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ON EXISTENCE AND UNIQUENESS OF STATIONARY DISTRIBUTIONS WITHOUT CONTINUITY

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On Existence and Uniqueness of Stationary Distributions Without Continuity

Section 2. Propositions and Proofs

In this section we present the two main results of the paper and corollaries.

Theorem 1 establishes the existence of fixed points for monotone maps defined on a space of measures. Corollaries 1 and 2 are applications of Theorem 1 to Markov Processes. Theorem 2 gives sufficient conditions for the invariant measure guaranteed to exist by Theorem 1 to be unique.

The proof of Theorem 1 involves defining an antisymmetric order on the space of measures on a compact subset of R^n with minimal and maximal elements. The order is shown to have adequate continuity properties so that every linearly ordered subset has a sup. The proof of the theorem is then a direct application of the Knaster-Tarski fixed point theorem (see Dugundji, pg. 14).

<u>Definition:</u> for x,y members of R^n, let $x < y$ if $x_i < y_i$ $i = 1, \ldots, n$. Let S be a compact subset of R^n with minimal and maximal elements a, b, i.e., for all $x \in S$, $a < x < b$. A function $f: S \to R$ is said to be <u>monotone</u> if whenever $x < y$, $f(x) < f(y)$. For $E \subset S$, let $E_{>}$ $(E_{<}) = \{y \in S: y > x (y < x)$ for some $x \in E\}$.

<u>Lemma:</u> If E is a closed subset of S and G an open subset of S, then $E_{<}$ and $E_{>}$ are closed and $G_{<}$ and $G_{>}$ are open.

<u>pf</u>: Let $y_n \to y$ and $y_n \in E_>$. Then there exist $x_n \in E$ such that $x_n < y_n$. Since E is compact, $x_{n_k} \to x \in E$ for some subsequence. But by continuity of $<$, $x < y$ so $y \in E_>$.

Let $y \in G_<$. Then there exists an element $x \in G$ such that $y < x$. Let $U = (G-(x-y)) \cap S$. Then U is open, contains y and if $y' \in U$, then $y' = x'-(x-y)$ for some $x' \in G$ and since $(x-y) > 0$, $x' = y' + (x-y) > y'$. Thus $G_<$ is open.

Similar arguments can be used to show that $E_<$ is closed and $G_>$ open.

Let **M** be the class of bounded, monotone, measurable and nonnegative real valued functions on $(S,\mathbf{S})$, where **S** is the Borel σ-algebra of subsets of S.

Let $[a,b] \equiv \{x \in R^n\colon a<x<b\}$. Let $G \subset [a,b]$ be an open set (in the relative topology of $[a,b]$) with $G = G_<$. Let γ: $[a,b] \to [0,1]$ be a correspondence defined by:

$$\gamma(x) = \{\lambda \in [0,1]\colon \lambda a+(1-\lambda)x \in G^c\} \cup \{0\}.$$

Define ρ: $[a,b] \to [0,1]$ by $\rho(x) = \max\ \{\lambda\colon \lambda \in \gamma(x)\}$. γ is non-empty valued and G^c is compact. Hence ρ is well defined.

<u>Claim:</u> ρ is continuous.

<u>Proof:</u> It will suffice to show that γ is a continuous correspondence. The conclusion will follow from Berge (1959). Upper hemi-continuity can be easily checked noticing the graph of γ is closed. To see that γ is lower hemi-continuous, let V be an open set in $[0,1]$. Denote $g(\lambda,x) \equiv \lambda a + (1-\lambda)x$. The function g can

easily be checked to be continuous, non-increasing in λ and non-decreasing in x. Suppose $g(\lambda,x) \in G^c$ for some $\lambda \in V$. We will show that there exists an open set U in $[a,b]$ such that $x \in U$ and for $x' \in U$, $\gamma(x')\ \ V \neq \phi$. Without loss of generality we can assume $0 \notin V$. Then $(\lambda',\lambda) \in V$ for $\lambda' < \lambda$. Let $W = g((\lambda',\lambda)X\{x\})$. Since g is affine for fixed x, W is open in $[a,b]$. Also, $g(\cdot,x)$ is non-increasing and $G^c = (G^c)_>$ (as can be easily verified), so $W \subset G^c$. Since g is continuous, $g^{-1}(W)$ is open. Also $(\lambda',\lambda)\ X\{x\} \subset g^{-1}(W)$. Hence, there exists open sets U', U where $U' \times U \subset g^{-1}(W)$, $x \in U$ and $U' \cap (\lambda',\lambda) \neq \phi$. For any x' $\in U$, $g(U' \times \{x\}) \subset W \subset G^c$, and thus there exists some $\lambda \in V$ such that $g(\lambda,x') \in G^c$.

<u>Lemma</u>: ρ is increasing.

<u>pf</u>: Suppose $x > y$. It will suffice to show that $\gamma(y) \subset \gamma(x)$. Without loss of generality assume $x \notin G$ and $y \notin G$. Suppose $\lambda \in \gamma(y)$. Then $\lambda a + (1-\lambda)y \in G^c$. But since $y < x$, $\lambda a + (1-\lambda)y < \lambda a + (1-\lambda)x$ and hence $\lambda \in \gamma(x)$.

For E a closed subset of S abusing notation we will denote $\rho(E) = \min\ \{\rho(x) : x \in E\}$. Since E is compact and ρ is continuous, this is well defined. Let $C_M = \{f \in \mathbf{M}: f \text{ is continuous}\}$.

A misuse of notation!

<u>Proposition 1</u>: For any bounded measure μ on $(S,\mathbf{S})$, where $\mathbf{S}$ is the Borel σ-algebra of subsets of S, C_M is $L^1(\mu)$ dense in $\mathbf{M}$.

<u>pf</u>: <u>case 1</u>: Suppose g is a monotone indicator function. Then $g = \chi_A$ for some set A. Fix $\varepsilon > 0$. Since S is a metric space, μ is a regular measure. Thus there exist sets $O \subset S$ and $D \subset S$ closed

such that $C \subset A$, $D \subset A^c$ and $\mu(A \Delta C) + \mu(D \Delta A^c) < \varepsilon$. Let $E = C_>$. By previous lemma E is closed and that $E \cap D = \phi$ can be easily checked.

Since S is a normal space, there exist disjoint open subsets U, V of S such that $D \subset U$ and $E \subset V$. Let $G = U_<$ and $F = (V^c)_<$. Then G is open, F is closed and $G \subset F$. Furthermore, $F \cap E = \phi$ can be easily checked. Define $f\colon [a,b] \to [0,1]$ by

$$f(x) = \min \{1, \frac{\rho(x)}{\rho(E)}\}$$

We need to check that $\rho(E) > 0$. For this purpose, let $x \in E$. Since $E \cap F = \phi$, there exists an open set W such that $x \in W$ and $W \cap F = \phi$. Since g is continuous, $g^{-1}(W)$ is open and $(0,x) \in g^{-1}(W)$. But then there exists some $\lambda > 0$ such that $g\ (\lambda,x) \in g^{-1}(W) \subset F^c \subset G^c$. Hence, for $x \in E$, $\rho(x) > 0$. Since ρ is continuous and E compact, $\rho(E) > 0$ and the definition of f is justified.

For $x \in E$, $f(x) = 1$ and for $x \in G$, $f(x) = 0$. Thus $f|_{D \cup C} = g|_{D \cup C}$. Furthermore, f is continuous since ρ is continuous. We will now show that f is monotone. Choose $x \in S$, $y \in S$ such that $y < x$. If $y \in E$ then $x \in E$ and $f(x) = f(y)$. Also, if $x \in G$, $f(x) = f(y)$. If $x \in E$ or $y \in G$ then $f(x) > f(y)$. So assume y and x are not in $E \cup G$. Since $y < x$ and G satisfies assumptions of previous lemma, $\rho(y) < \rho(x)$, so $f(y) < f(x)$. The restriction of f to S, $f_S\colon S \to [0,1]$ is also continuous and monotone. Finally

$$\|f_s - g\|_{L^1(\mu)} = \int |f_s - g|\, d\mu < \int_{C \cup D} |f_s - g|\, d\mu$$

$$+ \int_{(C \cup D)^c} |f_s - g|\, d\mu$$

$$< \mu((C \cup D)c) < \varepsilon.$$

Case 2: Suppose g is a monotone simple function, i.e., $g(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x)$. Without loss of generality assume $c_i < c_j$ whenever $i < j$, and $A_i \cap A_j = \phi$ for $i \neq j$. Write

$$g(x) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{i} a_j \Big) \chi_{A_i}(x), \text{ where } a_j = c_j - c_{j-1} \text{ and } c_0 = 0.$$

Let $\varepsilon > 0$. For each $i = 1, \ldots, n$ let $g_i(x) = \chi_{\bigcup_{j>i} A_j}$. Let $y \varepsilon \bigcup_{j>i} A_j$ and $y < x$. Then $g(x) > g(y)$ and thus $y \varepsilon \bigcup_{j>i} A_j$. So the functions g_i are monotone. Also,

$$g = \sum_{i=1}^{n} \Big(\sum_{j=1}^{i} a_j \Big) \chi_{A_i} = \sum_{i=1}^{n} a_i \sum_{j>i} \chi_{A_j} = \sum_{i=1}^{n} a_i g_i .$$

By previous step, there exist continuous and monotone $f_i : S \to [0,1]$ with $\| f_i - g_i \|_{L_1(u)} < \frac{\varepsilon}{nc_n}$. Let $f = \sum a_i f_i$. Since $a_i > 0$ f_i is monotone, continuous and non-negative. Finally

$$\| f-g \|_{L_{1}(\mu)} = \| \sum a_i f_i - \sum a_i g_i \| = \| \sum a_i (f_i - g_i) \|$$

$$< \sum a_i \| f_i - g_i \| < nc_n \frac{\varepsilon}{nc_n} = \varepsilon .$$

Case 3: Suppose $g \; \varepsilon \; \mathbf{M}$. Then by the standard proof of denseness of simple functions in $L_1(\mu)$ we can approximate g by a monotone simple function. This, together with results for the previous case show that g can be approximated by a continuous, monotone and non-negative function $\blacksquare$ (1)

Let Λ be the space of bounded measures on S. For (μ, ν) in $\Lambda \times \Lambda$ we will say that $\mu > \nu$ or μ is stochastically greater than ν, if $\int f d\mu > \int f d\nu$ for all $f \; \varepsilon \; \mathbf{M}$.

Proposition 2: $\mu > \nu$ if an only if $\int f d\mu > \int f d\nu$ for all $f \; \varepsilon \; C_M$.

pf: Necessity is immediate by definition. Suppose $g \in M$. By proposition 1 there exists a sequence $\{f_n\}$, $f_n \in C_M$, with $\|f_n - g\|_{L^1(\mu+\nu)} \to 0$ as $n \to \infty$. Notice that since f_n are positive and bounded, $\|f_n - g\|_{L^1(\mu)} \to 0$ and $\|f_n - g\|_{L^1(\nu)} \to 0$ as $n \to \infty$.

Suppose $\int g d\mu < \int g d\nu$. Then there exists some N such that for $n > N$, $\int f_n d\mu < \int f_n d\nu$, contradicting the fact that $\int f d\mu \geq \int f d\nu$ for all $f \in C_M$.

Let Gr $(\geq) = \{(\mu,\nu) \in \Lambda \times \Lambda : \mu \geq \nu\}$, the graph of $\geq$ in $\Lambda \times \Lambda$.

Lemma: Gr $(\geq)$ is weak-* closed.

pf: Suppose $\mu_n \geq \nu_n$ and $\mu_n \to \mu$, $\nu_n \to \nu$. Let $f \in C_M$. Then $|\int f d\mu_n - \int f d\mu| \to 0$ and $|\int f d\nu_n - \int f d\nu| \to 0$. Since $\int f d\mu_n \geq \int f d\mu$ for all n, $\int f d\mu \geq \int f d\nu$. By previous lemma, $\mu \geq \nu$.

Lemma: $\geq$ is an anti-symmetric partial order.

pf: Reflexivity and transitivity are immediate. Hence we need to prove that if $\mu \geq \nu$ and $\nu \geq \mu$ then $\mu = \nu$. Notice that $\mu(S) = \nu(S) < \infty$. Without loss of generality, assume $\mu(s) = 1$. Let F_μ and F_ν be the corresponding distribution functions of μ and ν. Since $\mu \geq \nu$,

$$F_\mu(x) = 1 - \int \chi_{[y<x]^c} d\mu \leq 1 - \int \chi_{[y<x]^c} d\nu = F_\nu(x),$$

since $\chi_{[y<x]^c}$ is a monotone, bounded and non-negative function. Since $\nu \geq \mu$, the reverse inequality is also true and hence $F_\mu(x) = F_\nu(x)$.

<u>Proposition 3</u>: Let C be a chain in Λ relative to $>$, i.e., a linearly ordered ($>$) subset of Λ. Then $\mu^* \equiv \sup C$ exists and C converges (as a net directed by itself) to μ^*.

<u>pf</u>: Since C is a chain, it is a net directed by itself and the identity function. S compact implies that Λ is weak-* compact. Hence there is a subnet C' that converges to μ^*. We will now show that $\mu^* = \sup C$.

We show first that μ^* is an upper bound for C. Let $\nu \in C$. Let $C'' = \{\mu \in C' : \mu > \nu\}$. C'' is a subnet of C' so it converges to μ^*. By previous lemma, the graph of $>$ is closed and thus $\mu^* > \nu$.

Finally, we need to show that μ^* is a least upper bound. Suppose towards a contradiction that ν^* is another upper bound and there exists $f \in C_M$ with $\int f d\nu^* < \int f d\mu^*$. Since C' converges to μ^*, there exists μ on C' with $\int f d\nu^* < \int f d\mu$, and hence ν^* is not an upper bound.

We will now show that c converges to μ^*. Suppose not. Then there exists an open set U, such that $\mu^* \in U$ and for every $\mu \in C$ there exists $\mu' > \mu$ with $\mu' \in U^c$. Let $C' = C \;\; U^c$. C' is a cofinal subset of C and can easily be verified to be a subnet of C (see Kelley, 1955, pg. 65-70). By compactness, C' has a further subnet that converges to some element $\mu' \in \Lambda$. As seen previously, μ' is a sup for C, and thus $\mu' < \mu^*$ and $\mu^* < \mu$ so by antisymmetry of $<$, $\mu' = \mu^*$. But this contradicts the definition of C'.

We are now in conditions to prove one of the main results of the paper.

Recall that S is a compact subset of R^n containing elements a, b such that if $x \varepsilon S$ then $a < x < b$, and Λ is the set of bounded measures on S.

<u>Theorem 1</u>: Let T: $\Lambda \rightarrow \Lambda$ be a monotone function relative to the stochastic dominance order. Then T has a fixed point, i.e., there exists $\mu \varepsilon M$ such that $\mu = T\mu$.

<u>pf</u>: By proposition 3, every chain in M has a sup. Let $\delta_a \varepsilon \Lambda$, denote the measure with:

$$\delta_a(A) = \begin{cases} 1 \text{ if } a \varepsilon A \\ 0 \text{ otherwise} \end{cases}.$$

Then $\delta_a < \mu$ for every $\mu \varepsilon \Lambda$. Thus $\delta_a < T\delta_a$, and by assumption T is monotone. Thus the hypothesis of Knaster-Tarski's fixed point theorem (see Dugundji, pg. 14.) are satisfied and the conclusion follows.

We will now state some corollaries that prove useful in many economic problems.

Let P: $S \times S \rightarrow [0,1]$ be a transition function for a Markov Process. We will say that P is <u>increasing</u> if $x \varepsilon S$, $y \varepsilon S$ and $y < x \Rightarrow P(x,\cdot) > P(y,\cdot)$ in the stochastic order sense.

The transition P induces a map T: $\Lambda \rightarrow \Lambda$ by:

$$T\mu(A) = \int P(x,A)\mu(dx)$$

<u>Corollary 1</u>: If S has the properties listed above and P is increasing, then T has a fixed point on Λ.

pf: We will show that T is monotone on S. Let μ' and μ be elements of Λ such that $\mu' > \mu$. Let $f \in M$. For $\lambda \in \Lambda$, denote $\langle T\lambda, f\rangle \equiv \int f d\lambda$. We will show $\langle T\mu', f\rangle > \langle T\mu, f\rangle$. Since monotone indicator functions are dense in the monotone functions of $L^1(T\mu)$ and $L^1(T\mu')$, we may assume without loss of generality that $f = \chi_A$, a monotone indicator function. For $x' > x$, $P(x', A) = \int \chi_A(s) P(x', ds) > \int \chi_A(s)\; P(x, ds) = P(x, A)$, since $\chi_A \in M$ and P is increasing. Hence $P(\cdot, A) \in M$. Thus if $\mu' > \mu$ then $\langle T\mu', f\rangle \equiv T\mu'(A) = \int P(x,A)\mu'(dx) = \int P(x,A)\mu(dx) = T\mu(A) = \langle T\mu, f\rangle$ and the proof is complete.

Many problems on economic dynamics have the following structure: Given a state space S and a random variable ε on the space (E, ε) with distribution μ, the evolution of the state is described by a mapping $g\colon S \times E \to S$, where if at time t the state is S_t and realization of ε is ε_t, then $s_{t+1} = g(s_t, \varepsilon_t)$. This structure induces a mapping $P\colon S \times S \to [0,1]$ defined by $P(s, A) = \mu\{\varepsilon\colon g(S, \varepsilon) \varepsilon A\}$, which under appropriate conditions is a transition function for a Markov Process.

Lemma: Suppose g is monotone for each $\varepsilon \in E$. Then P is increasing.

pf: Let $s' > s$. Let g be a monotone indicator function of a set A. Then we need to show that $P(s', A) > P(S, A)$. Let $E_s = \{\varepsilon\colon g(s, \varepsilon) \varepsilon A\}$ and define similarly $E_{s'}$. It suffices to show that $E_s \subset E_{s'}$. Let $\varepsilon \in E_s$. Then $g(s, \varepsilon) \in A$. But then $g(s, \varepsilon) < g(s', \varepsilon) \in A$ since A is a monotone set.

pf: See appendix.

Corollary 2: If g is measurable in $S \times E$ and monotone in S for each $\varepsilon \in E$, then g induces a Markov Process which has a stationary distribution.

pf: As shown in the appendix, g induces a transition function P. Since, as proved above, P is increasing, the process has an invariant measure by corollary 1.

Section 2.2 Uniqueness and Global Stability

In this section we will provide conditions under which the invariant distribution for the process is unique and globally stable.

We say that transition P satisfies the stability criterion if there exists a point s^* ε S, $\varepsilon > 0$ and N such that $P^N(b,[a,x^*]) > \varepsilon$ and $P^N(a,[x^*,b]) > \varepsilon$.

Theorem 2: Suppose P is increasing and satisfies the stability criterion. Then there is a unique and globally stable stationary distribution for P.

pf: Fix N as given by the stability criterion. Let δ_x denote the probability measure that concentrates all the mass on the set $\{x\}$. We will prove that the following inequality holds:

$$(1-\varepsilon)\delta_a + \varepsilon\delta_{s^*} < T^N\delta_a < T^N\delta_b < (1-\varepsilon)\delta_b + \varepsilon\delta_{s^*}$$

where ε is given by the stability criterion. For this purpose let $f \in M$.

$$\int f dT^N\delta_a = \int_{x<s^*} f dT^N\delta_a + \int_{x\geq s^*} f dT^N\delta_a$$

$$> f(a)\ (1-\varepsilon) + f(s^*)\varepsilon$$

$$= \int f(S)\ d\{(1-\varepsilon)\delta_a + \varepsilon\delta_s{*}\}.$$

The right hand side inequality can be proved in the same way. $T^N\delta_a < T^N\delta_b$ since T is increasing. The above inequality implies:

$$(1-\varepsilon)T^k\delta_a + \varepsilon T^k\delta_s{*} < T^{k+N}\delta_a < (1-\varepsilon)T^k\delta_b + \varepsilon T^k\delta_s{*}.$$

Since by Proposition 3 monotone sequences converge $T^k\delta_a \to \lambda_a$, $T^k\delta_b \to \lambda_b$, and if necessary along a subsequence, $T^k\delta_{S^*} \to \lambda_{S^*}$. Hence the following inequality holds:

$$(1-\varepsilon)\lambda_a + \varepsilon\lambda_s^* \leq \lambda_a \leq \lambda_b \leq (1-\varepsilon)\lambda_b + \varepsilon\lambda_s^*.$$

$$\lambda_a - (1-\varepsilon)\lambda_a \geq \varepsilon\lambda_s^* \text{ so } \lambda_a \geq \lambda_s^*$$

Similarly $\lambda_b \leq \lambda_S^*$. since $\lambda_a \leq \lambda_b$ and $\geq$ was shown to be transitive and autisymmetric, $\lambda_a = \lambda_s^* = \lambda_b$. Let $\mu \in \Lambda$. Then $T^k\delta a \leq T^k\mu \leq T^k\delta_b$ and thus $T^k\mu \to \lambda_a$. That λ_a is an invariant measure follows from the fact that there is an invariant measure μ^* and $T^k\mu^* \to \delta_s$. Uniqueness and global stability follow.

Notes

(1) In proving this proposition the only features of $S \subset R^n$ and $<$ being the order defined used, is the fact that $<$ is a continuous partial order, R^n is a normal space and $(R^n, <)$ is an ordered topological vector space, (Shafer, 1971) i.e., a topological vector space X with a partial order that satisfies the following conditions:

If $y < x$, then:

(i) for any $z \in X$, $y + z < x + z$

and

(ii) for any $\lambda > 0$, $\lambda y < \lambda x$.

Hence this approximation result can be extended to S being a compact subset of an ordered topological vector space X with d continuous order and such that there exist elements $a \in X$, $b \in X$ with $a < x < b$ for all $x \in S$.