problem, one can see that the only difference between the two are the terms

$$\frac{-A_1 n^2}{2}[aL(t) - I(t) + \sigma I(t-1)]^2$$

versus

$$-A_1 n^2[aL(t) - I(t) + \sigma I(t-1)]$$

or

(33) $\frac{-A_1}{2} n^2 S^2(t)$

versus

(43) $A_1 n^2 S^2(t)$

Hence, the monopolist incurs a larger penalty in his objective function for S(t). This is consistent with the standard static microeconomic result that a competitive industry produces more than a monopolistic

industry, which, in this dynamic context, can be translated to read that, other things equal, a competitive industry sells more output than a monopolistic industry.

A different way to view the above result is by considering investment in inventories of finished goods for a given level of aggregate employment. Since

$$S(t) = aL(t) - \Delta I(t) = aL(t) - I(t) + \sigma I(t - 1), \ S(t) > 0$$

and

$$\bar{S}^{C.E.}(t) > \bar{S}^{M}(t)$$

where $\bar{S}^{C.E.}(t)$ = total sales at time t of the competitive industry,

$\bar{S}^{M}(t)$ = total sales at time t of the monopoly industry

and by hypothesis

$$\bar{L}^{C.E.}(t) = \bar{L}^{M}(t)$$

where

$\bar{L}^{C.E.}(t)$ = the aggregate level of employment at t in the competitive industry

and

$\bar{L}^{M}(t)$ = the aggregate level of employment in the monopoly industry.

Assume

$$\Delta\bar{I}^{C.E.}(t) > 0, \ \Delta I^{C.E.}(t) > 0$$

then

$$a\bar{L}^{C.E.}(t) - \Delta\bar{I}^{C.E.}(t) > a\bar{L}^{M}(t) - \Delta I^{M}(t)$$

or

$$\Delta I^{M}(t) > \Delta I^{C.E.}(t)$$

If

$$\Delta \bar{I}^{C.E.}(t) < 0, \ \Delta \bar{I}^{M}(t) < 0$$

then

$$-\Delta \bar{I}^{C.E.}(t) > -\Delta \bar{I}^{M}(t)$$

or

$$|\Delta \bar{I}^{C.E.}(t)| > |\Delta \bar{I}^{M}(t)|$$

Threfore, if inventory investment is positive, the amount of inventory investment will be larger in the monopoly industry. However, when inventory investment is negative, the amount of disinvestment will be larger in the competitive industry.

II.4 The Nash Equilibrium

In Sections II.2 and II.3 we examined our problem in the contexts of perfect competition and monopolies. Here we analyze these interactions by defining a dynamic differential game and solving for the Nash equilibrium.

Again,

$$P(t) = A_0 - A_1\bar{S}(t) + U_s(t) \quad A_0, A_1 > 0 \tag{20}$$

$$\bar{S}(t) = \sum_{i=1}^{n} S^i(t)$$

or

$$P(t) = A_0 - A_1 \sum_{i=1}^{n} [aL^i(t) - I^i(t) + \sigma I^i(t-1)] + U_s(t)$$

Because we seek a Nash equilibrium we assume that agent i assumes that agent J's ($i \neq J$) decision rules are invariant to his choice of decision rules. Hence, the ith firm maximizes

$$\lim_{T\to\infty} E_0 \sum_{t=1}^{T} B^t\{[A_0 + U_s(t)][aL^i(t) - I^i(t) + \sigma I^i(t-1)]$$

$$- A_1[aL^i(t) - I^i(t) + \sigma I^i(t-1)]$$

$$\cdot \left[\sum_{J=1}^{n} (aL^J(t) - I^J(t) + \sigma I^J(t-1))\right]$$

$$- \frac{d}{2}[L^i(t) + \rho^i(t)]^2 - \frac{e}{2}[L^i(t) - L^i(t-1)]^2$$

$$- \omega(t)L^i(t) - \frac{f}{2}[I^i(t) + \psi^i(t)]^2 + b\sigma I^i(t-1)$$

$$- \frac{g}{2}[I^i(t) - I^i(t-1)]^2$$

subject to I^J_{-1}, L^J_{-1}, $J = 1, \ldots, n$ given and

$$\alpha(L)U_s(t) = V^s(t) \qquad (22)$$

$$\delta^i_\psi(L)\psi^i(t) = U^\psi_i(t) \qquad (23)$$

$$\delta^i_\rho(L)\rho^i(t) = U^\rho_i(t) \qquad (24)$$

and

$$\zeta(L)M(t) = V^M(t) \qquad (25)$$

The ith firm's information set Ω^i_t consists of at least $\{L^1(t-1), \ldots L^N(t-1), I^1(t-1), \ldots I^n(t-1), \bar{\psi}^1(t), \ldots \bar{\psi}^n(t), \bar{\rho}^1(t), \ldots \bar{\rho}^n(t), \bar{U}_s(t), \bar{M}(t)\}$. The parameters of the stochastic processes of (22) - (25) are known with certainty by the ith firm.

The first-order necessary conditions to problem (44) are:

$$\begin{aligned}
&BeL^i(t+1) - [B(1+e) + d + 2A_1a^2]L^i(t) + eL^i(t-1) \\
&\qquad + 2A_1aI^i(t) - 2A_1a\sigma I^i(t-1) = d\rho(t) + \omega(t) \\
&\qquad - a[A_0 + U_s(t)] \\
&\qquad + A_1a[\sum_{\substack{J=1 \\ i\neq J}}^{n} (aL^J(t) - I^J(t) + \sigma I^J(t-1))]
\end{aligned} \tag{45}$$

$$\begin{aligned}
&[B\sigma(g+2A_1)]I^i(t+1) - [(1+B\sigma^2)(2A_1+g) + f]I^i(t) \\
&\qquad + [\sigma(g+2A_1)]I^i(t-1) - 2Ba_1\sigma aL^i(t+1) + 2A_1aL^i(t) \\
&\qquad = A_0(1-B\sigma) + U_s(t) - B\sigma U_s(t+1) + Bb\sigma + f\psi(t) \\
&\qquad - A_1[\sum_{J=1}^{n} aL^J(t) - I^J(t) + \sigma I^J(t-1)] \\
&\qquad + BA_1\sigma[\sum_{\substack{J=1 \\ i\neq J}}^{n} aL^J(t+1) - I^J(t+1) + \sigma I^J(t)]
\end{aligned} \tag{46}$$

Hence in matrix notation the Euler equations for the ith firm are

$$G_1^iY^i(t+1) + G_0^iY^i(t) + G_{-1}^iY^i(t-1) = J^i(t)$$

where

$$Y^i(t) = \begin{bmatrix} L^i(t) \\ I^i(t) \end{bmatrix}$$

and

$$G_0 = \left[\begin{array}{c|c} -[B(1+e) + d + 2A_1a^2] & 2A_1a \\ \hline 2A_1a & -[(1+B\sigma^2)(2A_1+g) + f] \end{array}\right]$$

$$G_1 = \left[\begin{array}{c|c} Be & 0 \\ \hline -2BA_1\sigma a & B\sigma(g + 2A_1 \end{array}\right]$$

$$G_{-1} = \left[\begin{array}{c|c} e & -2A_1 a\sigma \\ \hline 0 & \sigma(g + 2A_1) \end{array}\right]$$

and

$$J^i(t) = \left[\begin{array}{c} d\rho(t) + \omega(t) - a[A_0 + U_s(t)] \\ \quad + A_1 a[\sum_{\substack{J=1 \\ i \neq J}}^{n} aL^J(t) - I^J(t) + \sigma I^J(t-1)] \\ \hline A_0(1 - B\sigma) + U_s(t) - B\sigma U_s(t+1) + Bb\sigma + f\psi(t) \\ \quad - A_1[\sum_{\substack{J=1 \\ i \neq J}}^{n} aL^J(t) - I^J(t) + \sigma I^J(t-1)] \\ \quad + BA_1\sigma[\sum_{\substack{J=1 \\ i \neq J}}^{n} aL^J(t+1) - I^J(t+1) + \sigma I^J(t)] \end{array}\right]$$

Before proceeding we note that the problem as stated is not well posed. We must ascribe to agent i views about agent J's ($i \neq J$) decision rules so that all the constraints that agent i's maximization is subject to are fully specified.

In general if $V^i(\cdot)$ denotes the return function of agent i, X(t) denotes the state variables of the system, Z(t) represents the random variables in the system and $\bar{Y}_i(\cdot)$ denotes the decision rule of agent i, then a Rational Expectations Nash Equilibrium is n functions $\bar{Y}_1(\cdot)$, ..., $\bar{Y}_n(\cdot)$ such that

(a) $\bar{Y}_i(\cdot)$ maximizes

$$\lim_{T\to\infty} E_0 \sum_{t=0}^{T} B_i^t V_i[Z_t, X_t^i, Y_i(t), \underset{J\neq i}{\bar{Y}_J}(Z_t, X_t^J)]$$

subject to

$$X(t + 1) = \bar{g}[Z(t), X^i(t), Y_i(t), \bar{Y}_J(Z(t), X^J(t))]$$

and

$$\bar{Y}_J(Z^J(t), X^J(t))$$

is given for all $i \neq J$, and

(b) $\bar{Y}_1(\cdot)$, $\bar{Y}_2(\cdot)$, ..., $\bar{Y}_N(\cdot)$ imply the aggregate law of motion

$$X(t + 1) = \bar{g}[Z(t), X(t), \bar{Y}_1(Z(t), X_1(t), \ldots, \bar{Y}_n(t), X_n(t))]$$

In order to facilitate computation of this non-trival mapping and to ease comparison of the equilibrium with those obtained in Sections II.2 and II.3 we assume that the firms in this industry are identical in all respects. Hence, $Y_i(t) = Y_J(t)$ $\forall$i, J and t. Substituting $I^i(t) = I^J(t) = I(t)$ and $L^i(t) = I^J(t) = L(t)$ into the Euler equations of the representative firm, (45) and (46), and solving for I(t) and L(t) we obtain

$$BeL(t+1) - [B(1+e) + d + A_1a^2(n+1)]L(t) + eL(t-1)$$
$$+ A_1a(n+1)I(t) - A_1a\sigma(n+1)I(t-1)$$
$$= d\rho(t) + \omega(t) - a[A_0 + U_s(t)] \qquad (47)$$

$$[B\sigma g + BA_1\sigma(n+1)]I(t+1) - [(1+B\sigma^2)g + f + A_1(n+1)$$
$$+ B\sigma^2A_1(n+1)]I(t) + [\sigma g + (n+1)A_1\sigma]I(t-1)$$
$$- (n+1)BA_1\sigma aL(t+1) + A_1a(n+1)L(t)$$
$$= A_0(1 - B\sigma) + U_s(t) - B\sigma U_s(t+1) + Bb\sigma + f\psi(t) \qquad (48)$$

Hence the solution of the following matrix Euler equation determines the equilibrium law of motion for the representative firms' employment and inventories, $Y^T(t) = [L(t), I(t)]$

$$\tilde{G}_1Y(t+1) + \tilde{G}_0Y(t) + \tilde{G}_{-1}Y(t-1) = \tilde{J}(t) \quad .$$

where

$$\tilde{G}_1 = \left[\begin{array}{c|c} Be & 0 \\ \hline -(n+1)BA_1\sigma a & B\sigma g + BA_1\sigma(n+1) \end{array}\right]$$

$$\tilde{G}_0 = \left[\begin{array}{c|c} B(1+e) + d + A_1a^2(n+1) & A_1a(n+1) \\ \hline A_1a(n+1) & (1+B\sigma^2)g + f + A_1(n+1)(1+B\sigma^2) \end{array}\right]$$

$$\tilde{G}_{-1} = \left[\begin{array}{c|c} e & -A_1\sigma a(n+1) \\ \hline 0 & \sigma g + (n+1)A_1\sigma \end{array}\right]$$

$$\tilde{J}(t) = \left[\begin{array}{c} d\rho(t) + \omega(t) - a[A_0 + U_s(t) \\ \hline A_0(1 - B\sigma) + U_s(t) - B\sigma U_s(t+1) + f\psi(t) + Bb\sigma \end{array}\right]$$

As in the competitive equilibrium we find a social planning problem whose solution gives us the Nash equilibrium laws of motion for L(t) and I(t).

Consider the social planning problem. Maximize

$$\lim_{T\to\infty} E_0 n \sum_{t=0}^{T} B_n^t\{[A_0 + U_s(t)][aL(t) - I(t) + \sigma I(t-1)]$$

$$- \frac{A_1}{2}(n+1)[aL(t) - I(t) + \sigma I(t-1)]^2 - \omega(t)L(t)$$

$$- \frac{d}{2}[L(t) + \rho(t)]^2 - \frac{e}{2}[L(t) - L(t-1)]^2$$

$$- \frac{f}{2}[I(t) + \psi(t)]^2 + b\sigma I(t-1) - \frac{g}{2}[I(t) - I(t-1)]^2\} \tag{49}$$

subject to

$$\alpha(L)U_s(t) = V^s(t) \tag{22}$$

$$\delta_\psi(L)\psi(t) = U^\psi(t) \tag{23}$$

$$\delta_\rho(L)\rho(t) = U^\rho(t) \tag{24}$$

$$\zeta(L)M(t) = V^M(t) \tag{25}$$

The Euler equations for this problem are just (47) and (48). Hence, solving (49) for $L(t)$ and $I(t)$, and multiplying $L(t)$ and $I(t)$ by n, the number of firms in the industry, yields the Nash equilibrium for industry employment and inventories of finished goods. Numerical examples are presented in Appendix B.

Notice that if one compares the social planning problems (33), (43) and (49) corresponding to the different types of industry structures, they are identical except for the terms

$$-\frac{A_1}{2} N^2[aL(t) - I(t) + \sigma I(t-1)]^2 \qquad \text{(competitive equilibrium)}$$

versus

$$-A_1 N^2[aL(t) - I(t) + \sigma I(t-1)]^2 \qquad \text{(n-plant monopolist)}$$

versus

$$-\frac{A_1}{2} N(N+1)[aL(t) - I(t) + \sigma I(t-1)]^2 \qquad \text{(Nash equilibrium)}$$

or

$$-\frac{A_1}{2} N^2 S^2(t) \qquad (33)$$

$$-A_1 N^2 S^2(t) \qquad (43)$$

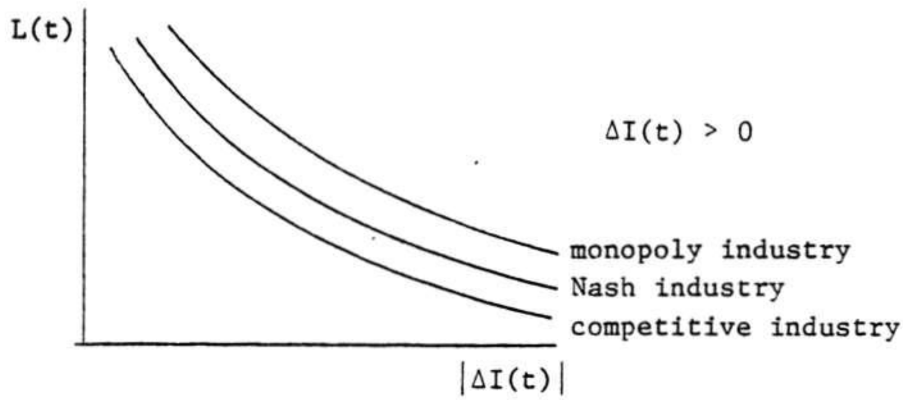
and

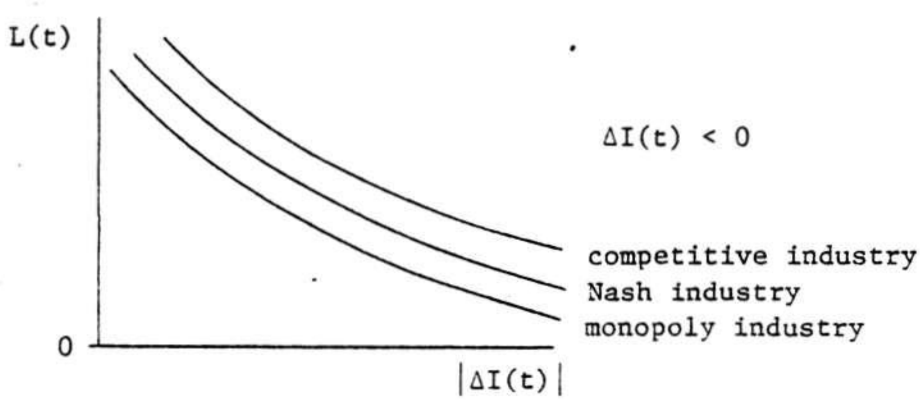
$$-\frac{A_1}{2} N(N+1) S^2(t) \qquad (49)$$

Hence, for $N > 1/2$, the monopolist incurs a larger penalty in his objective function for $S(t)$ than the Nash industry who in turn incurs larger penalties for $S(t)$ than the competitive industry. Hence, all other things equal, a competitive industry sells more than the Nash industry which seels more than the monopolistic industry, or

$$\bar{S}(t)^{c.e.} > \bar{S}(t)^{Nash} > \bar{S}(t)^{M}.$$

If as the data suggest, inventory and employment are negatively related, one could expect the following types of behavior.





Alternatively, one could say that, for a given amount of employment, the monopolist builds up inventory stocks more quickly, while the competitive industry depletes stocks of inventories more quickly.

III. Estimation

III.1 Exogeneity and Granger Causality

Recall that the system to be estimated is

$$\begin{bmatrix} \bar{L}(t) \\ \bar{I}(t) \end{bmatrix} = \begin{bmatrix} \bar{F}_L & \bar{F}_I \\ \bar{G}_L & \bar{G}_I \end{bmatrix} \begin{bmatrix} \bar{L}(t-1) \\ \bar{I}(t-1) \end{bmatrix} + \begin{bmatrix} F_M(L) & F_{u_s}(L) \\ G_M(L) & G_{u_s}(L) \end{bmatrix} \begin{bmatrix} M(t) \\ U_s(t) \end{bmatrix} + \begin{bmatrix} \Sigma_L(t) \\ \Sigma_I(t) \end{bmatrix} \tag{50}$$

$$\alpha(L)U_s(t) = V^S(t)$$

$$\zeta(L)M(t) = V^M(t)$$

and

$$\begin{bmatrix} \Sigma_L(t) \\ \Sigma_I(t) \end{bmatrix} = \begin{bmatrix} F_\psi(L) & F_\rho(L) \\ G_\psi(L) & G_\rho(L) \end{bmatrix} \begin{bmatrix} \psi(t) \\ \rho(t) \end{bmatrix}$$

where

$\alpha(L) = \sum_{J=0}^{r_s} \alpha_J L^J$, α_J is a scalar for all J

$\zeta(L) = \sum_{J=0}^{q} \zeta_J L^J$, ζ_J is a (PxP) matrix for all J,

M(t) is a (Px1) vector for all t,

$F_M(L) = \sum_{J=0}^{q-1} F_{M_J} L^J$, F_{M_J} is a (1xP) vector for all J,

$F_{u_s}(L) = \sum_{J=0}^{r_s-1} F_{u_{s_J}} L^J$, $F_{u_{s_J}}$ is a scalar for all J,

$G_M(L) = \sum_{J=0}^{q-1} G_{M_J} L^J$, G_{M_J} is a (1xP) vector for all J,

$$G_{u_s}(L) = \sum_{J=0}^{r_s-1} G_{u_{s_J}} L^J \ , \ G_{u_{s_J}} \text{ is a scalar for all } J,$$

$$F_\psi(L) = \sum_{J=0}^{r_\psi-1} F_{\psi_J} L^J \ , \ F_{\psi_J} \text{ is a scalar for all } J,$$

$$F_\rho(L) = \sum_{J=0}^{r_\rho-1} F_{\rho_J} L^J \ , \ F_{\rho_J} \text{ is a scalar for all } J,$$

$$G_\psi(L) = \sum_{J=0}^{r_\psi-1} G_{\psi_J} L^J \ , \ G_{\psi_J} \text{ is a scalar for all } J,$$

and

$$G_\rho(L) = \sum_{J=0}^{r-1} G_{\rho_J} L^J \ , \ G_{\rho_J} \text{ is a scalar for all } J.$$

We rewrite (50) as

$$Y(t) = \Pi_1 Y(t-1) + \Pi_2(L)Z(t) + \Sigma(t) \tag{50}$$

$$\zeta(L)M(t) = V^M(t)$$

$$\alpha(L)U_s(t) = V^s(t)$$

$$\Sigma(t) = \Pi_3(L)\phi(t)$$

where

$$\Sigma^T(t) = [\Sigma_L(t), \Sigma_I(t)]$$

$$\phi^T(t) = [\psi(t), \rho(t)]$$

$$\Pi_1 = \begin{bmatrix} \bar{F}_L & \bar{F}_I \\ \bar{G}_L & \bar{G}_I \end{bmatrix}, \quad Z(t) = \begin{bmatrix} M(t) \\ U_s(t) \end{bmatrix}$$

is a (P+1)x1 vector for all t.

$$\Pi_2(L) = \begin{bmatrix} F_M(L) & F_{u_s}(L) \\ G_M(L) & G_{u_s}(L) \end{bmatrix}$$

is a 2x(P+1) matrix polynomial in L of order $\max\{q - 1, r_s - 1\}$ and

$$\Pi_3(L) = \begin{bmatrix} F_\psi(L) & F_\rho(L) \\ G_\psi(L) & G_\rho(L) \end{bmatrix}$$

is a (2x2) matrix polynomial in L of order $\max\{r_\psi - 1, r_\rho - 1\}$.

From (50) we have that

$$Y(t) = [I - \Pi_1 L]^{-1}\Pi_2(L)\ Z(t) + [I - \Pi_1 L]^{-1}\Pi_3(L)\phi(t) \tag{51}$$

Notice that

$$M(t) = \zeta^{-1}(L)V^M(t)\ ,\quad U_s(t) = \alpha^{-1}(L)V^s(t)\ ,\ \psi(t) = \delta_\psi^{-1}(L)U^\psi(t)$$

and

$$\rho(t) = \delta_\rho^{-1}(L)U^\rho(t).$$

Hence, we define

$$\delta(L) = \begin{bmatrix} \delta_\psi^{-1}(L) & 0 \\ 0 & \delta_\rho^{-1}(L) \end{bmatrix} \text{ and } U^T(t) = [U^\psi(t),\ U^\rho(t)]$$

we have that

$$\phi(t) = \delta(L)U(t) \tag{52}$$

where $\delta(L)$ is a (2x2) matrix polynomial in L, and $\phi(t)$ is a (2x1) vector for all t.

Similarly, if we let

$$\Phi(L) = \begin{bmatrix} \zeta^{-1}(L) & 0 \\ 0 & \alpha^{-1}(L) \end{bmatrix}$$

which is a (P+1)x(P+1) matrix polynomial in L, and

$$V(t) = \begin{bmatrix} V^M(t) \\ V^s(t) \end{bmatrix}$$

which is a (P+1)x1 vector for all t, we have

$$Z(t) = \Phi(L)V(t) \qquad (53)$$

which implies that (51) may be written as

$$Y(t) = [I - \Pi_1 L]^{-1} \Pi_2(L)\Phi(L)V(t) + [I - \Pi_1 L]^{-1}\Pi_3(L)\delta(L)U(t) \qquad (54)$$

Recall that $[V(t), U(t)]$ are the innovations in the joint $[Z(t), \phi(t)]$ process. In particular,

$$V(t) = Z(t) - E[Z(t)\ Z(t - 1),\ Z(t - 2),\ \ldots,\ \phi(t - 1),\ \phi(t - 2)\ldots]$$

and

$$U(t) = \phi(t) - E[\phi(t)\ Z(t - 1),\ Z(t - 2),\ \ldots,\ \phi(t - 1),\ \phi(t - 2)\ldots]$$

Hence, $V(t)$ and $U(t)$ are serially uncorrelated and $EU(t)V^T(t - J) = 0\ \forall J \neq 0$. However, we cannot rule out contemporaneous correlation between $V(t)$ and $U(t)$.

We now introduce a new process

$$C(t) = U(t) - \lambda V(t)$$

where $C(t)$ is a (2x1) vector for all t, λ is a 2x(P+1) matrix, and $EC(t)V^T(t) = 0$. This defines $\lambda V(t)$ as the linear least squares predictor of $U(t)$ given $V(t)$. Notice that if $U(t)$ and $V(t)$ are uncorrelated, λ is equal to $\underset{2x(P+1)}{[0]}$ and $C(t) = U(t)$.

Substituting into (54),

$$Y(t) = [I - \Pi_1 L]^{-1}[\Pi_2(L)\Phi(L) + \Pi_3(L)\delta(L)\lambda]V(t)$$

$$+ [I - \Pi_1 L]^{-1}\Pi_3(L)\delta(L)C(t) \tag{55}$$

Define the new disturbance term

$$d(t) = [I - \Pi_1 L]^{-1}\Pi_3(L)\delta(L)C(t) \tag{56}$$

Because $\Pi_3(Z)$ may not be invertible, $C(t)$ may not be fundamental for $d(t)$. However, using the transformation with Blashke factors described in Hansen and Sargent [9], there exists a $\underset{2x2}{\Theta}(L)$ such that

$$\Pi_3(Z)\Pi_3(Z^{-1}) = \Theta(Z)\Theta(Z^{-1}) \quad \text{for } |Z| = 1$$

where $\Theta(Z)$ does not have any zeroes inside the unit circle.[14] We may therefore define a new serially uncorrelated process $V^d(t)$ which is fundamental for $d(t)$, i.e., lies in the space spanned by square-summable linear combinations of current and lagged d's,

$$d(t) = [I - \Pi_1 L]^{-1}\Theta(L)\delta(L)V^d(t) \tag{57}$$

Since $E[C(t)V^T(t - J)] = [0]$ for all J, we have that $Ed(t)V^T(t - J)] = [0]$ for all J which implies that $EV^d(t)V^T(t - J) = [0]$ for all J.

Substituting (57) and (56) into (55), we obtain

$$\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} \begin{bmatrix} [I-\Pi_1(L)]^{-1}\phi(L)\delta(L) & [I-\Pi_1(L)]^{-1}[\Pi_2(L)\Phi(L)+\Pi_3(L)\delta(L)\lambda] \\ 0 & \Phi(L) \end{bmatrix} \begin{bmatrix} V^d(t) \\ V(t) \end{bmatrix} \tag{58}$$

which expresses [Y(t), Z(t)] as one-sided square summable moving averages of the serially uncorrelated processes $V^d(t)$ and V(t) which satisfy $EV(t)V^{dT}(t - J) = [0]$. Because the joint $[V^d(t), V(t)]$ process is fundamental for the joint [Y(t), Z(t)] process, (58) is a Wold moving average representation of the joint [V(t), Z(t)] process.

Theorem I

Let {X(t), Y(t)} be a jointly covariance, stationary strictly indeterministic process with mean zero. Then {Y(t)} fails to Granger cause {X(t)} if and only if there exists a vector moving average representation

$$\begin{bmatrix} Y(t) \\ X(t) \end{bmatrix} = \begin{bmatrix} C^{11}(L) & C^{12}(L) \\ 0 & C^{22}(L) \end{bmatrix} \begin{bmatrix} \Sigma(t) \\ U(t) \end{bmatrix}$$

where $\Sigma(t)$ and U(t) are serially uncorrelated processes with means zero and $E\Sigma_t U_s = 0$ for all t and s, and where the one step ahead linear least squares prediction errors

$$X(t) - E(X(t)[X(t - 1), \ldots, Y(t - 1), \ldots])$$

and

$$Y(t) - E(Y(t)\ [X(t - 1), \ldots, Y(t - 1), \ldots])$$

are each linear combinations of $\Sigma(t)$ and U(t). Sims [35].

Theorem II

Y(t) can be expressed as a distributed lag of current and past X's with a disturbance process that is orthogonal to past, present, and future X's if and only if Y does not Granger cause X. Sims [35].

Hence, the triangular character of (58) and Theorem I imply that $Y(t)$ fails to Granger cause $Z(t)$. Therefore, by Theorem II there exists a representation of the form

$$Y(t) = n(L)Z(t) + F(t) \tag{59}$$

$$n(L) = \sum_{J=0}^{\infty} n_J L^J$$

where $F(t)$ is a covariance stationary process such that $EF(t)Z^T(t - J) = [0]$ for all J, which is to say that $Z(t)$ is strictly exogenous in (59). One candidate for the representation guaranteed by Sim's theorem is the closed loop system representing the rational expectations competitive equilibrium laws of motion, (54)

$$Y(t) = [I - \Pi_1 L]^{-1}\Pi_2(L)Z(t) + [I - \Pi_1 L]^{-1}\Pi_3(L)\delta(L)U(t) \tag{54}$$

However, (54) need not be that representation in which $Z(t)$ is strictly exogenous. Substituting $V(t) = \Phi^{-1}(L)Z(t)$ and $d(t) = [I - \Pi_1 L]^{-1}\Pi_3(L)\Theta(L)\delta(L)V^d(t)$ into (58) we have that

$$Y(t) = [I - \Pi_1 L]^{-1}[\Pi_2(L) + \Pi_3(L)\delta(L)\lambda\Phi^{-1}(L)]Z(t) + d(t) \tag{60}$$

Since $Ed(t)V^T(t - J) = 0$ for all J, $E[d(t)Z^T(t - J)] = 0$ for all J. Let $d(t) = F(t)$ and $n(L) = [I - \Pi_1 L]^{-1}[\Pi_2(L) + \Pi_3(L)\delta(L)\lambda\Phi^{-1}(L)]$; we see that (55) is the representation insured by Sims. Comparing (60) and (54) we see that (60) is the rational expectations equilibrium law of motion if and only if $\lambda = [0]$ which as we saw is equivalent to the condition that $V(t)$ and $U(t)$ are uncorrelated. Hence, the hypothesis of strict exogeneity of $Z(t)$ is equivalent to the hypothesis that $\lambda = [0]$ in the Wold moving average representation (58). It is important to notice that consistent

estimates of the model's parameters may be obtained by imposing only Granger non-causality of $Z(t)$ by $Y(t)$, leaving λ unrestricted. Hence, we may test the null hypothesis that $\lambda = [0]$ by re-estimating the model assuming $\lambda = [0]$. Under the null hypothesis that $\lambda = [0]$, the likelihood ratio statistic is asymptotically distributed with $2(P+1)$ degrees of freedom.

III.2 Estimation of the Model Parameters

To summarize, the system to be estimated is

$$Y(t) = [I - \Pi_1 L]^{-1}[\Pi_2(L) + \Pi_3(L)\delta(L)\lambda\Phi^{-1}(L)]Z(t) + [I - \Pi_1 L]^{-1}\Pi_3(L)\delta(L)C(t) \tag{61}$$

$$\Phi^{-1}(L)Z(t) = V(t)$$

$$EC(t)C^T(t - J) = 0, \quad EV(t)V^T(t - J) = 0 \text{ for } J \neq 0$$

and

$$EC(t)V^T(t - J) = 0 \text{ for all } t \text{ and } J.$$

The underlying parameters which are to be estimated are λ, the parameters of $\delta(L)$ and $\Phi^{-1}(L)$ as well as the other parameters appearing in the objective function of the social planner of Section II.2. For the sake of simplicity the constant term has been dropped.

As discussed in Hansen and Sargent [9], system (61) may be estimated via maximum likelihood with a normal density function. As Whittle [ ] points out, even if the underlying stochastic processes are not Gaussian, the resulting estimates will have the desired properties of maximum likelihood estimates: consistency and asymptotic efficiency.

Assume that we have a sample of size T for $[Y(t), Z(t)]$, $t = 1, \ldots, T$.

Let $\bar{Y}_T = [Y_1, \ldots, Y_T]'$ and $\bar{Z}_T = [Z_1', \ldots, Z_T']$. If we define

$$B(L) = \begin{bmatrix} [I - \Pi_1 L]^{-1}\phi(L)\delta(L) & [I - \Pi_1]^{-1}[\Pi_2(L)\phi(L) + \Pi_3(L)\delta(L)\lambda] \\ 0 & \Phi(L) \end{bmatrix}$$

we may write the moving average representation of our system as

$$\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} B_1(L) & B_2(L) \\ 0 & B_4(L) \end{bmatrix} \begin{bmatrix} V_t^d \\ V_t \end{bmatrix} \tag{62}$$

and

$$E \begin{bmatrix} V_t^d \\ \\ V_t \end{bmatrix} \begin{bmatrix} V_t^{d^T} & V_t^T \end{bmatrix} = \begin{bmatrix} D_{11} & 0 \\ \\ 0 & D_{22} \end{bmatrix} = D$$

where D_{11} is a (2x2) positive definite matrix, and D_{22} is a (P+1)x(P+1) positive definite matrix, and by construction, $EV_t^d V_t^T = 0$.

Let the covariance matrix of $(\bar{Y}_T', \bar{Z}_T)$ be

$$\Gamma_T = E \begin{bmatrix} \bar{Y}_T \\ \bar{Z}_T \end{bmatrix} \cdot [\bar{Y}_T' \quad \bar{Z}_T'] , \quad \Gamma_T \text{ is } T(P+2) \times T(P+2)$$

where the mean of $(\bar{Y}_T', Z_T')$ is zero because the means have been subtracted off.

If $(V_{(t)}^d, V_{(t)}')$ are jointly normal, then the normal log likelihood function for $(\bar{Y}_T, Z_T')$ is

$$\overset{*}{L}_T = -\frac{1}{2} T(P+2)\log 2\Pi - \frac{1}{2}\log|\Gamma_T| - \frac{1}{2}[\bar{Y}_T' \; Z_T']\Gamma_T^{-1} \begin{bmatrix} \bar{Y}_T \\ \bar{Z}_T \end{bmatrix} \tag{58}$$

It is immediately evident that directly maximizing $\overset{*}{L}_T$ is difficult

computationally because Γ_T is highly nonlinear in the structural parameters of the model. Furthermore, Γ_T must be inverted each time $\overset{*}{L}_T$ is evaluated.

Instead, we consider estimators which have, asymptotically, the same properties as the maximum likelihood estimator in the strict Gausian case. In particular, Hannan [ ] establishes the normality of the estimates assuming that $(V^d(t), V'(t))$ are serially independent and identically distributed with finite variance.

Because of the large dimensionality of the matrix Γ_T, Hannan [ ] suggests an approximation for the term

$$[\bar{Y}_T', Z_T']\Gamma_T^{-1}\begin{bmatrix}\bar{Y}_T \\ \bar{Z}_T\end{bmatrix}.$$

We know that the theoretical spectral density matrix of the $[Y_t, Z_t]$ process is given by

$$S(\omega) = B(e^{-i\omega})DB(e^{i\omega})'$$

where ' denotes complex conjugation as well as transposition. Let $I(\omega_J)$ be the periodogram at frequency $\omega_J = 2\Pi J/T$.

Making the substitutions

$$[\bar{Y}_T', Z_T']\Gamma_T^{-1}\begin{bmatrix}Y_T \\ Z_T\end{bmatrix} \simeq \sum_{J=1}^{T} \text{trace } [S(\omega_J)^{-1}I(\omega_J)]$$

and

$$\log (\det \Gamma_T) \simeq \sum_{J=1}^{T} \log \{\det [S(\omega_J)]\}$$

and substituting into the log likelihood function (58), we have

$$\bar{L}_T^* = -\frac{1}{2} T(P+2) \log 2\Pi - \frac{1}{2} \sum_{J=1}^{T} \log \{\det [S(\omega_J)]\}$$
$$- \frac{1}{2} \sum_{J=1}^{T} \text{trace } [S(\omega_J)^{-1}I(\omega_s)] \qquad (59)$$

which is to be minimized over the free parameters of the spectral density matrix, i.e., B(L) and D, by means of one of several acceptable iterative methods.

IV. Conclusion

The business cycle is defined by particular patterns of serial correlation and cross-serial correlation among sets of variables. Among the salient characteristics of the data are the cyclical relationships between inventories, employment, and prices. In this paper we have constructed an equilibrium model of an industry which produces and sells storable output. The firms in this industry are perfect competitors who know the actual probability distribution of the stochast processes which directly or indirectly affect the expected present value of profits. Moreover, the decisions of firms reflect the optimal use of such knowledge. The equilibrium laws of motion for production, inventories, and prices are capable of generating the patterns of own-serial correlation and cross-serial correlation which permeate the actual data. Despite the assumptions of perfect competition and cleared markets, inventories of finished goods and employment are correlated, both contemporaneously and over time. Both the data and the examples suggest that inventories of finished goods respond negatively to shocks in aggregate demand and that employment responds positively to shocks in demand.

An important conclusion of the paper is that considerations of equilibrium often force the modeler to reject well accepted exclusion restrictions. In the context of this paper, the fact that inventories of

finished goods influence prices forces the modeler to attribute to agents well defined views about the behavior of the aggregate shock of inventories over time. When these expectations are rational, the decomposition theorem, which is essentially an exclusion restriction, fails.

APPENDIX A

In this section we show that the problem of Section II.2 can be formulated as a well behaved infinite time optimal linear regulator problem. As the theorems used are fairly standard, they are stated without proof. The interested reader is referred to the Kwakernaak and Sivan [13] or Sargent [32].

<u>Theorem I</u>

Consider the optimal linear regulator problem:

Max

$$E_{t_0}\{\sum_{t=t_0}^{t_1-1}(X^1 + RX_t + X_t^T QX_t + V_t^T QV_t + X_{t_1}^T P_{t_1} X_{t_1} \tag{1}$$

subject to X_0 given, $R \leq 0$, $Q < 0$, $P_{t_1} \leq 0$ where

$$X(t+1) = AX(t) + BV(t) + \Sigma(t+1) \tag{2}$$

where $\Sigma(t+1)$ is a vector white noise with $E\Sigma\Sigma^T = \bar{V}_t$.

The maximization of (1) is carried out over the parameters of feedback rules F_t in

$$V_t = -F_t X_t,\ t = t_0,\ t_0 + 1,\ \ldots,\ t_1 - 1$$

For an arbitrary $\{F_t\}_{t=t_0}^{t_1-1}$ sequence, the value of the criterion (1) is $X_{t_0}^T P_{t_0} X_{t_0} + d_{t_0}$ where P_{t_0} and d_{t_0} are the solutions to the difference equations

$$P_{t-1} = (A - BF_{t-1})^T P_t + (A - BF_{t-1}) + R + F^T_{t-1} Q F_{t-1}$$

$$d_{t-1} = d_t + \mathrm{tr}\, \bar{V}_t P_t$$

with terminal conditions P_{t_1} and $d_{t_1} = 0$ given. The optimal choice of the F_t's is given by

$$F^0_t = (B^T P^0_{t+1} B + Q)^{-1} B^T P^0_{t+1} A \qquad t = t_0, t_0 + 1, \ldots, t_1 - 1 \quad (3)$$

where P^0_t is the solution of the matrix Riccati difference equation

$$P^0_{t-1} = A^T P^0_t A + R - A^T P^0_t B [B^T P^0_t B + Q]^{-1} B^T P^0_t A \quad (4)$$

with terminal condition P_{t_1} given.

The matrices P^0_t are negative semi-definite. When the optimal feedbacks are used, the criterion function attains the value

$$X^T_{t_0} P^0_{t_0} X_{t_0} + d_{t_0}$$

where $P^0_{t_0}$ maximizes $P^0_{t_0}$ with respect to F_t, $t = t_0, \ldots, t - 1$ over the class of all matrices P^0_t that satisfy (4) with terminal condition P_{t_1}. given.

Theorem II

Consider the linear optimal regulator problem where (A, B) is stabilizable. Without loss of generality, let (A, B) be in controllability canonical form, so that

$$\begin{bmatrix} X_1(t+1) \\ X_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} V(t)$$

where (A_{11}, B_1) is controllable and A_{22} is a stable matrix. Write the criterion function in the form

$$E_{t_0} \sum_{t=t_0}^{t_1-1} \{[X_1^T(t) \quad X_2^T(t)] \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \nu^T(t)Q\nu(t)$$

$$+ \begin{bmatrix} X_1(t_1) \\ X_2(t_1) \end{bmatrix}^T P(t_1) \begin{bmatrix} X_1(t_1) \\ X_2(t_1) \end{bmatrix}$$

where $P(t_1)$ and R are negative semi-definite and Q is negative definite. Let the rank of the negative semi-definite matrix R_{11} be $r \leq m$ where m is the dimension of the controllable sulspace. Let $-R_{11} = G^TG$ where G is r x m. Assume that the pair (A_{11}, G) is detectable. Then

(i) iterations on the matrix Riccati equation converge to a unique negative semi-definite matrix that is independent of the terminal matrix $P(t_1)$.

(ii) The optimal closed loop system matrix

$$(A - BF) = \begin{bmatrix} A_{11} - B_1F_1 & A_{12} - B_2F_2 \\ 0 & A_{22} \end{bmatrix}$$

is stable.

Theorem III

Consider the optimal linear regulator problem described in Theorem III. Assume that (A, B) is stabilizable, $P_{11}(t_1)$ and R_{11} are negative semi-definite and (A_{11}, G) is detectable where $G^TG = -R_{11}$. Otherwise,

R_{12}, R_{22}, $P_{12}(t_1)$ and $P_{22}(t_1)$ are arbitrary sequences. Then

(i) interations on the matrix Riccati equation converge to a unique matrix independent of $P(t_1)$. The limit matrix $\lim_{t\to-\infty} P(t)$ is <u>not</u> necessarily negative semi-definite, although $\lim_{t\to-\infty} P_{11}(t)$ is negative semi-definite.

(ii) The optimal stationary closed loop system matrix $(A - BF)$ is stable.

(iii) Partitioning $F = (F_1 F_2)$ conformably with the partitioning of X, F_1 is independent of R_{12} and R_{22}, while F_2 is independent of R_{22}.

Now consider the social planning problem of Section II.2, with $B = 1$.[15]

Maximize

$$\begin{aligned}\lim_{N\to\infty} E_0 \sum_{t=0}^{N} \{ & n[A_0 + U_s(t)][aL(t) - I(t) + \sigma I(t-1)] \\ & - \frac{A_1 n^2}{2}[aL(t) - I(t) + \sigma I(t-1)]^2 \\ & - \omega(t) n L(t) - \frac{dn}{2}(L(t) + \rho(t))^2 \\ & - \frac{fn}{2}[L(t+1) - \psi(t)]^2 + b\sigma I(t-1) \\ & - \frac{gn}{2}[I(t+1) - \sigma I(t)]^2\end{aligned}$$

subject to I_{-1} and L_{-1} given and

$$\alpha(L) U_s(t) = V^s(t) \qquad (22)$$

$$\delta_{\psi}(L)\psi(t) = U^{\psi}(t) \qquad (23)$$

$$\delta_{\rho}(L)\rho(t) = U^{\rho}(t) \qquad (24)$$

$$\zeta(L)M(t) = V^{M}(t) \qquad (25)$$

and the information set $\bar{\Omega}(t)$ consists of at least $\{L(t), I(t), \bar{\psi}(t), \bar{U}_s(t), \bar{\rho}(t)\}$. The parameters of the stochastic processes of (22), (23), (24) and (25) are known with certainty by the social planner.

We now write the above problem as a linear regulator problem, when

$$\delta_{\psi}(L) = 1 - \delta_{\psi_1} L, \; \omega(t) = V_{\omega}\omega(t-1) + V_{\omega}(t)$$

$$\delta_{\rho}(L) = 1 - \delta_{\rho_1} L, \; \alpha(L) = 1 - \alpha_1 L$$

define

$$X^T(t+1) = [L(t+1), I(t+1), I(t), 1, \omega(t+1), \rho(t+1), \psi(t+1), U_s(t+1)]$$

$$V^T(t) = [L(t+1) - L(t), I(t+1) - \sigma I(t)]$$

$$\underset{3x3}{A_{11}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underset{5x3}{A_{21}} = A_{21}^T = [0]$$

$$\underset{5x3}{A_{22}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & V_{\omega} & 0 & 0 & 0 \\ 0 & 0 & \delta_{\rho_1} & 0 & 0 \\ 0 & 0 & 0 & \delta_{\psi_1} & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 \end{bmatrix}$$

$$\underset{3x2}{B_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \underset{5x2}{B_2} = [0]$$

Define

$$\underset{3x3}{R_{11}} = \begin{bmatrix} \frac{-A_1na^2 - d}{2} & \frac{A_1na}{2} & \frac{-A_1na\sigma}{2} \\ \frac{A_1na}{2} & \frac{-A_1n - f}{2} & \frac{A_1n\sigma}{2} \\ \frac{-A_1na\sigma}{2} & \frac{A_1n\sigma}{2} & \frac{A_1n\sigma}{2} \end{bmatrix}$$

$$R_{12} = R_{21}^T = \begin{bmatrix} \frac{aA_0}{2} & -\frac{1}{2} & -\frac{d}{2} & 0 & \frac{a}{2} \\ \frac{-A_0}{2} & 0 & 0 & -\frac{f}{2} & -\frac{1}{2} \\ \frac{\sigma}{2}[A_0+b] & 0 & 0 & 0 & \frac{\sigma}{2} \end{bmatrix}$$

$$R_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{f}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{2x2}{\bar{Q}} = \begin{bmatrix} -\frac{e}{2} & 0 \\ 0 & -\frac{g}{2} \end{bmatrix}$$

Notice that the system will then be in the form

Maximize

$$\lim_{T\to\infty} E_0 \sum_{t=0}^{T} B^t[X^T(t)RX(t) + V^T(t - 1)QV(t - 1)] \qquad (A.1)$$

subject to

$$X(t + 1) = AX(t) + BV(t) + \Sigma(t + 1)$$

But because V(-1) is inherited by the planner and does not affect his decision rule, we may rewrite (A.1) as

Maxmize

$$\lim_{T\to\infty} E_0 \sum_{t=0}^{\infty} B^t[X^T(t)RX(t) + V^T(t)QV(t)]$$

subject to

$$X(t + 1) = AX(t) + BV(t) + \Sigma(t + 1)$$

where

$$Q = B\bar{Q}$$

Since the rank of

$$[B_1, A_{11}B_1, A_{11}^2B] = 3$$

the pair (A_{11}, B_1) is completely controllable and our system is in controllability cannonical form.

$\underset{2x2}{Q}$ is trivially negative definite. Consider R_{11} which must be negative semi-definite.

$$-R_{11} = \begin{bmatrix} A_1na^2 + d & -A_1na & A_1na\sigma \\ -A_1na & n(A_1 + f) & -A_1n\sigma \\ A_1na\sigma & -A_1n\sigma & A_1n\sigma^2 \end{bmatrix}$$

A necessary and sufficient condition for $-R_{11}$ to be positive semi-definite is that all of the principal minors be nonnegative.

$$A_1na^2 + d \geq 0$$

$$n(A_1na^2 + d)(A_1 + f) - A_1^2n^2a^2 \geq 0$$

or

$$A_1n^2a^2f + ndA_1 + dA_1f \geq 0$$

and

$$|-R_{11}| \geq 0$$

or

$$(A_1na^2 + d)[A_1n^2\sigma^2f] + (A_1na)(-A_1^2n^2a\sigma^2 + A_1^2n^2a\sigma^2]$$
$$+ A_1na\sigma[A_1^2n^2a\sigma - A_1n^2a\sigma(A_1 + f)]$$

or

$$(A_1na^2 + d)[A_1n^2\sigma^2f] - A_1na\sigma[A_1n^2a\sigma f]$$

$$= A_1^2n^3a^2\sigma^2f + A_1n^2\sigma^2fd - A_1^2n^3a^2\sigma^2f$$

$$= A_1n^2\sigma^2fd \geq 0$$

Hence, R_{11} is negative semi-definite.

Now the assumptions on our stochastic processes imply that (A_1B)

is stabilizable. Note that given that A_{11}, B_1) is completely controllable and that $0 < B < 1$, this only requires that the eigenvalues of A_{22} are less that $1/\sqrt{B}$ in modulus which can be confirmed by inspection, at least in the simple specification of the stochastic processes used in this appendix. Therefore, all that remains to show in order to apply Theorem III is that (A_{11}, G) is detectable.

Consider the system

$$\begin{bmatrix} X_1(t+1) \\ X_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} V(t) \qquad \text{(a)}$$

$$Y(t) = [G \quad 0] \underset{nx1}{X}(t)$$

where $-R_{11} = G^T G$.

The dual of this system is

$$\bar{X}(t+1) = \begin{bmatrix} A_{11}^T & 0 \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \end{bmatrix} + \begin{bmatrix} G^T \\ 0 \end{bmatrix} U(t) \qquad \text{(b)}$$

$$\bar{Y}(t) = [B_1^T \quad 0] X(t)$$

<u>Theorem</u>

Kwakernaak and Sivan [p. 466]. The system (a) is detectable if and only if its dual, the system (b) is stabilizable.

In system (b) $A_{12}^T = 0$. Therefore,

$$\begin{bmatrix} \bar{X}_1(t+1) \\ X_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11}^T & 0 \\ 0 & A_{11}^T \end{bmatrix} \begin{bmatrix} \bar{X}_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} G^T \\ 0 \end{bmatrix} U(t)$$

Since the eigenvalues of A_{22}^T are all less than $1/\sqrt{B}$ in modulus, we need only establish that the pair (A_{11}^T, G^T) is completely controllable, which is true if and only if the rank of $[G^T, A_{11}^T G^T (A_{11}^2)^T G^T] = 3$, which can easily be established.

Appendix B - Numerical Examples

We now consider various numerical examples of the competitive equilibrium emerging from the model of Section II.2.

As we indicated in Section II, in order to compute the equilibrium laws of motion for $[\vec{L}(t) = \vec{I}(t)] = [nL(t), nI(t)]$, we solve the following social planning problem; maximize

$$E_0 \sum_{t=0}^{\infty} B^t [n[A_0+U_s(t)][aL(t)-I(t+1)+\sigma I(t)] \tag{33}$$

$$- \frac{A_1 n^2}{2}[aL(t)-I(t+1)+\sigma I(t)]^2$$

$$- w(t)nL(t) - \frac{dn}{2}[L(t)+\rho(t)]^2 - \frac{en}{2}[L(t) - L(t-1)]^2 + nb\sigma I(t-1)$$

$$- \frac{fn}{2}[I(t)+\Psi(t)]^2 - \frac{gn}{2}[I(t) - \sigma I(t-1)]^2$$

subject to

$$\alpha(L)U_s(t) = V^s(t) \tag{22}$$

$$\delta_\Psi(L)\Psi(t) = U^s(t) \tag{23}$$

$$\delta_\rho(L)\rho(L) = U^\rho(t) \tag{24}$$

$$\zeta(L)M(t) = V^m(t) \tag{25}$$

where $w(t)$ is the first element of the (Px1) vector process $M(t)$. At time t the social planner knows $[L(t-1),\ldots,I(t-1)]$ and $\{M(t),M(t-1),\ldots,U_s(t),U_s(t-1),\ldots,\rho(t),\rho(t-1),\ldots,\Psi(t),\Psi(t-1),\ldots\}$, as well as the parameters of (22), (23), (24), (25), and those of the demand schedule, A_0 and A_1.

The maximization is over linear contingency plans for setting $[L(t), I(t)]$ as functions of the elements of the planner's information set at time t. Given the optimal decision rule for $[L(t+1),I(t+1)]$, the equilibrium laws of

motion for $[\bar{L}(t), \bar{I}(t)]$ is obtained by using $[\bar{L}(t), \bar{I}(t) = n[L(t), I(t)]$.
For all the examples, solutions are arrived at by iterating on the matrix Riccati difference equation until the convergence criterion is fulfilled. In particular, successive iterations were performed on the feedback law

$$F_t = \beta[Q+\beta B^1 P_t B]^{-1} B^1 P_t A$$

where iterations on the matrix Riccati difference equation

$$P_{t+1} = \beta A^1 P_t A + R - \beta^2 A^1 P_t B[Q+\beta B^1 P_t B]^{-1} B^1 P_t A$$

were started from $P_0 = 0$. Convergence was claimed when the norm, defined as the maximum absolute value over the elements of $(F_{t+1}-F_t)$, was less than 10^{-5}.

For all of the examples we assumed $\beta = .7$, $n = 1{,}000$, $a = .8$, $d = 1.5$, $e = 1.4$, $f = 1.2$. We also set $A_0 = 0$, which is equivalent to setting constant terms in the equilibrium $[\bar{L}(t), \bar{I}(t)]$ equal to zero. As such, the equilibrium describes variables measured in deviations from the mean.

(I) $A_1 = .010$, $g = 1.3$, $\sigma = 0.0$ or a depreciation rate of 100 percent.

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .33750 & 0 \\ .20678 & 0 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} -.09568 & -.17696 & -.05481 & .01021 \\ -.05862 & -.10842 & -.07034 & -.01672 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(II) $A_1 = .010$, $g = 1.3$, $\sigma = .2$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .33550 & -.02521 \\ .20511 & .16661 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.09442 & -.17432 & -.05696 & .01031 \\ -.05732 & -.10565 & -.07345 & -.01666 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(III) $A_1 = .010,\ g = 1.3,\ \sigma = .5$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .33603 & -.05845 \\ .20700 & +.42522 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.09361 & -.17230 & -.06218 & .01019 \\ -.05661 & -.10371 & -.08064 & -.01692 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(IV) $A_1 = .010,\ g = 1.3,\ \sigma = .9$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .36381 & -.02064 \\ .24159 & .87532 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.10340 & -.19117 & -.08547 & .00811 \\ -.06817 & -.12573 & -.10970 & -.01955 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(V) $A_1 = .001,\ g = 1.3,\ \sigma = 0.0$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .41677 & 0 \\ .08218 & 0 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.12199 & -.22734 & -.02233 & .04130 \\ -.02405 & -.04483 & -.12271 & -.06580 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(VI) $A_1 = .001,\ g = 1.3,\ \sigma = .2$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .41615 & -.01037 \\ .08021 & .13823 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} -.12167 & -.22668 & -.02267 & .04169 \\ -.02290 & -.04246 & -.12689 & -.06474 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(VII) $A_1 = .001,\ g = 1.3,\ \sigma = .5$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .41483 & -.02911 \\ .07540 & .33669 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} -.12102 & -.22535 & -.02660 & .04253 \\ -.02038 & -.03725 & -.13095 & -.06184 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(VIII) $A_1 = .001,\ g = 1.3,\ \sigma = .9$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .41219 & -.06546 \\ .06450 & .55782 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} -.11980 & -.22280 & -.02073 & .04428 \\ -.01507 & -.02624 & -.12885 & -.05484 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(IX) $A_1 = .010,\ g = .3,\ \sigma = .9$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .42445 & -.02823 \\ .31959 & .86650 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.12307 & -.22848 & -.11481 & .00343 \\ -.09188 & -.17011 & -.14633 & -.02571 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(X) $A_1 = .010,\ g = .1,\ \sigma = .9$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .44195 & -.03050 \\ .34190 & .86389 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.12888 & -.23995 & -.12337 & .00210 \\ -.09882 & -.18317 & -.15692 & -.02746 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

(XI) $A_1 = .010,\ g = .00001,\ \sigma = .9$

$$\begin{bmatrix} L(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} .45171 & -.03178 \\ .35431 & .86242 \end{bmatrix} \begin{bmatrix} L(t) \\ I(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -.13215 & -.24580 & -.12817 & .00135 \\ -.10272 & -.19051 & -.16284 & -.02843 \end{bmatrix} \begin{bmatrix} w(t) \\ \rho(t) \\ \Psi(t) \\ U_s(t) \end{bmatrix}$$

While we report the following regularities observed in the specific examples calculated, no claims are made for their robustness in the face of alternative specifications for equations (22), (23), (24), (25), and the other parameters of the model.

$$\frac{\partial L(t+1)}{\partial L(t)} > 0,\ \frac{\partial L(t+1)}{\partial I(t)} < 0,\ \frac{\partial I(t+1)}{\partial L(t)} > 0,\ \frac{\partial I(t+1)}{\partial L(t)} > 0,\ \frac{\partial I(t+1)}{\partial I(t)} > 0$$

$$\frac{\partial I(t+1)}{\partial \Psi(t)} < 0,\ \frac{\partial I(t+1)}{\partial U_s(t)} < 0,\ \frac{\partial L(t+1)}{\partial \rho(t)} < 0,\ \frac{\partial L(t+1)}{\partial U_s(t)} > 0$$

$$\frac{\partial L(t+1)}{\partial w(t)} < 0, \quad \frac{\partial L(t+1)}{\partial \Psi(t)} < 0, \quad \frac{\partial I(t+1)}{\partial \rho(t)} < 0, \quad \frac{\partial I(t+1)}{\partial w(t)} < 0$$

$$\frac{\partial[\partial I(t+1)/\partial I(t)]}{\partial \sigma} > 0, \quad \frac{\partial[|\partial L(t+1)/\partial w(t)|]}{\partial \sigma} < 0$$

$$\frac{\partial[|\partial I(t+1)/\partial U_s(t)|]}{\partial \sigma} > 0$$

$$\frac{\partial[\partial L(t+1)/\partial L(t)]}{\partial A_1} < 0$$

$$\frac{\partial[|\partial L(t+1)/\partial I(t)|]}{\partial A_1} > 0, \quad \frac{\partial[\partial I(t+1)/\partial I(t)]}{\partial A_1} > 0$$

$$\frac{\partial[\partial I(t+1)/\partial L(t)]}{\partial A_1} > 0, \quad \frac{\partial[|\partial L(t+1)/\partial w(t)|]}{\partial A_1} < 0$$

$$\frac{\partial[|\partial L(t+1)/\partial U_s(t)|]}{\partial A_1} < 0, \quad \frac{\partial[|\partial I(t+1)/\partial w(t)|]}{\partial A_1} > 0$$

$$\frac{\partial[|\partial I(t+1)/\partial U_s(t)|]}{\partial A_1} < 0$$

$$\frac{\partial[\partial L(t+1)/\partial L(t)]}{\partial g} < 0, \quad \frac{\partial[|\partial L(t+1)/\partial I(t)|]}{\partial g} < 0$$

$$\frac{\partial[\partial I(t+1)/\partial L(t)]}{\partial g} < 0, \quad \frac{\partial[\partial L(t+1)/\partial I(t)]}{\partial g} > 0$$

FOOTNOTES

[1]For a more detailed discussion of multi-agent statistical decision theory, see Prescott and Townsend [28].

[2]Agents care about these stochastic processes in as much as they influence the value of their objective functions.

[3]In Hansen and Sargent [9], the error term used for estimation purposes emerges from shocks to the productivities of factors of production which are observed by agents but not by the econometrician. A different model of the error term emerges from shocks to the costs of adjustment.

[4]An (m x 1) vector white noise $\Sigma(t)$ is fundamental for an (m x 1) vector process $\ell(t)$ if the vector of one-step-ahead linear least squares errors in predicting $\ell(t)$ from past ℓ's can be written as a linear combination of the $\Sigma(t)$'s.

[5]For most of this paper, this assumption can be relaxed and replaced by assuming $\rho(t)$, $\psi(t)$ and $Z(t)$ are of mean exponential order less than $1/\sqrt{B}$. A stochastic process $\{Y(t)\}$ is said to be of mean exponential order less than $1/B$ if for some $K > 0$ and $1 < X < 1/B$, $E_t Y(t + J) < K(X)^{t+J}$ for all t and J greater than zero.

[6]Both problems I and II are of the form of those considered in Sargent [32] and Hansen and Sargent [9], the latter of which deals with a general class of dynamic stochastic optimization problems.

[7]The above is modelled after a similar proof in Sargent [32], pp. 197-198.

[8]Despite the presence of L^{-1} in (10), the expression for

$$\sum_{J=0}^{\infty} \delta^J E + Z(t + J + 1)$$ depends only on information that the agent has

at time t. This implies that in equation (11) all of the right-hand-side expression are known to the firm at time t.

[9]Hansen and Sargent discuss the role of Granger Causality and econometric exogeneity in the context of dynamic linear rational expectations models more fully in Hansen and Sargent [9] and [10]. For a general discussion of Granger Causality in natural rate models, see Sargent [29].

10 A sufficient condition for the contingency plans to be linear is that the least squares predictions of the uncontrollable stochastic processes be linear functions of the elements of the representative firm's information set $\Omega(t)$.

11 It can also be shown that the transversality conditions of the two problems are the same.

12 As noted in the text, knowledge of these parameters enables us to work backwards and determine the parameters of the decision rule of the representative firm.

13 See Appendix A of Hansen and Sargent [10].

14 If $A_1, \ldots, Z_k$ are the zeroes of $\Pi(Z)$ that lie inside the unit circle, multiplying $\Pi(Z)$ by Blaschke factors we obtain

$$\theta(Z) = \Pi(Z)\,\frac{(1 - Z_1 Z) \ldots (1 - Z_k Z)}{(Z - Z_1) \ldots (Z - Z_k)}$$

15 Setting $B = 1$ involves no loss of generality since the discounted problem may always be formulated as an undiscounted problem. If the original system is

$$\text{Maximize} \lim_{N\to\infty} E_0 \sum_{t=0}^{N} B^t\{X^T(t)RX(t) + V^T(t)QV(t)\}$$

subject to

$$X(t + 1) = AX(t) + BV(t) + \varepsilon(t + 1)$$

define the transformed variables

$$\tilde{X}(t) = B^{t/2}X(t) \text{ and } \tilde{V}(t) = B^{t/2}V(t)$$

The discounted problem is then equivalent to the undiscounted linear regulator problem:

Maximize

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} \{\tilde{X}^T(t)R\tilde{X}(t) + \tilde{V}^T(t)Q\tilde{V}(t)$$

subject to

$$\tilde{X}(t + 1) = B^{1/2}A\tilde{X}(t) + B^{1/2}\tilde{V}(t) + B^{\frac{t+1}{2}}\varepsilon(t + 1)$$

by choice of control law of the form

$$\tilde{V}(t) = -\tilde{F}\tilde{X}(t)$$

from which the optimal control law of the original problem, $V(t) = \tilde{F}X(t)$ can be calculated.

[16] The constants will be equal to zero if the parameter A_0 of the industry demand curve is set equal to zero. The resulting equilibrium should then be thought of as describing variables measured in deviations from their means.

[17] Let $\overset{*}{\theta}$ be a (2 x 1) vector of true parameters. We start with a given point θ, known as the initial guess, and generate a sequence of points of θ_2, θ_3, ... which hopefully converges to the point $\overset{*}{\theta}$ at which the likelihood function $L(\theta)$ is minimum. If $L(\theta_{i+1}) < L(\theta_i)$ for all i, then the iterative method is said to be acceptable.

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