Economic Stabilization Policy: A Survey

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ECONOMIC STABILIZATION POLICY: A SURVEY

Introduction

This paper reviews selected studies in the theory of macroeconomic stabilization policy and summarizes their key findings. Although this theory still is relatively new, much has already been learned. It is the purpose of this survey, then, to make the existing body of knowledge more accessible to policymakers and students alike by consolidating it and presenting it in a unified fashion.[1/]

The problem generally posed in stabilization theory is to determine values of policy variables which maximize an objective function subject to laws governing the economy's motion. The arguments of the objective function, called goal variables, are assumed to be certain aggregate variables, such as the unemployment rate and inflation rate, and the motion of the economy is represented by a stochastic difference equation. Optimal policy is then derived as a rule which describes how policy variables should be set based on current information.

The validity of this seemingly straightforward approach to the policymaking problem has been challenged based on recent developments in general equilibrium theory. Robert E. Lucas, Jr., (1972) has constructed a general equilibrium model of the business cycle, and Neil Wallace (1980) has augmented that model to allow open market operations. These models suggest that economic stabilization policy may not be desirable.

In general equilibrium models individuals optimize and markets clear. Policies can be evaluated according to the Pareto criterion, so there is no need to posit an arbitrary policymaker objective function. And what is damaging is that active stabilization policies do not in general yield Pareto optimal outcomes in these models.

Business cycles emerge in general equilibrium models by assuming the economy is hit with uncorrelated random shocks which are transformed into serially correlated movements in output. The transformation can be due to costs of adjustment, imperfect information, or changes in the capital stock.

While government policies may be able to smooth the cycle, they are unlikely to make people better off, because people are assumed to be reacting optimally to the underlying shocks. The role of government then becomes limited to traditional concerns. Government spending and tax policies become concerned with public goods and income distribution, and monetary policy becomes concerned with the provision of fiat money and the structure of financial intermediation.

With this said, why then do this survey? There are at least three reasons. First, stabilization policy is being actively conducted today, and as long as it is, it should be made as well as current knowledge allows. Many issues are debated still which this theory easily resolves. Second, even in the context of general equilibrium models, some have speculated that fiat money would be most useful if its stock were adjusted over time in order to stabilize the price level around an announced path.[2/] Thus, in this case the solution to a stabilization theoretic model would be Pareto optimal. Finally, the studies in this survey have applications outside the realm of stabilization policy. Much of the theory of decision making under uncertainty was developed in the context of stabilization policy, but this theory applies just as well to individuals and firms.

Organization of Text

A simple model is constructed which includes all surveyed models as special cases. The simple model provides a standard framework for analysis and makes the relationships among the surveyed models straightforward to ascertain. All solutions are derived and described step by step.

The text first discusses in general terms the assumptions of the simple model and the method of solution. The solution next is illustrated in terms of a certainty model. Selected models then are described and analyzed roughly in order of their complexity. The models assume the following situations:

I. Random shocks to the economy with

   A. As many policy variables as goal variables,

   B. Fewer policy variables than goal variables,

   C. Multiple candidates for policy variables, or

   D. Information lags.

II. Uncertainty about the effects of policy on goal variables, where the uncertainty is

   A. Inherent,

   B. Inherent and characterized by long and variable policy lags, or

   C. Due to estimation.

Summary of Findings

Tinbergen (1952) is usually credited with constructing the first formal policymaking model. Tinbergen showed in a certainty setting how the policymaking problem can be viewed as the reverse of conditional forecasting. In conditional forecasting future time paths of policy variables are first assumed and then forecasts of goal variables are generated. In the policymaking problem, desired time paths of goal variables are first specified and then values of policy variables are found which generate goal variable paths closest (in terms of utility) to the desired paths. If there are as many policy variables as goal variables and the economic model's system of equations is invertible, there exist values of the policy variables which allow all the desired values of the goal variables to be achieved exactly.

Simon (1956) and Theil (1965) separately extended Tinbergen's model by allowing uncertainty to enter in a very simple way. They assumed that the economic model could be written as a system of equations, where each equation is composed of a deterministic function plus a random disturbance term. This implies that there is no uncertainty about the effects of policy variable changes on goal variables and that the variance of goal variables is independent of policy choice. They also assumed policymakers seek to maximize the expected value of a quadratic objective function, which implies that policymakers care only about the means and variances of goal variables. They found that the optimal setting of policy variables in the current period can be determined as in Tinbergen's certainty model with all random terms set at their expected values conditional on current information.[3/] The correspondence between their model and Tinbergen's has lead to theirs being referred to as the Simon-Theil "certainty equivalence" model.

When new information becomes available, the whole problem is solved anew to determine the optimal setting of policy variables in the next period. Unless new information comes in exactly as expected, conditional means of goal variables in future periods will change and cause the optimal setting of policy variables to differ from the values expected on the basis of the previous period's information. The fact that new information causes policy plans to change carries a simple message to policymakers: Do not commit yourselves unnecessarily into the future. If optimal policy requires reacting to new information, it would be silly to announce time paths for policy variables into the future and then stick to them regardless of surprises in the data.

The certainty equivalence models contain another important message for policymakers: you must take into account how the current policy choice restricts attainable outcomes in the future. In a special case when the number of policy

and goal variables are the same and the model's system of equations is invertible, policymakers can be myopic; that is, they can proceed one period at a time and ignore the consequences of their current decisions on future choices. In this case current decisions do not affect what is attainable in future periods. Policymakers cannot afford to be myopic, however, when there are more goal variables than policy variables. This more general case has been examined by many, including Simon (1956), Theil (1965), and Chow (1972).

Policymakers generally have a choice about which variable to control. Kareken (1970) and Poole (1970) studied in a one-period model how monetary authorities should determine whether to control an interest rate or a monetary aggregate. Their procedure is to determine optimal policy first under one policy control variable and then under the other. The decision on which variable to control is made by choosing the one which, when it is set optimally, implies the greater value of the objective function. The desirability of any policy must be judged in terms of how well it allows policy goals to be achieved.

So far we have discerned just two types of economic variables: goal variables and policy control variables. Kareken, Muench, and Wallace (1973) examined in a certainty equivalance setting the role of the remaining type of economic variables--variables in a model which are neither goal nor policy control variables. The authors dubbed this remaining type "information" variables. In general, differences in observed values of these variables from what was previously expected provides information on how the estimated relationship between goal variables and policy control variables has changed. Optimal policy requires revising the settings of policy control variables based on the new information in order to reflect the changed relationship. The important message is that the discrepancies between the observed values of information variables from their expected values provide information on which to adjust policy control

variables. There is a loss associated with attempting to control an information variable (make it an intermediate target), because that procedure fails to take advantage of new information.

Brainard (1967) added a second type of uncertainty to the policymaking model of the previously mentioned studies. Brainard assumed policymakers are faced with inherent uncertainty about the effects of policy on goal variables as well as with uncertainty due to random, additive shocks. When there is uncertainty about the effects of policy, optimal policy cannot be made myopically--even when there are as many policy control variables as goal variables. That is because the present policy choice restricts the attainable mean-variance combinations of goal variables in future periods. Nor can policy be made as in a certainty equivalence model with all random terms set at their conditional means. That is because the variances of goal variables now depend on the settings of policy control variables, and that relationship must be recognized in making policy. An important implication of Brainard's model is that the more uncertain are the effects of policy on goal variables, the less active policy should be. That is, policy should respond less to differences in desired values of goal variables from values expected conditional on the historical average policy.

Some, such as Friedman (1969), have argued that due to long and variable lags the effects of policy are so uncertain that it should not respond at all to perceived gaps in goal variables from their desired values. Fischer and Cooper (1973) analyzed the effects of long and variable lags on policymaking in an extended version of Brainard's model. They found that as long as the variance of policy lags is finite, some response to perceived gaps is warranted.

Chow (1975), Prescott (1972), and Zellner (1971) haved considered models in which the uncertainty about the effects of policy is due to estimation

error. As current values of policy control variables move away from their historical averages, the variances of current period goal variables grow in their models--just as in the Brainard model. But now the greater dispersion in outcomes allows the relationship between goal and policy control variables to be estimated with more accuracy, and this can lead to better policy in the future. The implication is that it can be useful to conduct policy allowing some "learning by doing." While the current economic situation may not be improved, increased knowledge about how policy works could improve economic conditions in the future.

A Framework for Analysis

Policymaking models are built from three classes of assumptions. The first class of assumptions concerns the nature of the policymaker's objective function, the second class concerns the specification of the economic process, and the third class concerns the availability of economic information.

In all of the models discussed below it is assumed that the policymaker maximizes expected utility, where utility is a quadratic function of certain uncontrolled variables. The arguments of the utility function are called goal variables, and utility at time 0 is given by

$$U(X_1,X_2) = -V_1\cdot(X_1-\hat{X}_1)^2 - V_2\cdot(X_2-\hat{X}_2)^2, \tag{1a}$$

where

X_t is the value of the uncontrolled variable X in period t,

$\hat{X}_t$ is the target value of X in period t,

V_t is the weight given to the (squared) deviation of the goal variable from its target value in period t, and the tth period is assumed to run from time t-1 to time t, t=1,2.[4/]

For concreteness, we can think of X as real GNP and $\hat{X}$ as, perhaps, potential real GNP.

A quadratic utility function is assumed for mathematical convenience. In a stochastic setting it permits expected utility to be expressed as a function of just the first two moments of goal variables, and this is not the case with more general utility functions. It is legitimate to ask, therefore, if this assumption severely limits the usefulness of results derived from our policy-making models. Although a definitive answer cannot be given, it is argued that this assumption is of secondary importance.

The crucial feature of a quadratic utility function is that it has a maximum or "bliss" point. If, for the time being, we suppose that $V_2 = 0$ in (1a), utility is a maximum at $X_1 = \hat{X}_1$. In other words, what has been called a target value of the goal variable is really a bliss point for the utility function. Utility decreases as X_1 moves away from $\hat{X}_1$ in either direction.

If utility were a function of a variable such as real GNP, it would seem reasonable to expect that more would always be preferred to less, and quadratic utility would seem a poor assumption. However, there are at least two ways out of this problem. First, it may be bliss is not attainable, so that only the portion of the utility function for values of real GNP less than bliss is relevant. In this case utility increases as real GNP increases but at a declining rate, and the quadratic utility function is able to serve as a reasonable approximation to more general utility functions. Second, it may be that we can regard the quadratic utility function in real GNP as an approximation to a general utility function which contains more arguments. Suppose, for instance, that the policymaker's utility function depends on real GNP, denoted by X, and the rate of inflation, denoted by π, and we assume only:

$U = u(X, -\pi)$, where u is defined for $X \geqq 0$ and $\pi \geqq 0$, and

$u_1, u_2 > 0$, $u_{11}, u_{22} < 0$ and $u_{11}u_{22} - u_{12}^2 > 0$.

Now, suppose the policymaker believes there exists a trade-off between real GNP and inflation given by:

$$-\pi = g(X), \text{ where } g'(X) < 0.$$

We can then write $U = u(X,g(X)) \equiv v(X)$. If there exists a real GNP-inflation rate pair $<\hat{X},\hat{\pi}>$ which maximizes $u(X,-\pi)$ subject to $-\pi = g(X)$, $v(X)$ will have a bliss point at $\hat{X}$ and might reasonably be approximated by a quadratic function.

In all but one of the models to be discussed below there is assumed to be a single goal variable, so that utility at time zero can be written without loss of generality as

$$\text{(1b)} \qquad U(X_1,X_2) = -(X_1-\hat{X}_1)^2 - \gamma(X_2-\hat{X}_2)^2, \text{ where } \gamma\varepsilon(0,\infty).^{5/}$$

For most models two cases will be analyzed. In the first case, $\gamma = 0$, the policymaker cares about the effects of policy actions taken today on the outcome for current real GNP only. In the second case, $\gamma \neq 0$, the policymaker is concerned also with the outcome for future real GNP.

The second class of assumptions underlying a policymaking model concerns the specification of the economic process. For the models discussed below the economic process can be written as a linear difference equation

$$\text{(2)} \qquad X_t = \theta_1(t)\cdot X_{t-1} + \theta_2(t)\cdot P_t + \theta_3(t) \qquad t=1,2$$

where P_t is the value of the policy control variable in period t and $\theta(t) = (\theta_1(t),\theta_2(t),\theta_3(t))$ are values of coefficients in period t.[6/] Again, for concreteness we can think of P as the Fed's portfolio of securities.

The only restriction involved in writing this reduced form equation is that the lagged value of the goal variable X_{t-1} and the current value of the

policy control variable P_t enter linearly in determining the current value of the goal variable X_t. Otherwise, the reduced form of any structural model can be written in this form.

To give economic meaning to (2) it is necessary to place restrictions on θ. These restrictions normally would be derived from theory and estimation of the underlying economic structure. For most of the models described below, however, the economic structure will not even be considered, and θ will be specified as a stochastic process with certain assumed properties. Each set of assumed properties for θ defines a class of reduced form models, and optimal policy will be derived for each class. Any class of reduced form models is consistent with a variety of economic theories. In the random shocks model, for example, we assume θ_1 and θ_2 are known constants and θ_3 is a serially uncorrelated random variable with mean $\bar{\theta}_3$ and variance σ_3^2. The implied reduced form model is consistent with versions of both Keynesian and classical economic theories. Under the adaptive expectations version of Keynesian theory it follows that $\theta_2 > 0$, and in this case an activist policy turns out to be optimal. The natural rate-rational expectations version of classical theory, meanwhile, implies $\theta_2 = 0$, and in this case any deterministic policy is seen to be as good as any other.

The generality of the reduced form models we consider is both a virtue and a vice. On the positive side implications from these models have wide applicability for policymaking. On the negative side none of the implications is specific enough to provide a policymaker with numerical guides. That requires an estimated reduced form.

A number of reduced forms can be consistent with observed economic time series, but not all of them will be invariant to changes in the policy rule. In the policy models which follow it is assumed that whether we think of θ as a

vector of estimated coefficients or a vector of "true" coefficients, the true coefficients are independent of the choice of policy rule. This assumption rules out fully dynamic rational expectations models for which there do not exist reduced forms having coefficients independent of policy rule.

The final class of assumptions underlying a policymaking model relates to the kinds of economic data available and the frequency with which they are observed by the policymaker. This class of assumptions is necessarily related to the other two. Both the economic process and the policymaker's objective function must be expressed in terms of observed variables. Any model of the economic process estimated from observed variables could be generated by an infinite number of models having a finer time dimension. That is, without further assumptions there is no way to identify even the reduced form of the economic structure for time intervals smaller than those for which data are observed. Similarly, if the policymaker's goal were to control the continuous time path for real GNP, it would have to be recognized that the best which could be done is to control its quarterly, observable path. The policymaker's preferences would then have to be transformed and stated in terms of quarterly real GNP.

For all but the information lags model it is assumed that values of economic variables for period t are known by the policymaker at the beginning of period t+1. In the information lags model it is assumed instead that values of some economic variables are reported more frequently than others.

The policymaking problem in each model we explore is to maximize expected utility (as of time zero) E_0U, subject to the assumed economic process and information structure. This formulation of the policy choice problem under uncertainty is motivated by Arrow's (1971) theory of decision making under uncertainty. The maximizers are chosen from the set F of functions which indicate how the policy control variable is to be set each period based on information

available at the beginning of that period. Let I_t be the information (observed values of economic variables over time) available at time t. Then the maximizers to the policymaking problem $<\tilde{f}_1,\tilde{f}_2>$ are included in the set of paired functions $F = \{<f_1,f_2>|f_1:\{I_0\} \rightarrow \mathbb{R} \text{ and } f_2:\{I_1\} \rightarrow \mathbb{R}\}$, where $\mathbb{R} = (-\infty,\infty)$ and for arbitrary I_0, I_1 we identify $<P_1,P_2> = <f_1(I_0),f_2(I_1)>$. Thus, for arbitrary values of economic variables observed at time t-1, the optimal t^{th} period policy rule $\tilde{f}_t$ indicates how the policy control variable should be set in period t: $\tilde{P}_t = \tilde{f}_t(I_{t-1})$ t=1,2.

For the models we explore, I_0 is a known vector and $\tilde{P}_1 = \tilde{f}_1(I_0)$ is a known quantity at time t=0.[7/] If the economic process is not deterministic, I_1 may contain realizations of economic variables which will not be known (observed) until t=1. In this case I_1 contains variables which are random as of t=0. Since $\tilde{P}_2 = \tilde{f}_2(I_1)$, the optimal setting of the policy control variable in the second-period $\tilde{P}_2$ is not known at t=0, even though the policy rule $\tilde{f}_2$ is. Instead $\tilde{P}_2$ is a random variable at t=0.

Maximizing over the set of paired functions F is more general and thus allows a wider range of policy choices than does maximizing over the set of paired scalars $\{<P_1,P_2>|P_1 \varepsilon \mathbb{R} \text{ and } P_2 \varepsilon \mathbb{R}\}$. Maximization over this latter set is equivalent to maximization over the set of paired functions $G = \{<g_1,g_2>|g_1:\{I_0\} \rightarrow \mathbb{R},\ g_2:\{I_0\} \rightarrow \mathbb{R}\}$, and we identify $P_1 = g_1(I_0)$ and $P_2 = g_2(I_0)$ for given I_0.[8/] It is straightforward to show that $G \subseteq F$.[9/]

A pair of functions $<g_1,g_2>$ which does not make use of new information is called a nonfeedback policy rule, while a pair $<f_1,f_2>$ which does is called a feedback policy rule. The set G contains only nonfeedback rules, while the set F contains both feedback and nonfeedback rules. Information has value when

$$\max_F E_0U = E_0\Psi(\tilde{f}_1(I_0),\tilde{f}_2(I_1)) > E_0\Psi(\tilde{g}_1(I_0),\tilde{g}_2(I_0)) = \max_G E_0U$$

where $\Psi(\cdot,\cdot) = U(X_1,X_2)$ with the substitutions:

$$X_t = \theta_1(t)\cdot X_{t-1} + \theta_2(t)\cdot P_t + \theta_3(t) \quad t=1,2 \text{ and}$$

$$\langle P_1,P_2\rangle = \langle f_1(I_0),f_2(I_1)\rangle \text{ or } \langle g_1(I_0),g_2(I_0)\rangle.$$

An interesting question which will be addressed subsequently is under what conditions will $\tilde{f}_1 = \tilde{g}_1$?

The policymaking model can be stated formally as

Objective function

(1) $$\max_{P_1,P_2} -E_0[V_1\cdot(X_1-\hat{X}_1)^2+E_1V_2\cdot(X_2-X_2)^2], \text{ where } E_s(\cdot) = E(\cdot|I_s) \quad s=0,1$$

subject to

Economic process

(2) $$X_t = \theta_1(t)\cdot X_{t-1} + \theta_2(t)\cdot P_t + \theta_3(t) \quad t=1,2$$

where for s=0,1 and t,t'=1,2

Stochastic specification

(2a) $$E_s\theta(t) \equiv E[\theta(t)|I_s] = \overline{\theta(t)}_s,$$

(2b) $$E_s\{[\theta(t)-E_s\theta(t)]'[\theta(t'))-E_s\theta(t')]\} \equiv$$

$$E\{[\theta(t)-E\theta(t)]'[\theta(t')-E\theta(t')]|I_s\} = \sum(t,t')_s,$$

and $\overline{\theta(t)}_s$ and $\sum(t,t')_s$ are known matrices, and where

Information structure

(3) $$I_0 = \langle X_0,P_0\rangle,\ I_1 = \langle X_0,\underline{X}_1,P_0,P_1\rangle, \text{ and}$$

$$\underline{X}_1 = \langle X_{i1}\rangle \text{ such that } i\varepsilon\{1,\ldots,n\} \text{ and } X_{i1} \text{ is observed at } t=1.$$ [10]

The nested, conditional expectations formulation of the objective function assures that optimal policy is being selected from the set of feedback rules $P_t = f_t(I_{t-1})$ t=1,2.

The policymaking problem in each model is solved using the Bellman dynamic programming principle. This is an iterative technique which involves solving backwards through time. In our two-period models the first step is to maximize $-E_1\{V_2\cdot(X_2-\hat{X}_2)^2\}$ with respect to P_2 in order to obtain $\tilde{f}_2(I_1)$. The intuitive idea here is that no matter what the economic outcome in the first period, second-period policy should be made to yield on average the best possible second-period outcome. We can write

$$-E_1\{V_2\cdot(X_2-\hat{X}_2)^2\} = -V_2\cdot E_1\{(X_2-E_1X_2+E_1X_2-\hat{X}_2)^2\}$$

$$= -V_2\cdot[E_1(X_2-E_1X_2)^2+(E_1X_2-\hat{X}_2)^2]$$

$$= -V_2\cdot(\sigma^2_{X_2}+(\overline{X}_2-\hat{X}_2)^2), \text{ where } E_1X_2 \equiv \overline{X}_2.$$

Thus, based on information available at t=1, the objective in the first step of the Bellman technique is to choose P_2 in order to minimize a weighted sum of (a) the variance(s) of the goal variable(s) in the second-period $\sigma^2_{X_2}$ and (b) the squared deviation(s) of the mean(s) of the goal variable(s) in the second period from the target(s) $(\overline{X}_2-\hat{X}_2)^2$. From the economic process, $X_2 = \theta_1(2)\cdot X_1 + \theta_2(2)\cdot P_2 + \theta_3(2)$, and the above relationship we have:

$$-E_1\{V_2\cdot(X_2-\hat{X}_2)^2\} = -V_2\cdot\{\sigma^2_{X_2}+(\overline{X}_2-\hat{X}_2)^2\} \equiv U_2(P_2,I_1).$$

Maximizing $U_2(P_2,I_1)$ with respect to P_2 for arbitrary I_1 yields

$\tilde{P}_2 = \tilde{f}_2(I_1)$ and we can write $\tilde{U}_2(I_1) = U_2(\tilde{P}_2,I_1)$.

If I_1 contains realizations of economic variables observed at t=1 but not at t=0, $\tilde{U}_2(I_1)$ is in general a random variable at t=0.

In the second step of the solution optimal first-period policy is found by maximizing $-E_0\{V_1\cdot(X_1-\hat{X}_1)^2\} + E_0\tilde{U}_2(I_1)$, with respect to P_1 in order to obtain $\tilde{P}_1 = \tilde{f}_1(I_0)$. Analogous to the first step, we can write

$$-E_0\{V_1\cdot(X_1-\hat{X}_1)^2\} + E_0\tilde{U}_2(I_1) = -V_1\cdot(\sigma^2_{X_1}+(\bar{X}_1-\hat{X}_1)^2) + E_0\tilde{U}_2(I_1),$$

where the expectations and variances are conditional on I_0. As the objective function indicates, $\tilde{f}_1$ is chosen with two considerations in mind. First, it determines the first-period mean-variance combination of goal variables from those which are attainable, and this is captured in the term $-V_1\cdot(\sigma^2_{X_1}+(\bar{X}_1-\hat{X}_1)^2)$. Second, it may restrict the set of mean-variance combinations of goal variables which are attainable in the second period. This effect is captured in the term $E_0\tilde{U}_2(I_1)$ if that term depends on P_1. Maximal expected utility as of t=0 is then the expression $-E_0\{V_1\cdot(X_1-\hat{X}_1)^2\} + E_0\tilde{U}_2(I_1)$ evaluated at $\tilde{P}_1 = \tilde{f}_1(I_0)$.

Certainty Model

In order to illustrate some of the concepts which have been discussed, let us consider a two-period nonstochastic model with one policy control variable and one goal variable (n=1). Whether or not X_1 is observed at t=1 is not important, since knowledge of P_1 and X_0 allows X_1 to be computed exactly. This certainty model can be specified by equations (1)-(3) with

(1) $\gamma = \frac{V_2}{V_1}$

(2a) $\overline{\theta(t)}_s = (\bar{\theta}_1,\bar{\theta}_2,\bar{\theta}_3)\ (s,t)$

(2b) $\sum(t,t')_s = 0_{3x3}\ (s,t,t')$ and

(3) $\underline{X}_1 = X_1$.

By (2a) and (2b) we can write (2) as

$$X_t = \bar{\theta}_1 X_{t-1} + \bar{\theta}_2 P_t + \bar{\theta}_3 \quad t=1,2 \text{ where } \bar{\theta}_1, \bar{\theta}_2, \text{ and } \bar{\theta}_3 \text{ are known.}$$

The first step in the solution routine is to maximize $-\gamma E_1(X_2-\hat{X}_2)^2$ or equivalently $-E_1(X_2-\hat{X}_2)^2$ with respect to P_2 in order to determine $\tilde{P}_2 = \tilde{f}_2(I_1)$. Since $E_1X_2 = X_2$, we have

$$-E_1(X_2-\hat{X}_2)^2 = -E_1(X_2-E_1X_2)^2 - (E_1X_2-\hat{X}_2)^2 = -(\bar{\theta}_1X_1+\bar{\theta}_2P_2+\bar{\theta}_3-\hat{X}_2)^2.$$

Thus, maximization of $-E_1(X_2-\hat{X}_2)^2$ with respect to P_2 yields as a first-order condition

$$\tilde{P}_2 = \frac{\hat{X}_2-\bar{\theta}_3-\bar{\theta}_1X_1}{\bar{\theta}_2} \equiv \tilde{f}_2(I_1), \ (\bar{\theta}_2 \neq 0).$$

The second-order condition for maximization is clearly met. Under $\tilde{f}_2$ we have

$$\tilde{U}_2(I_1) = -\gamma E_1(\bar{\theta}_1X_1+\bar{\theta}_2\tilde{P}_2+\bar{\theta}_3-\hat{X}_2)^2 = 0,$$

indicating that for arbitrary initial conditions $\langle X_1,P_1\rangle$, $\tilde{f}_2$ allows the target for X_2 to be hit exactly and with certainty.

The second step in the solution routine is to maximize $-E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1) = -E_0(X_1-\hat{X}_1)^2$ with respect to P_1. This is precisely the same maximization problem as in the first step except the subscripts are moved back one period. First-period optimal policy is given by

$$\tilde{P}_1 = \frac{\hat{X}_1-\bar{\theta}_3-\bar{\theta}_1X_0}{\bar{\theta}_2} \equiv \tilde{f}_1(I_0), \ (\bar{\theta}_2 \neq 0),$$

and for arbitrary initial conditions $\langle X_0,P_0\rangle$, $\tilde{f}_1$ allows the target for X_1 to be hit exactly and with certainty. Thus, maximal expected utility as of t=0, $-E_0(\bar{\theta}_1X_0+\bar{\theta}_2\tilde{P}_1+\bar{\theta}_3-\hat{X}_1)^2$, is zero.

In the certainty model it follows that as long as $\overline{\theta}_2 \neq 0$ the targets for the goal variable can be hit exactly and with certainty period by period. Since $X_1 = \hat{X}_1$ with certainty, we can write

$$\tilde{P}_2 = \frac{\hat{X}_2 - \overline{\theta}_3 - \overline{\theta}_1 \hat{X}_1}{\overline{\theta}_2} \equiv \tilde{g}_2(I_0).$$

The fact that $\tilde{g}_2(I_0) = \tilde{f}_2(I_1)$ for any observed pair $\langle X_1, P_1 \rangle$ is another way of saying that new information has no value, and that is because the policymaker knows with certainty the outcome for X_1 given P_1 and X_0. Thus, at time zero optimal policy in this case can be described equivalently as a feedback rule $\langle \tilde{f}_1, \tilde{f}_2 \rangle$ or as a nonfeedback rule

$$\langle \tilde{g}_1, \tilde{g}_2 \rangle = \left\langle \frac{\hat{X}_1 - \overline{\theta}_3 - \overline{\theta}_1 X_0}{\overline{\theta}_2}, \frac{\hat{X}_2 - \overline{\theta}_3 - \overline{\theta}_1 \hat{X}_1}{\overline{\theta}_2} \right\rangle.$$

In the special case where $\overline{\theta}_2 = 0$, one choice of policy rules is as good as any other.

I. Random Shocks Model

A. As Many Policy Control Variables as Goal Variables

We will now consider a stochastic model with one goal variable and one policy control variable. The coefficients of the model are assumed to be known with certainty, but serially independent random shocks to the economy cause there to be uncertainty about the intercept of the reduced form. At the beginning of each period, the policymaker observes the value of the goal variable in the previous period. This is a simple version of Theil's (1965) first-period certainty-equivalence model, and the meaning of Theil's label will soon become apparent.

The random shocks model can be specified by equations (1)-(3) with

(1) $$\gamma = \frac{V_2}{V_1}$$

(2a) $$\overline{\theta(t)}_s = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3) \quad t > s$$

(2b) $$\sum(t,t')_s = \begin{cases} 0_{3x3} & t \neq t' \text{ or } t \leqq s \\ \left(\begin{array}{cc|c} & & 0 \\ & 0 & 0 \\ \hline 0 & 0 & \sigma_3^2 \end{array}\right) & , \sigma_3^2 > 0 \quad t=t' > s \end{cases}$$

(3) $$\underline{X}_1 = X_1.$$

Except for (2b) this specification is precisely the same as the certainty model. Let us examine the solution to the policymaking problem in both the one-period and two-period cases.

1. One-period horizon (γ=0).

The problem is

(1) $$\max_{P_1} -E_0(X_1-\hat{X}_1)^2 = -\sigma_{X_1}^2 - (E_0X_1-\hat{X}_1)^2$$

subject to

(2) $$X_1 = \bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \theta_3(1)$$

where by (2a) and (2b)

$$E_0\theta_3(1) = \bar{\theta}_3 \text{ and } E_0(\theta_3(1)-\bar{\theta}_3)^2 = \sigma_3^2.$$

We have by (2)

(i) $$E_0 X_1 = \bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \bar{\theta}_3.$$

Substracting (i) from (2) we have

$$X_1 - E_0 X_1 = \theta_3(1) - \bar{\theta}_3$$

so that

(ii) $$\sigma_{X_1}^2 \equiv E_0(X_1 - E_0 X_1)^2 = E_0(\theta_3(1)-\bar{\theta}_3)^2 \equiv \sigma_3^2.$$

Substituting (i) and (ii) into (1) the problem is simply

(iii) $$\max_{P_1} -\sigma_3^2 - (\bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \bar{\theta}_3 - \hat{X}_1)^2.$$

The first-order condition for maximization yields

$$\tilde{P}_1 = \frac{\hat{X}_1 - \bar{\theta}_3 - \bar{\theta}_1 X_0}{\bar{\theta}_2} \quad (\bar{\theta}_2 \neq 0).$$

Notice optimal policy in the single-period random shocks model is precisely the same as in the certainty model; the lone exception being that the expected value of the random disturbance term in the former model replaces the known value of the intercept in the latter model. That is a reason why the random shocks model is sometimes called the certainty-equivalence model. More formally, notice that in the single-period random shocks model the same optimal policy is

derived whether we maximize with respect to P_1, $E_0U(X_1) \equiv -E_0(X_1-\hat{X}_1)^2$ or $U(E_0X_1) \equiv -(E_0X_1-\hat{X}_1)^2$. It is this interchangeability of the expectations operator which yields the certainty equivalence result; namely, that optimal policy in the stochastic model is the same as in the certainty model with all stochastic terms set at their conditional means. Sufficient conditions for the certainty equivalence result to obtain in a one-period, n-goal variable, m-policy control variable model are:[11/]

a. Utility is a quadratic function of goal variables and policy control variables.

b. The economic process is separable: $X_t = R(X_{t-1}, X_{t-2}, \ldots, P_t, P_{t-1}, P_{t-2}, \ldots) + \mu_t$, where R is any well-defined function and μ_t is a random vector conformable to X with finite mean and finite variance-covariance matrix independent of P.

Maximum utility can be found by substituting $\tilde{P}_1$ into (iii) which yields $E_0\tilde{U} = -\sigma_3^2$. This result can be interpreted as follows. For arbitrary P_1 expected utility as expressed in (iii) is the negative of the sum of (a) the variance of X_1 which in this case is σ_3^2 and (b) the squared deviation of $\bar{X}_1$ from the target $\hat{X}_1$ which is $(\bar{\theta}_1X_0+\bar{\theta}_2P_1+\bar{\theta}_3-\hat{X}_1)^2$. Notice the variance of X_1 in this case is independent of the setting of the policy instrument P_1. Thus, the optimal value of P_1 is the one which minimizes (b). The optimal value $\tilde{P}_1$ sets the expected value of X_1 equal to its target so that (b) is zero and the negative of expected utility is equal to the irreducible variance of X_1, σ_3^2.

Let us now turn to the two-period case.

2. Two-period horizon ($\gamma \neq 0$).

The problem is

$$(1) \qquad \max_{P_1, P_2} \; -E_0[(X_1-\hat{X}_1)^2+\gamma E_1(X_2-\hat{X}_2)^2]$$

subject to

(2) $$X_1 = \bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \theta_3(1)$$

$$X_2 = \bar{\theta}_1 X_1 + \bar{\theta}_2 P_2 + \theta_3(2)$$

where by (2a) and (2b) for s=0,1 and t > s, $E_s\theta_3(t) = \bar{\theta}_3$ and

$$E_s(\theta_3(t)-\bar{\theta}_3)(\theta_3(t')-\bar{\theta}_3) = \begin{cases} 0 & t \neq t' \\ \sigma_3^2 & t = t'. \end{cases}$$

By the Bellman principle $\tilde{P}_2$ is found by maximizing $-\gamma E_1(X_2-\hat{X}_2)^2$ with respect to P_2 subject to $X_2 = \bar{\theta}_1 X_1 + \bar{\theta}_2 P_2 + \theta_3(2)$. But this problem is precisely the same as the one-period problem with the subscripts moved up one period. Thus, we have

$$\tilde{P}_2 = \frac{\hat{X}_2-\bar{\theta}_3-\bar{\theta}_1 X_1}{\bar{\theta}_2} \qquad (\bar{\theta}_2 \neq 0)$$

and

$$\tilde{U}_2(I_1) \equiv -\gamma \cdot \{\sigma_{X_2}^2 + (\bar{\theta}_1 X_1 + \bar{\theta}_2 \tilde{P}_1 + \bar{\theta}_3 - \hat{X}_2)^2\} = -\gamma\sigma_3^2.$$

Note that $\tilde{P}_2$ is a random variable as of t=0 since its value depends on X_1, and X_1 is a linear function of the random variable $\theta_3(1)$. However, $\tilde{U}_2(I_1)$ is not random. It is a known constant independent of P_1. Thus, in the second step of the solution, optimal first-period policy is found by maximizing with respect to P_1

$$-E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1)$$

or equivalently by maximizing with respect to P_1

$$-\sigma_{X_1}^2 - (E_0 X_1 - \hat{X}_1)^2$$

subject to

$$X_1 = \bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \theta_3(1).$$

Again this is the same problem as in the one-period case so that we have

$$\tilde{P}_1 = \frac{\hat{X}_1-\bar{\theta}_3-\bar{\theta}_1X_0}{\bar{\theta}_2} \qquad (\bar{\theta}_2\neq 0)$$

and

$$E_0U = -E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1) = -\sigma_3^2 - \gamma\sigma_3^2 = -(1+\gamma)\sigma_3^2.$$

Note the following:

(a) Optimal first-period policy is independent of second-period parameters. In this multiperiod random shocks model with as many policy control variables as goal variables, policy can be set myopically. That is, optimal policy can be made each period by ignoring future periods. In our example at time t=0, $\tilde{P}_1$ could be found by maximizing $-E_0(X_1-\tilde{X}_1)^2$ with respect to P_1 subject to $X_1 = \bar{\theta}_1X_0 + \bar{\theta}_2P_1 + \theta_3(1)$. At time t=1 the optimal value of P_2 could be found by maximizing $-E_1(X_2-\hat{X}_2)^2$ with respect to P_2 subject to $X_2 = \bar{\theta}_1X_1 + \bar{\theta}_2P_2 + \theta_3(2)$. Thus, the policymaker need not consider the consequences in future periods of the present policy choice.

(b) Optimal first-period policy can be found by the first-period certainty equivalence method, that is, by maximizing with respect to P_1 and P_2

$$U(E_0X_1,E_0X_2) = -(E_0X_1-\hat{X}_1)^2 - \gamma(E_0X_2-\hat{X}_2)^2$$

subject to (2). The maximizers to this problem are

$$\tilde{\tilde{P}}_1 = \tilde{g}_1(I_0) = \frac{\hat{X}_1-\bar{\theta}_3-\bar{\theta}_1X_0}{\bar{\theta}_2}$$

$$\tilde{\tilde{P}}_2 = \tilde{g}_2(I_0) = \frac{\hat{X}_2-\bar{\theta}_3-\bar{\theta}_1E_0X_1}{\bar{\theta}_2}.$$

In this model we obtain the first-period certainty equivalence result; that is, optimal first-period policy is the same whether we maximize with respect to P_1 and P_2, $E_0U(X_1,X_2)$ or $U(E_0X_1,E_0X_2)$. Hence, in the random shocks model $\tilde{f}_1(I_0) = \tilde{g}_1(I_0)$. Sufficient conditions for the first-period certainty equivalence result to obtain in a T-period, n-goal variable, m-policy control variable model are:[12/]

(i) Utility is a quadratic function of dated goal variables and dated policy control variables.

(ii) The economic process is given by: $X = RP + \mu$, where

$$\underset{nTx1}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_t \\ \vdots \\ X_T \end{pmatrix}; \quad \underset{nx1}{X_t} = \begin{pmatrix} X_{1t} \\ \cdot \\ \cdot \\ \cdot \\ X_{nt} \end{pmatrix}; \quad \underset{mTx1}{P} = \begin{pmatrix} P_1 \\ \vdots \\ P_t \\ \vdots \\ P_T \end{pmatrix}; \quad \underset{mx1}{P_t} = \begin{pmatrix} P_{1t} \\ \cdot \\ \cdot \\ \cdot \\ P_{mt} \end{pmatrix};$$

$$\underset{nTx1}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_t \\ \vdots \\ \mu_T \end{pmatrix}; \quad \underset{nx1}{\mu_t} = \begin{pmatrix} \mu_{1t} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{nt} \end{pmatrix};$$

μ is a vector of random elements with finite mean, finite variance-covariance matrix, and distribution independent of P;

$$\underset{nTxmT}{R} = \begin{pmatrix} R_{11} & 0 & \cdots & 0 \\ R_{21} & R_{22} & \cdots & 0 \\ \vdots & & & \vdots \\ R_{T1} & R_{T2} & \cdots & R_{TT} \end{pmatrix}$$

R is a lower triangular matrix of fixed and known coefficients, and the R_{ij} submatrix is nxm. The assumed form of R indicates the outcome for X in any given period is independent of the choice of P in future periods.

B. <u>More Goal Than Policy Control Variables</u>

This model is a simple variant of the previous one; the only difference being there are now two goal variables instead of one. With more goal variables than policy control variables, the policymaker faces contemporaneous trade-offs among goal variables as well as trade-offs over time. It is no longer optimal to make policy myopically.

This model can be specified by equations (1)-(3) with

(1) $$V_1 = (V_{11}, V_{21}),\ V_2 = (V_{12}, V_{22})$$

(2a) $$\overline{\theta(t)}_s = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3) = \begin{pmatrix} \bar{\theta}_{11} & 0 & \bar{\theta}_{13} & \bar{\theta}_{14} \\ 0 & \bar{\theta}_{22} & \bar{\theta}_{23} & \bar{\theta}_{24} \end{pmatrix} \equiv \begin{pmatrix} \bar{\theta}_{1R} \\ \bar{\theta}_{2R} \end{pmatrix} \quad t > s$$

(2b) $$\sum(t,t')_s \equiv E_s \begin{pmatrix} (\theta_{1R}(t)-\bar{\theta}_{1R})' \\ (\theta_{2R}(t)-\bar{\theta}_{2R})' \end{pmatrix} (\theta_{1R}(t')-\bar{\theta}_{1R}, \theta_{2R}(t')-\bar{\theta}_{2R})$$

$$\equiv \begin{pmatrix} \sum_{11}(t,t')_s & \sum_{12}(t,t')_s \\ \sum_{21}(t,t')_s & \sum_{22}(t,t')_s \end{pmatrix} \quad t > s$$

$$= \begin{cases} 0 \quad t \neq t' \text{ or } t \leqq s \\ \left(\begin{array}{c|c|c|c} 0 & 0 & & \\ \hline 0 & \sigma_{14}^2 & 0 & \\ \hline & & 0 & 0 \\ \hline 0 & & 0 & \sigma_{24}^2 \end{array}\right), \ \sigma_{14}^2, \sigma_{24}^2 > 0 \quad t = t' > s \end{cases}$$

(3) $$\underline{X}_1 = \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix},$$

where use has been made of notation in footnotes 4, 6, and 10.

Let us suppose X_{1t} and X_{2t} are the rates of unemployment and inflation, respectively, at time t and make the substitutions $X_t = X_{1t}$ and $\pi_t = X_{2t}$. Similarly let $r_t = V_{1t}$ and $d_t = V_{2t}$. We can then write our model as:

$$(1) \qquad \max_{P_1,P_2} \; -E_0\{r_1(X_1-\hat{X}_1)^2+d_1(\pi_1-\hat{\pi}_1)^2+E_1\{r_2(X_2-\hat{X}_2)^2+d_2(\pi_2-\hat{\pi}_2)^2\}\}$$

subject to

$$(2) \qquad \begin{aligned} X_t &= \bar{\theta}_{11}X_{t-1} + \bar{\theta}_{13}P_t + \theta_{14}(t) \\ \pi_t &= \bar{\theta}_{22}\pi_{t-1} + \bar{\theta}_{23}P_t + \theta_{24}(t), \end{aligned} \qquad t=1,2$$

where by (2a) and (2b) for s=0,1 and t > s

$$E_s\begin{pmatrix}\theta_{14}(t)\\ \theta_{24}(t)\end{pmatrix} = \begin{pmatrix}\bar{\theta}_{14}\\ \bar{\theta}_{24}\end{pmatrix}$$

and

$$E_s\begin{pmatrix}\theta_{14}(t)-\bar{\theta}_{14}\\ \theta_{24}(t)-\bar{\theta}_{24}\end{pmatrix}(\theta_{14}(t')-\bar{\theta}_{14},\theta_{24}(t')-\bar{\theta}_{24})$$

$$= \begin{cases} 0 & t\neq t' \\ \begin{pmatrix}\sigma_{14}^2 & 0\\ 0 & \sigma_{24}^2\end{pmatrix}, & t=t' \end{cases}$$

We will again look at solutions to both the one-period and two-period cases.

1. One-period horizon ($r_2=d_2=0$).

The problem is to maximize with respect to P_1

$$(1) \qquad -E_0(r_1(X_1-\hat{X}_1)^2+d_1(\pi_1-\hat{\pi}_1)^2) =$$

$$-r_1\sigma_{X_1}^2 - d_1\sigma_{\pi_1}^2 - r_1(E_0X_1-\hat{X}_1)^2 - d_1(E_0\pi_1-\hat{\pi}_1)^2$$

subject to

(2) $$X_1 = \bar{\theta}_{11}X_0 + \bar{\theta}_{13}P_1 + \theta_{14}(1)$$

$$\pi_1 = \bar{\theta}_{22}\pi_0 + \bar{\theta}_{23}P_1 + \theta_{24}(1)$$

where

$$E_0\begin{pmatrix}\theta_{14}(1)\\ \theta_{24}(1)\end{pmatrix} = \begin{pmatrix}\bar{\theta}_{14}\\ \bar{\theta}_{24}\end{pmatrix} \text{ and } E_0\begin{pmatrix}\theta_{14}(1)-\bar{\theta}_{14}\\ \theta_{24}(1)-\bar{\theta}_{24}\end{pmatrix}(\theta_{14}(1)-\bar{\theta}_{14},\theta_{24}(1)-\bar{\theta}_{24})$$

$$= \begin{pmatrix}\sigma_{14}^2 & 0\\ 0 & \sigma_{24}^2\end{pmatrix}.$$

Thus, we have

$$E_0X_1 = \bar{\theta}_{11}X_0 + \bar{\theta}_{13}P_1 + \bar{\theta}_{14}$$

$$\sigma_{X_1}^2 \equiv E_0(X_1-E_0X_1)^2 = E_0(\theta_{14}(1)-\bar{\theta}_{14})^2 = \sigma_{14}^2$$

$$E_0\pi_1 = \bar{\theta}_{22}\pi_0 + \bar{\theta}_{23}P_1 + \bar{\theta}_{24}$$

$$\sigma_{\pi_1}^2 \equiv E_0(\pi_1-E_0\pi_1)^2 = E_0(\theta_{24}(1)-\bar{\theta}_{24})^2 = \sigma_{24}^2,$$

and our problem is simply to maximize with respect to P_1

$$-r_1\sigma_{14}^2 - d_1\sigma_{24}^2 - r_1(\bar{\theta}_{11}X_0+\bar{\theta}_{13}P_1+\bar{\theta}_{14}-\hat{X}_1)^2 - d_1(\bar{\theta}_{22}\pi_0+\bar{\theta}_{23}P_1+\bar{\theta}_{24}-\hat{\pi}_1)^2.$$

The first-order condition for maximization yields

$$\tilde{P}_1 = \frac{r_1\bar{\theta}_{13}(\hat{X}_1-\bar{\theta}_{11}X_0-\bar{\theta}_{14})+d_1\bar{\theta}_{23}(\hat{\pi}_1-\bar{\theta}_{22}\pi_0-\bar{\theta}_{24})}{r_1\bar{\theta}_{13}^2+d_1\bar{\theta}_{23}^2}.$$

The terms in parentheses represent deviations of the unemployment rate and inflation rate from their targets which are expected conditional on $P_1 = 0$ (i.e., conditional on policy being set according to historical trend). In the random shocks model, when there are as many policy control variables as goal variables, optimal policy totally closes the gaps between the expected values of goal variables and their targeted values. When there are more goal variables than policy control variables, all the gaps cannot be closed. The expression above for $\tilde{P}_1$ indicates that optimal policy is a linear combination of the policy which totally closes the unemployment gap and the policy which totally closes the inflation gap, where the linear weights depend on the relative importance to the policymaker of hitting each target (r_1, d_1) and on the relative potency of policy in affecting the goal variables $(\bar{\theta}_{13}, \bar{\theta}_{23})$. That is, let

$$\tilde{P}_X = \frac{\hat{X}_1 - \bar{\theta}_{11} X_0 - \bar{\theta}_{14}}{\bar{\theta}_{13}}$$

and

$$\tilde{P}_\pi = \frac{\hat{\pi}_1 - \bar{\theta}_{22} \pi_0 - \bar{\theta}_{24}}{\bar{\theta}_{23}},$$

so that

$$E_0(X_1 | \tilde{P}_X) - \hat{X}_1 = \bar{\theta}_{11} X_0 + \bar{\theta}_{13} \tilde{P}_X + \bar{\theta}_{14} - \hat{X}_1 = 0$$

and

$$E_0(\pi_1 | \tilde{P}_\pi) - \hat{\pi}_1 = \bar{\theta}_{22} \pi_0 + \bar{\theta}_{23} \tilde{P}_\pi + \bar{\theta}_{24} - \hat{\pi}_1 = 0.$$

Optimal policy $\tilde{P}_1$ can be written as a linear combination of $\tilde{P}_X$ and $\tilde{P}_\pi$:

$$\tilde{P}_1 = \alpha\tilde{P}_X + (1-\alpha)\tilde{P}_\pi,$$

where

$$\alpha = \frac{r_1\bar{\theta}_{13}^2}{r_1\bar{\theta}_{13}^2+d_1\bar{\theta}_{23}^2}.$$

Maximum utility is found by substituting $\tilde{P}_1$ into (1) and by noting

$$\begin{aligned} E_0(X_1|\tilde{P}_1) &= \bar{\theta}_{11}X_0 + \bar{\theta}_{13}(\alpha\tilde{P}_X+(1-\alpha)\tilde{P}_\pi) + \bar{\theta}_{14} \\ &= \bar{\theta}_{11}X_0 + \bar{\theta}_{13}\tilde{P}_X + \bar{\theta}_{14} + (1-\alpha)\bar{\theta}_{13}(\tilde{P}_\pi-\tilde{P}_X) \\ &= \hat{X}_1 + (1-\alpha)\bar{\theta}_{13}(\tilde{P}_\pi-\tilde{P}_X) \end{aligned}$$

and

$$\begin{aligned} E_0(\pi_1|\tilde{P}_1) &= \bar{\theta}_{22}\pi_0 + \bar{\theta}_{23}(\alpha\tilde{P}_X+(1-\alpha)\tilde{P}_\pi) + \bar{\theta}_{24} \\ &= \bar{\theta}_{22}\pi_0 + \bar{\theta}_{23}\tilde{P}_\pi + \bar{\theta}_{24} + \alpha\bar{\theta}_{23}(\tilde{P}_X-\tilde{P}_\pi) \\ &= \hat{\pi}_1 + \alpha\bar{\theta}_{23}(\tilde{P}_X-\tilde{P}_\pi), \end{aligned}$$

which yields

$$\begin{aligned} E_0\tilde{U} = -r_1\sigma_{14}^2 - d_1\sigma_{24}^2 - \left(\frac{r_1d_1}{r_1\bar{\theta}_{13}^2+d_1\bar{\theta}_{23}^2}\right)(\bar{\theta}_{23}(\hat{X}_1-\bar{\theta}_{11}X_0-\bar{\theta}_{14}) \\ - \bar{\theta}_{13}(\hat{\pi}_1-\bar{\theta}_{22}\pi_0-\bar{\theta}_{24}))^2. \end{aligned}$$

Thus, in addition to the loss in expected utility caused by the irreducible variances of the goal variables, there is a loss caused by not being able to set the expected values of both goal variables at their target values.

This model satisfies the conditions for certainty equivalence--note on page 26 that $\sigma^2_{X_1}$ and $\sigma^2_{\pi_1}$ are independent of P_1--so that optimal policy also can be found by:

$$(1) \qquad \max_{P_1} UE_0(X_1,\pi_1) = -r_1(\bar{X}_1-\hat{X}_1)^2 - d_1(\bar{\pi}_1-\hat{\pi}_1)^2 \equiv U(\bar{X}_1,\bar{\pi}_1)$$

subject to

$$(2a) \qquad \bar{X}_1 = \bar{\theta}_{11}X_0 + \bar{\theta}_{13}P_1 + \bar{\theta}_{14}$$

$$(2b) \qquad \bar{\pi}_1 = \bar{\theta}_{22}\pi_0 + \bar{\theta}_{23}P_1 + \bar{\theta}_{24},$$

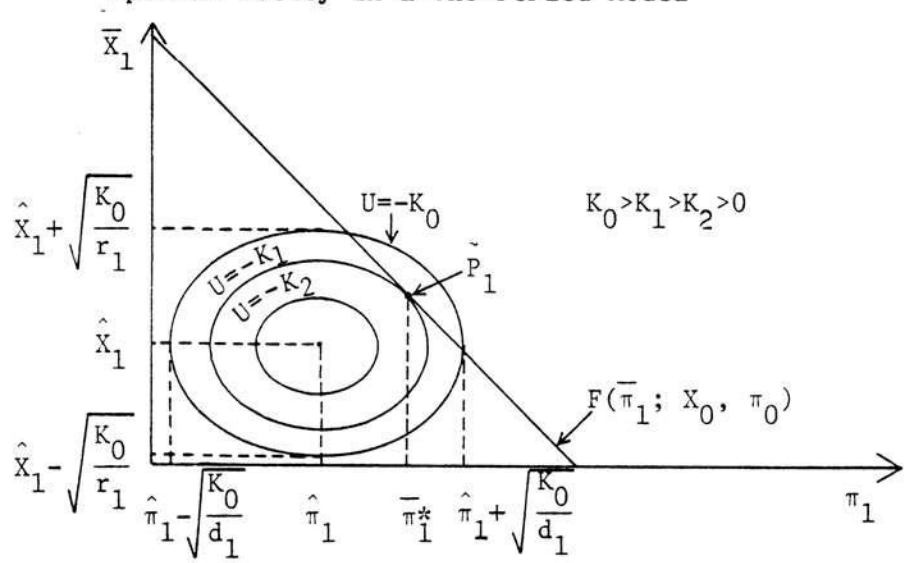
where $(\bar{\cdot}) \equiv E_0(\cdot)$. Solving for P_1 in (2b) and substituting the expression into (2a) yields a linear Phillips curve:

$$(2') \qquad \bar{X}_1 = \frac{\bar{\theta}_{13}}{\bar{\theta}_{23}}\bar{\pi}_1 + \bar{\theta}_{11}X_0 - \frac{\bar{\theta}_{13}\bar{\theta}_{22}}{\bar{\theta}_{23}}\pi_0 + \left(\bar{\theta}_{14} - \frac{\bar{\theta}_{13}\bar{\theta}_{24}}{\bar{\theta}_{23}}\right) \equiv F(\bar{\pi}_1;X_0,\pi_0).$$

Any point on this Phillips curve can be attained with an appropriate choice of P_1.

For $U(\bar{X}_1,\bar{\pi}_1) = -K_0 < 0$, an indifference curve is a rectangular ellipse with center $<\hat{X}_1,\hat{\pi}_1>$. Thus, in the $\bar{X}_1 - \bar{\pi}_1$ plane we have the following (assuming $<\hat{X}_1,\hat{\pi}_1>$ is southwest of F):

Figure 1
Optimal Policy in a One-Period Model



Optimal policy can be found by maximizing $U(\bar{X}_1,\bar{\pi}_1)$ with respect to $<\bar{X}_1,\bar{\pi}_1>$ subject to (2') and then by substituting the maximizing value of $\bar{\pi}_1$--call it $\bar{\pi}_1^*$--into (2b). The maximizing values of $\bar{X}_1$ and $\bar{\pi}_1$ are located where an indifference curve is tangent to the Phillips curve.

2. Two-period horizon ($r_2 \neq 0$ and/or $d_2 \neq 0$).

The model is stated on page 25. By the Bellman principle $\tilde{P}_2$ is found by maximizing

$$-E_1\{r_2(X_2-\hat{X}_2)^2+d_2(\pi_2-\hat{\pi}_2)^2\}$$

with respect to P_2 subject to

$$X_2 = \bar{\theta}_{11}X_1 + \bar{\theta}_{13}P_2 + \theta_{14}(2)$$

and

$$\pi_2 = \bar{\theta}_{22}\pi_1 + \bar{\theta}_{23}P_2 + \theta_{24}(2).$$

By our one-period model we know the solution to this problem is:

$$\tilde{P}_2 = \frac{r_2\bar{\theta}_{13}(\hat{X}_2-\bar{\theta}_{11}X_1-\bar{\theta}_{14})+d_2\bar{\theta}_{23}(\hat{\pi}_2-\bar{\theta}_{22}\pi_1-\bar{\theta}_{24})}{r_2\bar{\theta}_{13}^2+d_2\bar{\theta}_{23}^2},$$

and

$$\tilde{U}_2(I_1) = -r_2\sigma_{14}^2 - d_2\sigma_{24}^2 - \left(\frac{r_2d_2}{r_2\bar{\theta}_{13}^2+d_2\bar{\theta}_{23}^2}\right)(\bar{\theta}_{23}(\hat{X}_2-\bar{\theta}_{11}X_1-\bar{\theta}_{14})$$

$$- \bar{\theta}_{13}(\hat{\pi}_2-\bar{\theta}_{22}\pi_1-\bar{\theta}_{24}))^2.$$

(This is the one-period solution with time subscripts moved up one period.)

Optimal first-period policy is found by maximizing with respect to P_1

$$-E_0\{r_1(X_1-\hat{X}_1)^2+d_1(\pi_1-\hat{\pi}_1)^2\} + E_0\tilde{U}_2(I)$$

subject to

$$X_1 = \bar{\theta}_{11}X_0 + \bar{\theta}_{13}P_1 + \theta_{14}(1)$$

and

$$\pi_1 = \bar{\theta}_{22}\pi_0 + \bar{\theta}_{23}P_1 + \theta_{24}(1).$$

Notice that $E_0\tilde{U}_2(I_1)$ is a function of P_1, so that optimal first-period policy must take into account effects of P_1 on the attainable set of future unemployment and inflation rates as well as on the trade-off between unemployment and inflation in the current period.

Substituting from the constraints, we can write the objective function as:

$$E_0U = -(r_1+r_2)\sigma_{14}^2 - (d_1+d_2)\sigma_{24}^2 - r_1(\bar{\theta}_{11}X_0+\bar{\theta}_{13}P_1+\bar{\theta}_{14}-\hat{X}_1)^2$$

$$- d_1(\bar{\theta}_{22}\pi_0+\bar{\theta}_{23}P_1+\bar{\theta}_{24}-\hat{\pi}_1)^2$$

$$- \left(\frac{r_2d_2}{r_2\bar{\theta}_{13}^2+d_2\bar{\theta}_{23}^2}\right)\cdot\{[A^2+C^2(\sigma_{14}^2+\bar{\theta}_{14}^2)+D^2(\sigma_{24}^2+\bar{\theta}_{24})^2+2AC\bar{\theta}_{14}$$

$$+ 2AD\bar{\theta}_{24}+2CD\bar{\theta}_{14}\bar{\theta}_{24}]+2B(A+C\bar{\theta}_{14}+D\bar{\theta}_{24})P_1+B^2P_1^2\},$$

where

$$A \equiv \bar{\theta}_{23}(\hat{X}_2-\bar{\theta}_{11}^2X_0-\bar{\theta}_{14}) - \bar{\theta}_{13}(\hat{\pi}_2-\bar{\theta}_{22}^2\pi_0-\bar{\theta}_{24})$$

$$B \equiv \bar{\theta}_{13}\bar{\theta}_{23}(\bar{\theta}_{22}-\bar{\theta}_{11})$$

$$C \equiv -\bar{\theta}_{11}\bar{\theta}_{23}$$

$$D \equiv \bar{\theta}_{13}\bar{\theta}_{22}.$$

Differentiating E_0U with respect to P_1 and setting the expression equal to zero yields:

$$\tilde{P}_1 = \frac{r_1\bar{\theta}_{13}(\hat{X}_1-\bar{\theta}_{11}X_0-\bar{\theta}_{14})+d_1\bar{\theta}_{23}(\hat{\pi}_1-\bar{\theta}_{22}\pi_0-\bar{\theta}_{24})-R\cdot B\cdot E}{r_1\bar{\theta}_{13}^2+d_1\bar{\theta}_{23}^2+R\cdot B^2},$$

where

$$R \equiv \left(\frac{r_2d_2}{r_2\bar{\theta}_{13}^2+d_2\bar{\theta}_{23}^2}\right),$$

and

$$E \equiv A + C\bar{\theta}_{14} + D\bar{\theta}_{24}$$

$$\equiv \bar{\theta}_{23}(\hat{X}_2-\bar{\theta}_{11}^2X_0-\bar{\theta}_{14}) - \bar{\theta}_{13}(\hat{\pi}_2-\bar{\theta}_{22}^2\pi_0-\bar{\theta}_{24}) - \bar{\theta}_{11}\bar{\theta}_{14}\bar{\theta}_{23} + \bar{\theta}_{13}\bar{\theta}_{22}\bar{\theta}_{24}.$$

No attempt is made to evaluate $E_0\tilde{U}$; however, it is instructive to compare the optimal policy $\tilde{P}_1$ to the best myopic policy P_1^m:

$$P_1^m = \frac{r_1\bar{\theta}_{13}(\hat{X}_1-\bar{\theta}_{11}X_0-\bar{\theta}_{14})+d_1\bar{\theta}_{23}(\hat{\pi}_1-\bar{\theta}_{22}\pi_0-\bar{\theta}_{24})}{r_1\bar{\theta}_{13}^2+d_1\bar{\theta}_{23}^2}.$$

The two policies are the same whenever R or B are zero. R is zero if either of the second-period discount factors, r_2 or d_2, is zero. But if either of these is zero, the policymaker has only one goal variable in the second period, and the choice of first-period policy in no way restricts his ability to hit the second-period target. B is zero if either $\bar{\theta}_{13}$ or $\bar{\theta}_{23}$ is zero or if $\bar{\theta}_{11}$ is equal to $\bar{\theta}_{22}$. If $\bar{\theta}_{13}$ or $\bar{\theta}_{23}$ is zero, policy has no effect on one of the goal variables. The policymaker need only concern himself with the goal variable which he can influence, and we are back in the situation of having as many policy control variables as goal variables. When B is zero because $\bar{\theta}_{11}$ is equal to $\bar{\theta}_{22}$, first-period

policy does not affect the attainable pairs of unemployment and inflation rates in the second period. In all other cases where $R \neq 0$ and $B \neq 0$ there is a loss in expected utility caused by neglecting the effect of first-period policy on the attainable set of unemployment and inflation rate pairs in the second period. In other words, there is generally a welfare loss in following a myopic policy.

These results are, perhaps, better illustrated under the certainty equivalence method of solution. Optimal first-period policy can be found by maximizing with respect to P_1 and P_2

$$(1') \qquad UE_0(X_1, X_2, \pi_1, \pi_2) = -r_1(\bar{X}_1 - \hat{X}_1)^2 - r_2(\bar{X}_2 - \hat{X}_2)^2 - d_1(\bar{\pi}_1 - \hat{\pi}_1)^2 - d_2(\bar{\pi}_2 - \hat{\pi}_2)^2 \equiv U(\bar{X}_1, \bar{X}_2, \bar{\pi}_1, \bar{\pi}_2),$$

where $(\bar{\cdot}) = E_0(\cdot)$ subject to

$$(2a) \qquad \bar{X}_1 = \bar{\theta}_{11} X_0 + \bar{\theta}_{13} P_1 + \bar{\theta}_{14}$$

$$(2b) \qquad \bar{X}_2 = \bar{\theta}_{11}^2 X_0 + \bar{\theta}_{11}\bar{\theta}_{13} P_1 + \bar{\theta}_{11}\bar{\theta}_{14} + \bar{\theta}_{13} P_2 + \bar{\theta}_{14}$$

$$(2c) \qquad \bar{\pi}_1 = \bar{\theta}_{22} \pi_0 + \bar{\theta}_{23} P_1 + \bar{\theta}_{24}$$

$$(2d) \qquad \bar{\pi}_2 = \bar{\theta}_{22}^2 \pi_0 + \bar{\theta}_{22}\bar{\theta}_{23} P_1 + \bar{\theta}_{22}\bar{\theta}_{24} + \bar{\theta}_{23} P_2 + \bar{\theta}_{24}.$$

Let us write

$$(1') \qquad U(\bar{X}_1, \bar{X}_2, \bar{\pi}_1, \bar{\pi}_2) \equiv U_1(\bar{X}_1, \bar{\pi}_1) + U_2(\bar{X}_2, \bar{\pi}_2)$$

where

$$U_t(\bar{X}_t, \bar{\pi}_t) \equiv -r_t(\bar{X}_t - \hat{X}_t)^2 - d_t(\bar{\pi}_t - \hat{\pi}_t)^2$$

is the tth-period utility mapping. Substituting for P_1 in 2a and 2c and combining equations, we derive a linear first-period Phillips curve:

$$(3) \qquad \bar{X}_1 = \frac{\bar{\theta}_{13}}{\bar{\theta}_{23}}\bar{\pi}_1 + \bar{\theta}_{11}X_0 - \frac{\bar{\theta}_{11}\bar{\theta}_{22}}{\bar{\theta}_{23}}\pi_0 + \left(\bar{\theta}_{14} - \frac{\bar{\theta}_{13}\bar{\theta}_{24}}{\bar{\theta}_{23}}\right)$$

$$\equiv F^1(\bar{\pi}_1; X_0, \pi_0).$$

Similarly, by substituting for P_2 in 2b and 2d and combining equations, we derive a linear second-period Phillips curve:

$$(4) \qquad \bar{X}_2 = \frac{\bar{\theta}_{13}}{\bar{\theta}_{23}}\bar{\pi}_2 + \bar{\theta}_{11}^2 X_0 - \left(\frac{\bar{\theta}_{11}\bar{\theta}_{13}\bar{\theta}_{22}}{\bar{\theta}_{23}}\right)\pi_0 + \left(\frac{\bar{\theta}_{13}}{\bar{\theta}_{23}}\right)(\bar{\theta}_{11}-\bar{\theta}_{22})\bar{\pi}_1$$

$$+ [\bar{\theta}_{11}\bar{\theta}_{14} - \frac{\bar{\theta}_{13}\bar{\theta}_{24}}{\bar{\theta}_{23}}(1+\bar{\theta}_{11})]$$

$$\equiv F^2(\bar{\pi}_2; X_0, \pi_0, \bar{\pi}_1).$$ 13/

Notice the location of the second-period Phillips curve is affected by the first-period policy choice by the term for $\bar{\pi}_1$. The effect is zero only if $\bar{\theta}_{13}$ is zero or if $\bar{\theta}_{11} = \bar{\theta}_{22}$.

Optimal first-period policy can be found by maximizing

$$(1') \qquad U_1(\bar{X}_1, \bar{\pi}_1) + U_2(\bar{X}_2, \bar{\pi}_2)$$

with respect to $\bar{X}_1$, $\bar{\pi}_1$, $\bar{X}_2$, $\bar{\pi}_2$ subject to (3) and (4), and then by substituting the maximizing value of $\bar{\pi}_1$--call it $\bar{\pi}_1^*$--into 2c. (See figure 2.) The maximizing values of the policy control variable found by this method, $\tilde{\tilde{P}}_1 = \tilde{g}_1(I_0)$ and $\tilde{\tilde{P}}_2 = \tilde{\tilde{g}}_2(I_0)$, are related to the optimal feedback policies $\tilde{P}_1 = \tilde{f}_1(I_0)$ and $\tilde{P}_2 = \tilde{f}_2(I_1)$ by $\tilde{\tilde{P}}_1 = \tilde{P}_1$ and $\tilde{\tilde{P}}_2 = E_0\tilde{P}_2$. That is, the certainty equivalence solution is a nonfeedback policy which sets the policy control variable in the first period at the value implied by the optimal feedback policy and in the second period at the

expected value of the optimal feedback policy. Since $\tilde{P}_2$ is a unique maximizer to the policy problem, $\tilde{\tilde{P}}_2$ is optimal if, and only if, $\tilde{P}_2 = \tilde{\tilde{P}}_2$; that is, if and only if

$$\tilde{P}_2 - \tilde{\tilde{P}}_2 = \phi_1(X_1 - E_0X_1) + \phi_2(\pi_1 - E_0\pi_1) = 0, \qquad \text{(see page 30)}$$

where

$$\phi_1 \equiv \frac{-r_2\bar{\theta}_{11}\bar{\theta}_{13}}{r_2\bar{\theta}_{13}^2 + d_2\bar{\theta}_{23}^2} \qquad\qquad \phi_2 \equiv \frac{-d_2\bar{\theta}_{22}\bar{\theta}_{23}}{r_2\bar{\theta}_{13}^2 + d_2\bar{\theta}_{23}^2}.$$

Thus, $\tilde{\tilde{P}}_2$ is optimal, if and only if, the realized values of the goal variables in the first period are exactly as expected or the forecast errors weighted by ϕ_1 and ϕ_2 are exactly offsetting. Either of these events has zero probability of occurring, so in general, the certainty equivalence solution requires that second-period policy be revised based on new information according to the formula for $\tilde{P}_2 - \tilde{\tilde{P}}_2$. (See figure 3.)

An algorithm for finding the certainty equivalence solution involves the following steps:

(1) Based on information at time t=0, generate the set of all feasible forecasts over the entire horizon conditional on policy sequences $\langle P_1, P_2 \rangle$.

(2) From the set of feasible forecasts, choose the most desirable: $\langle \bar{X}_1^*, \bar{\pi}_1^*, \bar{X}_2^*, \bar{\pi}_2^* \rangle$.

(3) The optimizers in (2) imply values for the policy control variables $\langle \tilde{\tilde{P}}_1, \tilde{\tilde{P}}_2 \rangle$. $\tilde{\tilde{P}}_1$ is optimal $(=\tilde{P}_1)$.

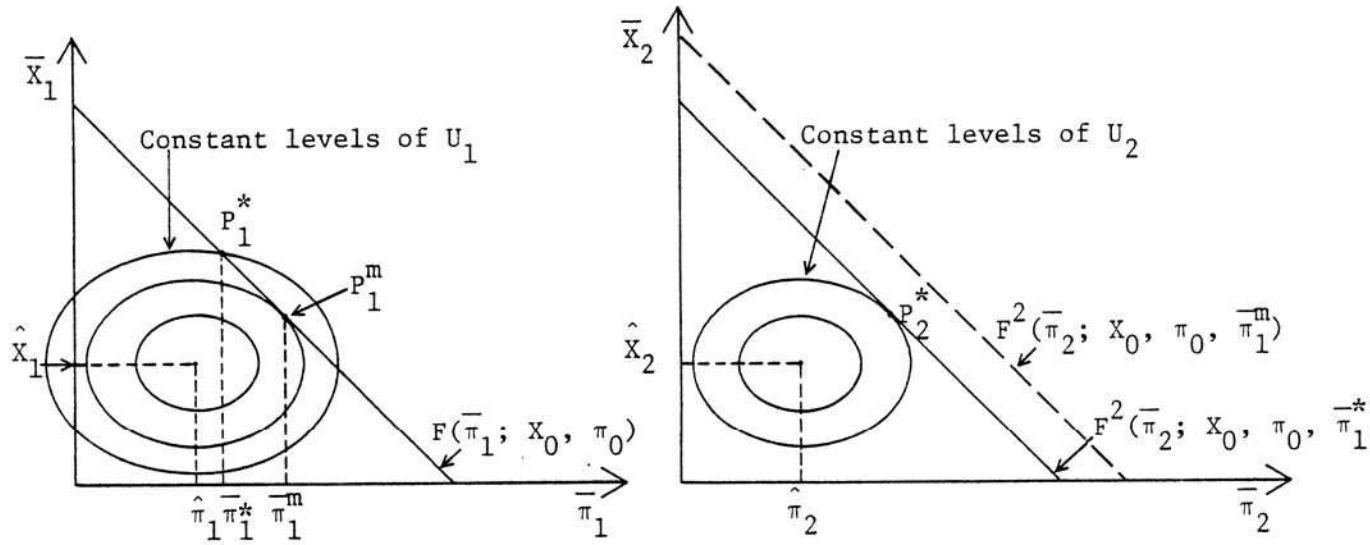
(4) When the second period comes, information is received on the realized values of X_1 and π_1. Based on this new information, generate the set of all feasible forecasts in period two conditional on P_2.

(5) From this new set of feasible forecasts, choose the most desirable: $\langle \bar{X}_2^{**}, \bar{\pi}_2^{**} \rangle$.

(6) The optimizers in (5) imply a value for P_2--call it $\tilde{\tilde{P}}_2'$. $\tilde{\tilde{P}}_2'$ is optimal $(=\tilde{P}_2)$.

This solution method easily generalizes for models with more than two periods.

Figure 2
Optimal First-Period Policy in a Two-Period Model

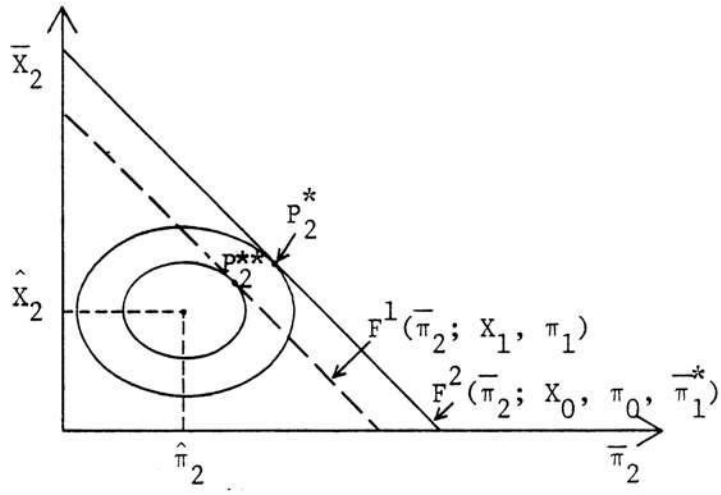


Note: Expectations are conditional on I_0. Constant levels of utility, defined by $U_t = -K_0 > 0$ t=1,2, are rectangular ellipses centered at $<\hat{\pi}_t, \hat{X}_t>$ with poles

$$\hat{\pi}_t \pm \sqrt{\frac{K_0}{d_t}} \text{ and } \hat{X}_t \pm \sqrt{\frac{K_0}{r_t}}.$$

The solution $<\tilde{\tilde{P}}_1, \tilde{\tilde{P}}_2>$ maximizes $U_1 + U_2$, while P_1^m maximizes U_1.

Figure 3
Optimal Second-Period Policy in a Two-Period Model



Note: For F^2 expectations are conditional on I_0, while for F^1 they are conditional on I_1.

C. Multiple Candidates for Policy Control Variables

Policymakers generally have a choice of policy control variables: they can choose to control either a quantity or a price. The Federal Reserve, for instance, can set the quantity of securities in its portfolio and let interest rates be determined by the market, or alternatively it can set the value of an interest rate and let its portfolio be market determined.[11]

When there are multiple candidates for policy control variables, the policy problem becomes more complicated. Its solution is still a rule, but one that specifies for each period both the choice of policy control variable and the value at which it is to be set--all as a function of information at the beginning of the period. The solution can be found as before by applying the Bellman principle, but for each period the economic process now is presented by one of two reduced forms. If a quantity variable is controlled, the economic process will be represented by one reduced form. And if a price variable is controlled, it will be represented by another.

The problem generally can be solved as follows. For the last period of the policy horizon, expected utility, conditional on information available at the beginning of the period, is maximized assuming first one choice of policy control variable and then the other. The solution to each maximization problem is a rule which describes how a given candidate for policy control variable should be set based on information available at the beginning of the period. Under each rule expected utility can be expressed as a function of initial information. The candidate for policy control variable which implies the greatest level of expected utility when set according to its rule is the one which should be chosen. In general, this choice will depend on initial conditions, that is, values of variables in the previous period.

Following the Bellman procedure, the solution routine is repeated for the period preceding the final period. And so on, until we reach the first period. In any given period, we solve for the optimal rule for each candidate for policy control variable. The rule which implies the highest level of expected utility indicates which candidate should be chosen. The choice, in general, depends on initial conditions.

To illustrate these concepts we will consider a one-period, one-goal variable, one-policy control variable, random shocks model. In this model the choice of policy control variable does not depend on initial conditions. This means that if the horizon were extended, the choice of policy control variable could be decided once and for all for all periods, and the optimal policy rule would be as described in IA.

Suppose the economic model in structural form can be written[15/]

$$\text{(IS)}\ X_t = aX_{t-1} - br_t + \varepsilon_t$$

(equilibrium in the goods market where X is income, r is the interest rate, and the underlying demand schedules for consumption C and investment I are given by

(a) $C_t = aX_{t-1}$

and

(b) $I_t = -br_t + \varepsilon_t)$

(LM) $r_t = AX_t - BM_t + \mu_t$

(equilibrium in the money market where M is the stock of money and the underlying demand schedule for money is given by

$$M_t = \frac{A}{B}X_t - \frac{1}{B}r_t + \frac{1}{B}\mu_t).$$

All coefficients are assumed to be positive. The random disturbance terms are assumed to have the following first and second moments:

$$E_s\begin{pmatrix}\varepsilon_t\\ \mu_t\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix},\ E_s\begin{pmatrix}\varepsilon_t\\ \mu_t\end{pmatrix}(\varepsilon_t,\mu_t) = \begin{pmatrix}\sigma_\varepsilon^2 & \sigma_{\varepsilon\mu}\\ \sigma_{\varepsilon\mu} & \sigma_\mu^2\end{pmatrix},$$

where $\sigma_\varepsilon^2 > 0$ and $\sigma_\mu^2 > 0$ for $s < t$.

The policymaker's objective is to maximize

(1) $E_0U = -E_0(X_1-\hat{X}_1)^2$

subject to (2) the economic process and (3) available information X_0. If r is the policy control variable, (IS) is the reduced form of the model, and we can write the economic process as

(2) $X_t = \theta_1(t)\cdot X_{t-1} + \theta_2(t)\cdot r_t + \theta_3(t)$

where

$$\theta(t) \equiv (\theta_1(t),\theta_2(t),\theta_3(t)) = (a,-b,\varepsilon_t),$$

(2a) $$\overline{\theta(t)}_s = (\bar{\theta}_1,\bar{\theta}_2,\bar{\theta}_3) = (a,-b,0) \quad t > s,$$

and

(2b) $$\sum(t,t')_s = \begin{cases} 0_{3x3} & t \neq t' \text{ or } t \leqq s \\ \left(\begin{array}{cc|c} & & 0 \\ & 0 & 0 \\ \hline 0 & 0 & \sigma_3^2 \end{array}\right) & , \sigma_3^2 = \sigma_\varepsilon^2 > 0 \quad t=t' > s. \end{cases}$$

From the results in IA we know that the optimal rule for r is given by

$$\tilde{r}_1 = \frac{\hat{X}_1-\bar{\theta}_3-\bar{\theta}_1 X_0}{\bar{\theta}_2} \equiv \frac{\hat{X}_1-aX_0}{-b}$$

and expected utility under this rule is given by

$$E_0\tilde{U}(r) = -\sigma_3^2 \equiv -\sigma_\varepsilon^2.$$

If M is the policy control variable, the reduced form of the model can be derived simply by substituting the expression for r_t from (LM) into (IS) to yield

$$X_t = (\frac{a}{1+Ab})X_{t-1} + (\frac{Bb}{1+Ab})M_t + (\frac{1}{1+Ab})(\varepsilon_t-b\mu_t).$$

Thus, when M is the policy control variable we can write the economic process as

(2) $$X_t = \theta_1'(t)\cdot X_{t-1} + \theta_2'(t)\cdot M_t + \theta_3'(t)$$

where

$$\theta'(t) \equiv (\theta_1'(t),\theta_2'(t),\theta_3'(t)) = (\frac{a}{1+Ab}, \frac{Bb}{1+Ab}, (\frac{1}{1+Ab})\ (\varepsilon_t-b\mu_t)),$$

(2a) $$\overline{\theta'(t)}_s = (\bar{\theta}_1',\bar{\theta}_2',\bar{\theta}_3') = (\frac{a}{1+Ab}, \frac{Bb}{1+Ab}, 0) \quad t > s,$$

and

$$\text{(2b)} \qquad \sum(t,t')_s = \begin{cases} 0_{3x3} & t \neq t' \text{ or } t \leqq s \\ \left(\begin{array}{cc|c} & & 0 \\ & 0 & \\ & & 0 \\ \hline 0 & 0 & \sigma_3^{2\prime} \end{array}\right) & , \sigma_3^{2\prime} > 0 \ t=t' > s. \end{cases}$$

The residual variance $\sigma_3^{2\prime} \equiv E_s(\theta_3'(t)-\overline{\theta_3'(t)})^2 = E_s(\theta_3'(t))^2$ $(s<t)$ in terms of the structural disturbances is given by

$$E_s[(\frac{1}{1+Ab})(\varepsilon_t-b\mu_t)]^2 = (\frac{1}{1+Ab})^2[\sigma_\varepsilon^2-2b\sigma_{\varepsilon\mu}+b^2\sigma_\mu^2].$$

Again, from IA we know that the optimal rule for M is given by

$$\tilde{M}_1 = \frac{\hat{X}_1-\bar{\theta}_3'-\bar{\theta}_1'X_0}{\bar{\theta}_2'} \equiv \frac{(1+Ab)\hat{X}_1-aX_0}{Bb}$$

and expected utility under this rule is given by

$$E_0\tilde{U}(M) = -\sigma_3^{2\prime} \equiv -(\frac{1}{1+Ab})^2[\sigma_\varepsilon^2-2b\sigma_{\varepsilon\mu}+b^2\sigma_\mu^2].$$

Thus, the interest rate should be the policy control variable with value set at $\tilde{r}_1$ if and only if

$$E_0\tilde{U}(r) \geqq E_0\tilde{U}(M) \iff$$

$$(*) \qquad -\sigma_\varepsilon^2 \geqq -(\frac{1}{1+Ab})^2[\sigma_\varepsilon^2-2b\sigma_{\varepsilon\mu}+b^2\sigma_\mu^2].$$

If this inequality is not satisfied, M should be the policy control variable and its value should be set at $\tilde{M}_1$.

Notice that the slope of the goods market equilibrium curve (Δr wrt ΔX) s_{IS} is $-\frac{1}{b}$ and that the slope of the money market equilibrium curve (Δr wrt ΔX) s_{LM} is A. We then can rewrite the expression (*) as

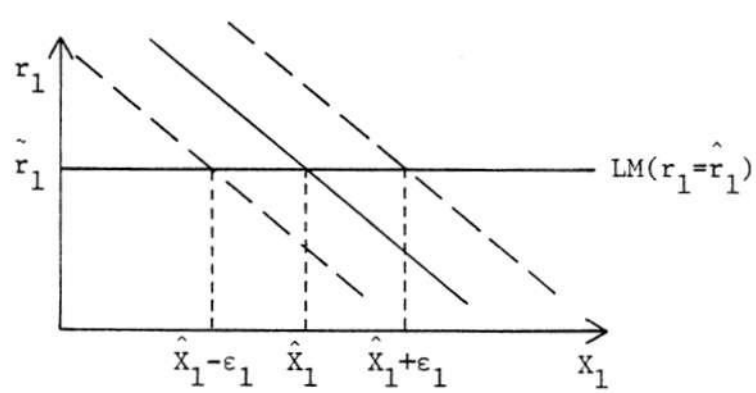
$$E_0\tilde{U}(r) \gtreqqless E_0\tilde{U}(M) \iff$$

$$\sigma_\varepsilon^2 \lesseqqgtr \left(\frac{1}{s_{LM}-s_{IS}}\right)^2 [s_{IS}^2\sigma_\varepsilon^2+2s_{IS}\sigma_{\varepsilon\mu}+\sigma_\mu^2].$$

Thus, the choice of policy control variable in this example depends only on the slopes of the market equilibrium curves and the variances and covariances of the structural disturbance terms; it does not depend on initial conditions X_0.

These results can be simply illustrated. Suppose r is the policy control variable:

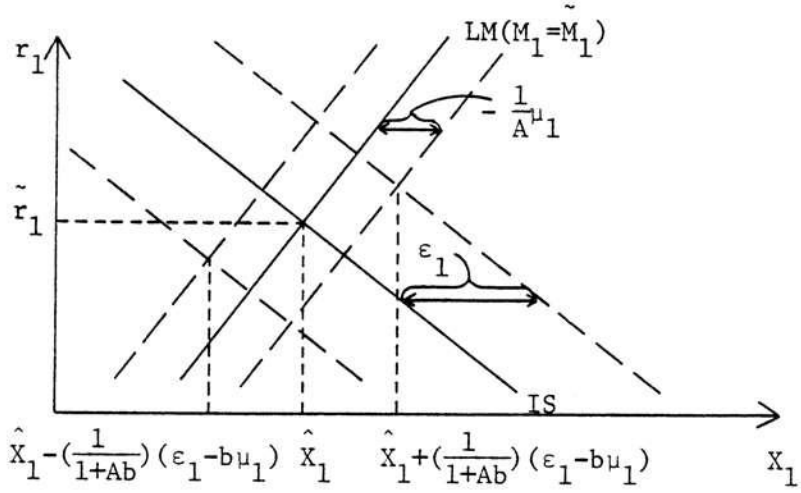
Figure 4



When r is fixed, random shifts in the investment schedule of magnitude ε_1 cause corresponding shifts in the IS curve and, thus, in equilibrium income. The expected squared deviation of income from its target given $\tilde{r}_1$ is then $E(\varepsilon_1^2) = \sigma_\varepsilon^2$.

Now suppose M is the policy control variable:

Figure 5



When M is fixed, random shifts in the investment schedule of magnitude ε_1 cause corresponding shifts in the IS curve as before. However, a shift in goods demand, say an increase, now causes money demand to increase and the interest rate to rise. This interest rate change moderates the effect on income of a random disturbance to investment demand. On the other hand, when M is fixed the interest rate can change also because of random disturbances to money demand. This type of disturbance then results in a change in equilibrium income, whereas it would have no effect if r were fixed.

As (*) and the diagrams illustrate, with everything else equal, the larger the variance of the LM curve relative to the IS curve (σ_μ^2 to σ_ε^2), the more likely r is a better choice of policy control variable than M. Similarly, the smaller the slope of the LM curve relative to a given slope of the IS curve, the more likely r is a better choice than M.

D. Information Lags

The models examined so far assume that at the beginning of each period policymakers receive a complete set of information on the economic outcome in the

previous period. Rarely in real life are policymakers so fortunate. The FOMC, for example, observes goal variables such as real GNP, the consumer price index, and the unemployment rate less frequently than financial variables such as interest rates and monetary aggregates.

How then should the FOMC or other agencies set policy when they have an incomplete set of information? The answer is the same as before. The settings of policy control variables should be based on the most recent observations of economic variables--no matter how incomplete that set of information is. Policymakers should not, as some have proposed, set target values for frequently observed variables which are not goals of policy.[16/]

We will consider a two-period, one-goal variable, one-policy control variable, random shocks model.[17/] We will assume that the economic structure is given essentially by the (IS) and (LM) curves in IC and that the interest rate has been shown to be the preferred policy control variable. This model differs from the two-period model in IA only with respect to the information assumption. In IA we assumed that the value of the goal variable in the first period X_1 is known by policymakers at the beginning of the second period. In the present model we assume that policymakers do not observe X_1 at that time but do observe M_1.

Our model can be written

(1) $$U = -(X_1-\hat{X}_1)^2 - \gamma(X_2-\hat{X}_2)^2,$$

(2.1) $$X_t = \theta_1(t)\cdot X_{t-1} + \theta_2(t)\cdot r_t + \theta_3(t)$$

(2.2) $$M_t = v_1(t)\cdot X_t + v_2(t)\cdot r_t + v_3(t)$$

(2a) $$E_s\begin{pmatrix}\theta(t)\\ v(t)\end{pmatrix} \equiv E_s\begin{pmatrix}\theta_1(t) & \theta_2(t) & \theta_3(t)\\ v_1(t) & v_2(t) & v_3(t)\end{pmatrix} = \begin{pmatrix}\bar{\theta}_1 & \bar{\theta}_2 & \bar{\theta}_3\\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3\end{pmatrix} t > s$$

(2b) $$E_s\begin{pmatrix}(\theta(t)-E_s\theta(t))'\\ (v(t)-E_s v(t))'\end{pmatrix}(\theta(t')-E_s\theta(t'), v(t')-E_s v(t'))$$

$$\equiv \begin{pmatrix}\Sigma_\theta(t,t') & \Sigma_{\theta v}(t,t')\\ \Sigma_{\theta v}(t,t') & \Sigma_v(t,t')\end{pmatrix} = 0 \ t \neq t' \text{ or } t \leqq s, \text{ and for } t>s$$

$$\Sigma_\theta(t,t) = \left(\begin{array}{cc|c} & & 0\\ & 0 & 0\\ \hline 0 & 0 & \sigma_\theta^2\end{array}\right)$$

$$\Sigma_{\theta v}(t,t) = \left(\begin{array}{cc|c} & & 0\\ & 0 & 0\\ \hline 0 & 0 & \sigma_{\theta v}\end{array}\right)$$

$$\Sigma_v(t,t) = \left(\begin{array}{cc|c} & & 0\\ & 0 & 0\\ \hline 0 & 0 & \sigma_v^2\end{array}\right), \text{ and}$$

(3) $$I_0 = \langle X_0, M_0, r_0\rangle, \ I_1 = \langle I_0; M_1, r_1\rangle.$$

Under this certainty equivalence, one-goal variable, one-policy control variable setup, we know from IA that in solving for optimal policy we can ignore variances and can proceed myopically. Thus, optimal policy is found by setting $E_0X_1 = \hat{X}_1$ and $E_1X_2 = \hat{X}_2$. For the first period we have

$$E_0X_1 = \bar{\theta}_1X_0 + \bar{\theta}_2\tilde{r}_1 + \bar{\theta}_3 = \hat{X}_1 \Rightarrow$$

$$\tilde{r}_1 = \frac{\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3}{\bar{\theta}_2}.$$

For the second period we have

$$E_1X_2 = \bar{\theta}_1E_1X_1 + \bar{\theta}_2\tilde{r}_2 + \bar{\theta}_3 = \hat{X}_2 \Rightarrow$$

$$\tilde{r}_2 = \frac{\hat{X}_2-\bar{\theta}_1E_1X_1-\bar{\theta}_3}{\bar{\theta}_2}.$$

Note that if I_1 includes X_1 as in IA, then

$$E_1X_1 \equiv E(X_1|I_1) = X_1$$

and the expression above for $\tilde{r}_2$ is the same as before.

In the present model we must forecast X_1 conditional on observations of X_0, $\tilde{r}_1$, and M_1. From (2.1) we have

$$E_1X_1 = \bar{\theta}_1X_0 + \bar{\theta}_2\tilde{r}_1 + E(\theta_3(1)|X_0,\tilde{r}_1,M_1).$$

Since $\theta_3(1)$ is distributed independently of X_0 and r_1, we need to determine $E(\theta_3(1)|M_1)$. This is the standard prediction problem, and by the orthogonality principle we have that

$$E(\theta_3(1)-E_0\theta_3(1)|M_1-E_0M_1) = \frac{\mathrm{cov}_0(\theta_3(1),M_1)}{\mathrm{var}_0M_1}\cdot(M_1-E_0M_1)$$

where

$$\mathrm{cov}_0(\theta_3(1),M_1) = E_0(\theta_3(1)-E_0\theta_3(1))(M_1-E_0M_1)$$

and

$$\mathrm{var}_0M_1 = E_0(M_1-E_0M_1)^2.$$

From (2.2) we have

$$M_1 - E_0M_1 = \bar{v}_1(X_1-E_0X_1) + (v_3(1)-\bar{v}_3),$$

which from (2.1) can be written

$$M_1 - E_0M_1 = \bar{v}_1(\theta_3(1)-\bar{\theta}_3) + (v_3(1)-\bar{v}_3).$$

Thus,

$$\begin{aligned}\mathrm{cov}_0(\theta_3(1),M_1) &\equiv E_0(\theta_3(1)-\bar{\theta}_3)(M_1-E_0M_1)\\ &= \bar{v}_1E_0(\theta_3(1)-\bar{\theta}_3)^2 + E_0(\theta_3(1)-\bar{\theta}_3)(v_3(1)-\bar{v}_3)\\ &= \bar{v}_1\sigma_\theta^2 + \sigma_{\theta v}\end{aligned}$$

and

$$\begin{aligned}\mathrm{var}_0M_1 &\equiv E_0(M_1-E_0M_1)^2\\ &= E_0[\bar{v}_1(\theta_3(1)-\bar{\theta}_3)+(v_3(1)-\bar{v}_3)]^2\\ &= \bar{v}_1^2\sigma_\theta^2 + 2\bar{v}_1\sigma_{\theta v} + \sigma_v^2.\end{aligned}$$

We then have

$$E(\theta_3(1)|X_0,\tilde{r}_1,M_1) = \bar{\theta}_3 + E(\theta_3(1)-\bar{\theta}_3|M_1-E_0M_1)$$

$$= \bar{\theta}_3 + \left(\frac{\bar{v}_1\sigma_\theta^2+\sigma_{\theta v}}{\bar{v}_1^2\sigma_\theta^2+2\bar{v}_1\sigma_{\theta v}+\sigma_v^2}\right)(M_1-E_0M_1)$$

$$\equiv \bar{\theta}_3 + \lambda(M_1-E_0M_1)$$

and

$$E_1X_1 = [\bar{\theta}_1X_0+\bar{\theta}_2\tilde{r}_1+\bar{\theta}_3] + [\lambda(M_1-E_0M_1)].$$

The forecast of X_1 conditional on I_1--E_1X_1--is simply the forecast of X_1 conditional on I_0--$\bar{\theta}_1X_0+\bar{\theta}_2\tilde{r}_1+\bar{\theta}_3$--plus an update term--$\lambda(M_1-E_0M_1)$--which is a constant times the prediction error in M_1 (defined as the actual value of M_1 less the value predicted at the beginning of the period).

We can now substitute our expression for E_1X_1 into the expression for $\tilde{r}_2$ to obtain

$$\tilde{r}_2 = \frac{\hat{X}_2-\bar{\theta}_1[\bar{\theta}_1X_0+\bar{\theta}_2\tilde{r}_1+\bar{\theta}_3+\lambda(M_1-E_0M_1)]-\bar{\theta}_3}{\bar{\theta}_2}$$

$$= \frac{\hat{X}_2-\bar{\theta}_1(\bar{\theta}_1X_0+\bar{\theta}_2\tilde{r}_1+\bar{\theta}_3)-\bar{\theta}_3-\bar{\theta}_1\left(\frac{\bar{v}_1\sigma_\theta^2+\sigma_{\theta v}}{\bar{v}_1^2\sigma_\theta^2+2\bar{v}_1\sigma_{\theta v}+\sigma_v^2}\right)(M_1-E_0M_1)}{\bar{\theta}_2}.$$

As long as $\bar{\theta}_1 \neq 0$ (current income is related to past income) and not both $\bar{v}_1 = 0$ and $\sigma_{\theta v} = 0$ (not both income elasticity of money demand is zero and disturbances to goods demand and money demand are uncorrelated), the expression above indicates that the prediction error $M_1 - E_0M_1$ should be used as information in setting $\tilde{r}_2$, and hence, M is referred to as an information variable. Note that optimal policy is not to set r_2 so that $E_1M_2 = E_0M_2$ or, in other words, to make M an intermediate target.

II. Stochastic Coefficients Model

We will turn now to models in which uncertainty about the economic structure plays an integral part in policy formulation. In the random shocks model uncertainty could essentially be ignored: optimal policy in the current period could be found by setting all stochastic terms at their means (conditional on current information) and solving the resulting deterministic maximization problem. This certainty equivalence result followed from two properties:

1. Expected utility depended only on first- and second-order moments of goal variables.
2. The second-order moments of goal variables were independent of settings of policy control variables.

Certainty equivalence does not extend to stochastic coefficients models, however. In these models the second-order moments of policy goal variables do depend on settings of policy control variables.

A. Inherent Uncertainty About the Effects of Policy on Goal Variables

We will consider a model with one goal variable and one policy control variable. As before, serially independent shocks to the economy cause there to be uncertainty about the intercept of the reduced form. However, we will assume now that there is also uncertainty about the value of the coefficient on the policy control variable. The distribution of the coefficient is assumed to be known and to be invariant over time. The invariance assumption precludes that the coefficient uncertainty is due to estimation, but instead requires it to be due to inherent randomness in the economic structure.

The stochastic coefficients model with inherent uncertainty can be expressed by equations (1)-(3) with[18/]

(1) $$\gamma = \frac{V_2}{V_1}$$

(2a) $$\overline{\theta(t)}_s = (\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3) \quad t > s$$

(2b) $$\sum(t,t')_s = \begin{cases} 0_{3x3} & t \neq t' \text{ or } t \leq s \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_2^2 & \sigma_{23} \\ 0 & \sigma_{32} & \sigma_3^2 \end{pmatrix}, \ \sigma_2^2, \ \sigma_3^2 > 0 & t = t' > s \end{cases}$$

(3) $$\underline{X}_1 = X_1.$$

This specification differs from the random shocks model in IA only in that the variance of θ_2 and the covariance of θ_2 and θ_3 now are not assumed to be zero.

Let us examine first the solution to the policymaking problem in the one-period case.

1. One-period horizon (γ=0).

The problem is

(1) $$\max_{P_1} -E_0(X_1 - \hat{X}_1)^2 = -\sigma_{X_1}^2 - (E_0 X_1 - \hat{X}_1)^2$$

subject to

(2) $$X_1 = \overline{\theta}_1 X_0 + \theta_2(1) P_1 + \theta_3(1)$$

where by (2a), (2b), and symmetry

$$E_0(\theta_2(1), \theta_3(1)) = (\overline{\theta}_2, \overline{\theta}_3)$$

and

$$E_0 \begin{pmatrix} \theta_2(1) - \overline{\theta}_2 \\ \theta_3(1) - \overline{\theta}_3 \end{pmatrix} (\theta_2(1) - \overline{\theta}_2, \theta_3(1) - \overline{\theta}_3) = \begin{pmatrix} \sigma_2^2 & \sigma_{23} \\ \sigma_{23} & \sigma_3^2 \end{pmatrix}.$$

We have by (2)

(i) $$E_0X_1 = \bar{\theta}_1X_0 + \bar{\theta}_2P_1 + \bar{\theta}_3.$$

Subtracting (i) from (2) we have

$$X_1 - E_0X_1 = (\theta_2(1)-\bar{\theta}_2)P_1 + (\theta_3(1)-\bar{\theta}_3),$$

so that

(ii) $$\sigma^2_{X_1} \equiv E_0(X_1-E_0X_1)^2 = E_0[(\theta_2(1)-\bar{\theta}_2)P_1+(\theta_3(1)-\bar{\theta}_3)]^2$$

$$= \sigma_2^2P_1^2 + 2P_1\sigma_{23} + \sigma_3^2.$$

Substituting (i) and (ii) into (1) the problem is simply

(iii) $$\max_{P_1} -\sigma_2^2P_1^2 - 2\sigma_{23}P_1 - \sigma_3^2 - (\bar{\theta}_1X_0+\bar{\theta}_2P_1+\bar{\theta}_3-\hat{X}_1)^2.$$

The first-order condition for maximization yields:

$$\tilde{P}_1 = \left(\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3) - \left(\frac{\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right).$$

It is instructive to compare this expression with the analogous expression in IA (p. 19). Both are of the form

$$\tilde{P}_1 = A\cdot(\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3) + B.$$

The term in parentheses is the gap between the target value of the goal variable and its expected value conditional on $P_1 = 0$. The coefficient A determines by how much the policy control variable should be adjusted in response to a unit increase in the gap. The intercept B determines the adjustment to P_1 which should be made regardless of the gap when there is nonzero covariance between θ_2 and θ_3.

The important point to note is that the absolute value of the coefficient A in the stochastic coefficients model is unambiguously smaller than the absolute value of the coefficient A in the random shocks model. More generally, the absolute value of A decreases continuously as the variance of θ_2 increases. This means that policymakers should become more cautious in responding to forecasted gaps in goal variables from their targets the larger is the uncertainty about the effects of policy.

The solution to the policymaking problem can be shown graphically in the mean-variance plane of X_1. Constant levels of expected utility (indifference curves) according to (1) are given by

$$E_0U = C = -\sigma^2_{X_1} - (\overline{X}_1-\hat{X}_1)^2,$$

where $\overline{X}_1 \equiv E_0X_1$. To derive the frontier to the policymakers' opportunity set in the mean-variance plane of X_1, we solve first for P_1 in (i) to get

$$P_1 = \alpha\overline{X}_1 + \beta, \text{ where } \alpha \equiv \frac{1}{\overline{\theta}_2} \text{ and } \beta = -\frac{(\overline{\theta}_1X_0+\overline{\theta}_3)}{\overline{\theta}_2}.$$

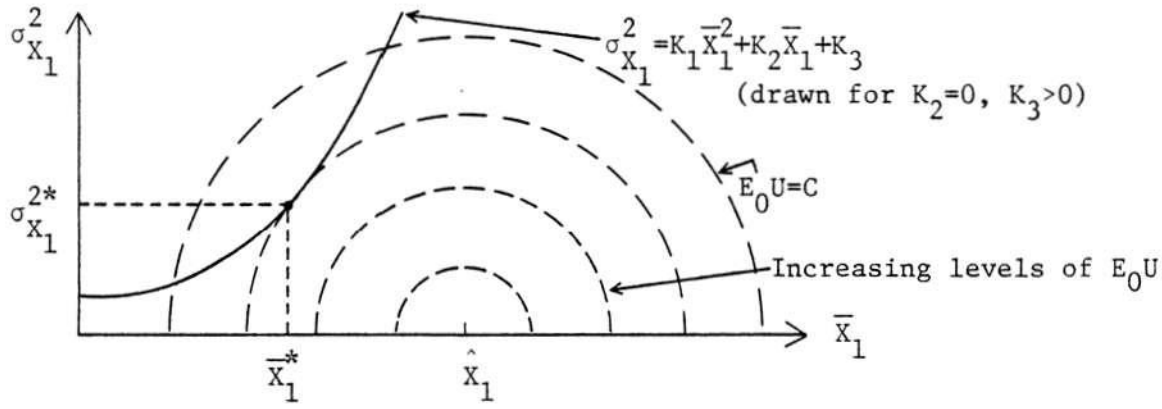
We then substitute this expression for P_1 into (ii) to get

$$\sigma^2_{X_1} = (\alpha^2\sigma^2_2)\cdot\overline{X}^2_1 + 2\alpha(\beta\sigma^2_2+\sigma_{23})\cdot\overline{X}_1 + (\beta^2\sigma^2_2+2\beta\sigma_{23}+\sigma^2_3)$$

$$\equiv K_1\overline{X}^2_1 + K_2\overline{X}_1 + K_3.$$

The expected utility maximizing mean-variance pair $(\overline{X}^*_1,\sigma^{2*}_{X_1})$ is located where the opportunity set frontier is tangent to an indifference curve. Given the maximizing value of $\overline{X}_1$, $\tilde{P}_1$ can be determined from (i).

Figure 6



In the random shocks model σ_2^2 and σ_{23} are zero so that the frontier to the mean-variance opportunity set is simply the horizontal line $\sigma^2_{X_1} = \sigma_3^2$. The tangency of the frontier and indifference curve is then at $\bar{X}_1 = \hat{X}_1$. As the variance of θ_2 grows, the frontier becomes steeper and the tangency point moves further away from $\hat{X}_1$.

Maximum utility is found by substituting the expression for $\tilde{P}_1$ into (iii) which yields

$$E_0\tilde{U} = -\left(\frac{\sigma_2^2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3)^2 - 2\left(\frac{\bar{\theta}_2\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_1-\bar{\theta}_1X_0-\bar{\theta}_3)$$

$$+\left(\frac{\sigma_{23}^2}{\sigma_2^2+\bar{\theta}_2^2} - \sigma_3^2\right).$$

When $\sigma_2^2 = 0$, the expression indicates that $E_0\tilde{U} = -\sigma_3^2$, which is the negative of the irreducible variance of X_1. When $\sigma_2^2 > 0$ and $\sigma_{23} = 0$, $E_0\tilde{U}$ is the sum of the first term and $-\sigma_3^2$, where the first term measures the expected loss from not closing the goal variable gap entirely.

Let us now turn to the two-period case.

2. Two-period horizon ($\gamma \neq 0$).

The problem is

$$(1) \qquad \max_{P_1, P_2} \; -E_0[(X_1-\hat{X}_1)^2+\gamma E_1(X_2-\hat{X}_2)^2]$$

subject to

$$(2) \qquad X_1 = \bar{\theta}_1 X_0 + \theta_2(1)\cdot P_1 + \theta_3(1)$$

$$X_2 = \bar{\theta}_1 X_1 + \theta_2(2)\cdot P_2 + \theta_3(2)$$

where by (2a) and (2b) for s=0, 1 and t>s

$$E_s[\theta_2(t),\theta_3(t)] = (\bar{\theta}_2,\bar{\theta}_3)$$

and

$$E_s\begin{pmatrix}\theta_2(t)-\bar{\theta}_2\\ \theta_3(t)-\bar{\theta}_3\end{pmatrix}(\theta_2(t')-\bar{\theta}_2,\theta_3(t')-\bar{\theta}_3)$$

$$= \begin{cases} 0_{3x3} & t\neq t' \\ \begin{pmatrix}\sigma_2^2 & \sigma_{23}\\ \sigma_{23} & \sigma_3^2\end{pmatrix} & t=t' \end{cases}$$

and (3) $\underline{X}_1 = X_1$.

By the Bellman principle $\tilde{P}_2$ is found by maximizing $-\gamma E_1(X_2-\hat{X}_2)^2$ with respect to P_2 subject to $X_2 = \bar{\theta}_1 X_1 + \theta_2(2)P_2 + \theta_3(2)$. As was the case in the random shocks model, this problem is precisely the same as the one-period problem with the subscripts moved up one period. Thus, it follows

$$\tilde{P}_2 = \left(\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2}\right)(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3) - \left(\frac{\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right)$$

and with $\sigma_{X_2}^2$ and E_1X_2 conditional on $\tilde{P}_2$,

$$\tilde{U}_2(I_1) \equiv -\gamma[\sigma_{X_2}^2+(E_1X_2-\hat{X}_2)^2]$$

$$= -\gamma[(1-H)(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)^2+2HK(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)+(\sigma_3^2-HK^2)],$$

where

$$H \equiv \frac{\bar{\theta}_2^2}{\sigma_2^2+\bar{\theta}_2^2}, \quad K \equiv \frac{\sigma_{23}}{\bar{\theta}_2}.$$

Optimal first-period policy is found by maximizing with respect to P_1

$$E_0U = -E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1)$$

subject to

$$X_1 = \bar{\theta}_1X_0+ \theta_2(1)P_1 + \theta_3(1)$$

and

$$\tilde{U}_2(I_1) = -\gamma[(1-H)(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)^2+2HK(\hat{X}_2-\bar{\theta}_1X_1-\bar{\theta}_3)+(\sigma_3^2-HK^2)].$$

As we found in the one-period case (expression (iii))

$$-E_0(X_1-\hat{X}_1)^2 = -\sigma_2^2P_1^2 - 2\sigma_{23}P_1 - \sigma_3^2 - (\bar{\theta}_1X_0+\bar{\theta}_2P_1+\bar{\theta}_3-\hat{X}_1)^2.$$

After some calculations and manipulations we find

$$E_0\tilde{U}_2(I_1) = -\gamma(a_0+2a_1P_1+a_2P_1^2)$$

where

$$a_0 = (1-H)[(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_3)^2-2\bar{\theta}_1\bar{\theta}_3(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_3)$$

$$+ \bar{\theta}_1^2(\sigma_3^2+\bar{\theta}_3^2)] + 2HK(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_1\bar{\theta}_3-\bar{\theta}_3) + (\sigma_3^2-HK^2),$$

$$a_1 = \bar{\theta}_1\{(1-H)[\bar{\theta}_1(\sigma_{23}+\bar{\theta}_2\bar{\theta}_3)-\bar{\theta}_2(\hat{X}_2-\bar{\theta}_1^2X_0-\bar{\theta}_3)]-HK\bar{\theta}_2\}$$

$$a_2 = (1-H)\bar{\theta}_1^2(\sigma_2^2+\bar{\theta}_2^2).$$

Differentiating E_0U with respect to P_1 and setting the resulting expression equal to zero yields

$$\tilde{P}_1 = \left(\frac{\bar{\theta}_2}{\sigma_2^2+\bar{\theta}_2^2}\right)\left[\frac{\hat{X}_1+\gamma(1-H)\bar{\theta}_1(\hat{X}_2-\bar{\theta}_3)}{1+\gamma(1-H)\bar{\theta}_1^2} - \bar{\theta}_1X_0 - \bar{\theta}_3\right]$$

$$- \left(\frac{\sigma_{23}}{\sigma_2^2+\bar{\theta}_2^2}\right)\left[1 - \frac{\gamma H\bar{\theta}_1}{1+\gamma(1-H)\bar{\theta}_1^2}\right].$$

It is interesting to compare the expression above with the expression for optimal policy in the one-period case. In general they are not the same. Thus, even with as many policy control variables as goal variables, policy cannot be made myopically when there is uncertainty about the effects of policy.

In our example optimal policy can be made myopically if $\hat{X}_2 = \bar{\theta}_1\hat{X}_1 + \bar{\theta}_3$ and $\sigma_{23} = 0$. There is no apparent reason why either of these equalities should hold. Suppose, however, that $\sigma_{23} = 0$. It then follows with $\bar{\theta}_2 > 0$ that the best myopic policy P_1^m is related to the best overall policy by $P_1^m \gtreqless \tilde{P}_1$ as $\hat{X}_2 \lesseqgtr \bar{\theta}_1\hat{X}_1 + \bar{\theta}_3$. Thus, the best myopic policy implicitly assumes that the dynamic structure of the economy is optimal: once the goal variable is on its target path, it assumes the economic process will keep it there. If the target for the goal variable in the future period is above (below) what the economic process could be expected to produce, the myopic policy will be too restrictive (expansionary) in the current period.

In general, policy cannot be made myopically because what is attainable in the second period depends on the value of the policy control variable in the first period. To ignore this dependence results in a loss in expected utility. From the one-period case we know that the mean-variance frontier in the second period conditional on X_1 can be written

$$\sigma^2_{X_2} = (\alpha^2\sigma_2^2)\cdot\overline{X}_2^2 + 2\alpha(\beta\sigma_2^2+\sigma_{23})\cdot\overline{X}_2 + (\beta^2\sigma_2^2+2\beta\sigma_{23}+\sigma_3^2),$$

where

$$\sigma^2_{X_2} = E_1(X_2-\overline{X}_2)^2$$

$$\overline{X}_2 = \overline{\theta}_1X_1 + \overline{\theta}_2P_2 + \overline{\theta}_3$$

$$\alpha \equiv \frac{1}{\overline{\theta}_2}$$

and

$$\beta \equiv -\frac{(\overline{\theta}_1X_1+\overline{\theta}_3)}{\overline{\theta}_2}.$$

The location of the frontier in the mean-variance plane depends on the level of X_1 as is seen in the expressions for $\overline{X}_2$ and β. The level of X_1, meanwhile, depends on P_1, although this dependence is random as of time zero. That is, the precise relationship which obtains between X_1 and P_1 depends on the realizations of $\theta_2(1)$ and $\theta_3(1)$. It is clear, however, that the location of the mean-variance frontier for X_2 which is expected conditional on I_0 depends on P_1. In order to maximize expected utility conditional on I_0, it is necessary, therefore, to recognize how the choice of P_1 restricts future policy choices.

B. Long and Variable Policy Lags

Milton Friedman (1969) argued that feedback policies are likely to be destabilizing when there are long and variable policy lags. There are at least

two interpretations of his argument. One is that he is referring to the likelihood of a policy drawn at random being destabilizing, while a second is that he is referring to an optimal policy.

Under the first interpretation, the argument is that because of insufficient knowledge or institutional constraints, feedback policies in practice are likely to deviate substantially from optimal policies.[19/] Since long and variable lags add variability to the economic structure, they then increase the likelihood that policies in practice will be destabilizing. If this is Friedman's argument, he is undoubtedly correct: the more variable the economic structure, the more likely a policy chosen at random will be destabilizing.

Under the second interpretation, the argument is that with sufficient variability in the economic structure even the best feedback policy will be destabilizing. The optimal policy then must be a nonfeedback rule, such as a constant growth rate for the Fed's portfolio. In this section we will examine the validity of this argument in a simple two-period version of a model by Fischer and Cooper (1973).

Fischer and Cooper first defined in the context of a particular model the notions "length of lag" and "variability of lag." They then used numerical solution techniques to evaluate different policies. They concluded that as long as the variances of lag weights are finite, feedback rules still dominate nonfeedback rules. However, the greater the variances in lag weights, the less policymakers should respond to new information.

Fischer and Cooper assumed an infinite horizon problem with lag weights on policy from a specific family of lag distributions:

$$\max_{P} E_0 U \equiv \max_{P_1, P_2, \ldots} E_0 \sum_{t=1}^{\infty} (X_t - \hat{X})^2$$

subject to

$$X_t = \beta X_{t-1} + \sum_{i=0}^{\infty} w_i(t) \cdot P_{t-i} + U(t)$$

where β is a known scalar and U is a random residual. They then defined the length of lag and the variability of lag with respect to the assumed family of lag distributions.[20/] While the model could be solved explicitly in the known lag case, it could not in the variable lag case.

The problems of definition and solution faced by Fischer and Cooper are not encountered in our two-period treatment of their model. We can express the model as:

$$(1) \qquad \max_{P_1, P_2} \; E_0 U = -E_0[(X_1 - \hat{X}_1)^2 + \gamma E_1 (X_2 - \hat{X}_2)^2]$$

subject to

$$(2) \qquad \begin{aligned} X_1 &= \theta_1(1) X_0 + \theta_2(1) P_1 + \theta_3(1) \\ X_2 &= \theta_1(2) X_1 + \theta_2(2) P_2 + \lambda P_1 + \theta_3(2) \end{aligned}$$

where

$$(2a) \qquad \overline{\theta(t)}_s = (\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3) \quad t>s$$

$$(2b) \qquad \sum(t,t')_s = \begin{cases} 0_{3x3} & t \neq t' \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix}, \; \sigma_2^2, \; \sigma_3^2 > 0 & t=t'>s, \end{cases}$$

λ has mean $\overline{\lambda}$, variance σ_λ^2, and is uncorrelated with the θ's, and

$$(3) \qquad \underline{X}_1 = X_1.$$

In order to make this model comparable to the one in IIA, we must substitute for P_1 in the second equation of (2) using the expression in the first equation of (2):

$$X_2 = \theta_1(2)X_1 + \theta_2(2)P_2 + \lambda\frac{(X_1-\theta_1(1)X_0-\theta_3(1))}{\theta_2(1)} + \theta_3(2)$$

or

$$X_2 = [\theta_1(2) + \frac{\lambda}{\theta_2(1)}]X_1 + \theta_2(2)P_2 + [\theta_3(2) - \lambda\frac{(\theta_1(1)X_0+\theta_3(1))}{\theta_2(1)}]$$

$$\equiv \theta_1'(2)X_1 + \theta_2'(2)P_2 + \theta_3'(2).$$

When we now compare this model to the one in IIA, we find two important differences. In general with $\bar{\lambda} \neq 0$ and $\sigma_\lambda^2 > 0$

(i) There is serial correlation in the residuals, $E_0\theta_3(1)\theta_3'(2) \neq 0$ and

(ii) The system lag $\theta_1'(2)$ becomes random even if $\theta_1(1)$ and $\theta_2(1)$ are known.

We are interested primarily in how the policy rule for $\tilde{P}_1$ changes as $\bar{\lambda}$ and σ_λ^2 change. We could solve the model in a general form allowing for serial correlation in residuals and a random system lag and then substitute specific terms for means and covariances using (2), (2a), and (2b). It is much simpler, however, to solve the model (1)-(3) directly.

We first solve for $\tilde{P}_2$ by maximizing $-\gamma E_1(X_2-\hat{X}_2)^2$ with respect to P_2 subject to

$$X_2 = \bar{\theta}_1X_1 + \theta_2(2)P_2 + \lambda P_1 + \theta_3(2).$$

$\tilde{P}_2$ must then minimize the sum $\sigma_{X_2}^2 + (\bar{X}_2-\hat{X}_2)^2$. From (2a) it follows

$$\bar{X}_2 = \bar{\theta}_1X_1 + \bar{\theta}_2P_2 + \bar{\lambda}P_1 + \bar{\theta}_3,$$

$$X_2 - \bar{X}_2 = (\theta_2(2)-\bar{\theta}_2)P_2 + (\lambda-\bar{\lambda})P_1 + (\theta_3(2)-\bar{\theta}_3),$$

and

$$\sigma^2_{X_2} = \sigma_2^2 P_2^2 + \sigma_\lambda^2 P_1^2 + \sigma_3^2.$$

Minimizing $\sigma^2_{X_2} + (\bar{X}_2 - \hat{X}_2)^2$ with respect to P_2 then yields:

$$\tilde{P}_2 = \left(\frac{\bar{\theta}_2}{\sigma_2^2 + \bar{\theta}_2^2}\right)(\hat{X}_2 - \bar{\theta}_1 X_1 - \bar{\lambda} P_1 - \bar{\theta}_3)$$

and

$$\tilde{U}_2(I_1) \equiv -\gamma E_1[X_2(\tilde{P}_2) - \hat{X}_2]^2 = -\gamma[\left(\frac{\sigma_2^2}{\sigma_2^2 + \bar{\theta}_2^2}\right)(\hat{X}_2 - \bar{\theta}_1 X_1 - \bar{\lambda} P_1 - \bar{\theta}_3)^2 + \sigma_\lambda^2 P_1^2 + \sigma_3^2].$$

We can now write

$$\begin{aligned} E_0 U &= -E_0(X_1 - \hat{X}_1)^2 + E_0 \tilde{U}_2(I_1) \\ &= -\sigma^2_{X_1} - (\bar{X}_1 - \hat{X}_1)^2 + E_0 \tilde{U}_2(I_1) \\ &= -\sigma_2^2 P_1^2 - \sigma_3^2 - (\bar{\theta}_1 X_0 + \bar{\theta}_2 P_1 + \bar{\theta}_3 - \hat{X}_1)^2 \\ &\quad -\gamma[(1-H)E_0(\hat{X}_2 - \bar{\theta}_1 X_1 - \bar{\lambda} P_1 - \bar{\theta}_3)^2 + \sigma_\lambda^2 P_1^2 + \sigma_3^2] \end{aligned}$$

where the expression for $E_0(X_1 - \hat{X}_1)^2$ is derived as in IIA and

$$H \equiv \frac{\bar{\theta}_2^2}{\sigma_2^2 + \bar{\theta}_2^2}.$$

Substituting for X_1 from (2), taking expected values in the expression for $E_0 \tilde{U}_2(I_1)$, and then maximizing the resulting expression for $E_0 U$ with respect to P_1 yields:

$$\tilde{P}_1 = \frac{\bar{\theta}_2(\hat{X}_1 - \bar{\theta}_1 X_0 - \bar{\theta}_3) + \gamma(1-H)(\bar{\theta}_1 \bar{\theta}_2 + \bar{\lambda})(\hat{X}_2 - \bar{\theta}_1^2 X_0 - \bar{\theta}_3 - \bar{\theta}_1 \bar{\theta}_3)}{\sigma_2^2 + \bar{\theta}_2^2 + \gamma(1-H)[\bar{\theta}_1^2(\sigma_2^2 + \bar{\theta}_2^2) + 2\bar{\lambda}\bar{\theta}_1 \bar{\theta}_2 + \bar{\lambda}^2] + \gamma\sigma_\lambda^2}.$$

We can write the rule for $\tilde{P}_1$ as

$$\tilde{P}_1 = AG_1 + BG_2$$

where

$$G_1 \equiv (\hat{X}_1 - \bar{\theta}_1 X_0 - \bar{\theta}_3)$$

is the first-period gap between the desired value of goal variable and its expected value with $P_1 = 0$,

$$G_2 \equiv (\hat{X}_2 - \bar{\theta}_1^2 X_0 - \bar{\theta}_3 - \bar{\theta}_1 \bar{\theta}_3)$$

is the second-period gap between the desired value of the goal variable and its expected value with $P_1 = P_2 = 0$,

$$A \equiv \frac{\bar{\theta}_2}{\text{Den}}$$

is the coefficient of policy response to the gap in X_1,

$$B \equiv \frac{\gamma(1-H)(\bar{\theta}_1 \bar{\theta}_2 + \bar{\lambda})}{\text{Den}}$$

is the coefficient of policy response to the gap in X_2, and

$$\text{Den} \equiv \sigma_2^2 + \bar{\theta}_2^2 + \gamma(1-H)[\bar{\theta}_1^2(\sigma_2^2 + \bar{\theta}_2^2) + 2\bar{\lambda}\bar{\theta}_1\bar{\theta}_2 + \bar{\lambda}^2] + \gamma\sigma_\lambda^2.$$

We observe first that A and B in general are not zero if $\sigma_\lambda^2 < \infty$. That is, as long as the variance of the policy lag is finite, a feedback rule dominates a nonfeedback rule.[21/]

We can now examine how the lag parameters $\bar{\lambda}$ and σ_λ^2 affect the policy response coefficients A and B. Note that Den can be written

$$\text{Den} \equiv \sigma_2^2 + \bar{\theta}_2^2 + \gamma(1-H)[\bar{\theta}_1^2\sigma_2^2 + (\bar{\theta}_1\bar{\theta}_2 + \bar{\lambda})^2] + \gamma\sigma_\lambda^2$$

so that clearly Den > 0. It follows immediately that

$$\frac{\partial|A|}{\partial\sigma_\lambda^2} < 0 \text{ and } \frac{\partial|B|}{\partial\sigma_\lambda^2} < 0.$$

This indicates that policy should respond less vigorously to perceived gaps in goal variables as the variance of the policy lag increases.[22/]

We next compute how A and B change as $\bar{\lambda}$ changes. We find first that

$$\frac{\partial A}{\partial\bar{\lambda}} = \frac{-2\gamma\bar{\theta}_2(1-H)(\bar{\theta}_1\bar{\theta}_2+\bar{\lambda})}{\text{Den}^2}.$$

It follows then that

$$\text{sgn}\left(\frac{\partial|A|}{\partial\bar{\lambda}}\right) = -\text{sgn}(\bar{\theta}_1\bar{\theta}_2+\bar{\lambda}).$$

The last term in parentheses is simply the effect of P_1 on the expected value of X_2:

$$E_0X_2 = \bar{\theta}_1^2X_0 + \bar{\theta}_2P_2 + (\bar{\theta}_1\bar{\theta}_2+\bar{\lambda})P_1 + \bar{\theta}_1\bar{\theta}_3 + \bar{\theta}_3$$

so that

$$\frac{\partial E_0X_2}{\partial P_1} = \bar{\theta}_1\bar{\theta}_2 + \bar{\lambda}.$$

Our result is then that

$$\frac{\partial|A|}{\partial\bar{\lambda}}\left\{\begin{smallmatrix}>0\\=0\end{smallmatrix}\right\} \Leftrightarrow \frac{\partial E_0X_2}{\partial P_1}\left\{\begin{smallmatrix}<0\\=0\end{smallmatrix}\right\}.$$

We next find

$$\frac{\partial B}{\partial\bar{\lambda}} = \frac{\gamma(1-H)(\sigma_2^2+\bar{\theta}_2^2+\gamma\sigma_\lambda^2)+\gamma^2(1-H)^2[\bar{\theta}_1^2\sigma_2^2-(\bar{\theta}_1\bar{\theta}_2+\bar{\lambda})^2]}{\text{Den}^2}.$$

Depending on parameter values, $\frac{\partial|B|}{\partial\bar{\lambda}}$ can be positive or negative, but the economic interpretation is not as apparent as for $\frac{\partial|A|}{\partial\lambda}$.

Our results would seem to generalize to multiperiod models as follows:

1. A given change in the shape of the policy lag (represented in our model as a change in $\bar{\lambda}$) can either increase or decrease the responsiveness of optimal policy to new information. The outcome depends on values of parameters of the economic structure as well as on values of parameters of the objective function.

2. An increase in the variability of the policy lag unambiguously reduces the responsiveness of optimal policy to new information. Unless the variance of the policy lag is infinite, however, some responsiveness is better than none.

C. Estimation Uncertainty About the Effects of Policy on Goal Variables

An important consideration in dynamic decision making under uncertainty is the opportunity to learn by doing, that is, to experiment.[23/] In some situations it is possible for the policymaker to learn more about the economic structure by taking an extreme action and then observing the outcome for the goal variable. This opportunity is present, for example, when uncertainty about coefficients is due to estimation. In these situations it may be beneficial to experiment early in the policy horizon, thereby sacrificing on near-term policy goals, in order to gain knowledge about the economic structure, thereby improving the ability to attain policy goals in the future. In this section we will analyze a simple learning-by-doing model and compare the optimal policy which obtains to that which obtains in the inherent uncertainty model.

Before turning to the learning-by-doing model, it is useful to contrast the inherent uncertainty and estimation uncertainty assumptions. Under either assumption the policymaker is assumed to know the distribution of the θ process. In two-period models this means the policymaker initially knows $E_0\theta(t)$, $t=1, 2$ and $E_0\theta(t)'\theta(t')$, $t,t'=1, 2$. With inherent uncertainty it is assumed that

information from the first period $\langle X_1, P_1 \rangle$ does not alter the distribution of second-period coefficients $\theta(2)$:

$$E_0\theta(2) = E_1\theta(2)$$

and

$$E_0\theta(2)'\theta(2) = E_1\theta(2)'\theta(2).$$

In Bayesian terms this says that the posterior distribution of $\theta(2)$ (the distribution of $\theta(2)$ conditional on I_1) is the same as the prior distribution of $\theta(2)$ (the distribution of $\theta(2)$ conditional on I_0). With estimation uncertainty these equalities do not hold, so that the prior and posterior distributions of $\theta(2)$ are different. Learning by doing, thus, involves setting P_1 to favorably alter the posterior distribution of θ_2.

As an example, suppose that uncertainty about θ is due to estimation and that the policymaker approaches the first period with ordinary least squares estimators of $\theta(1)$ based on T prior observations of X and P. At the beginning of the second period the OLS estimators of $\theta(2)$ will be based on the prior T observations of X and P and on X_1 and P_1. Thus, the choice of P_1 affects the estimated distribution of $\theta(2)$. By choosing an extreme value of P_1, the policymaker can generate an extreme observation which will improve the precision of the estimate of $\theta(2)$.[24/] Thus, there is an incentive to deviate from historical policy, which is not present when there is inherent uncertainty.

Our model of learning by doing captures the essence of this estimation example, but simplifies the problem to allow an explicit solution.[25/] We will assume that θ_1 is known, that θ_2 takes on one of the two values, 1 and 2, with priors p and 1-p, respectively, and that θ_3 is distributed uniformly on $[-1,1]$ and independently over time. We will choose specific targets for the goal variable, $\hat{X}_1 = \bar{\theta}_1 X_0 + 1$ and $\hat{X}_2 = \bar{\theta}_1 \hat{X}_1$, in order to facilitate computation of optimal policy under both types of uncertainty.[26/]

The nature of learning in this model is straightforward. Let P_1 be given. If $\theta_2 = 1$, then X_1 will be uniformly distributed on the interval $[\bar{\theta}_1 X_0+P_1-1, \bar{\theta}_1 X_0+P_1+1]$. Similarly, if $\theta_2 = 2$, then X_1 will be uniformly distributed on the interval $[\bar{\theta}_1 X_0+2P_1-1, \bar{\theta}_1 X_0+2P_1+1]$. Thus, for given P_1, X_1 will fall in one of three intervals:

$$A = \{y: \bar{\theta}_1 X_0+P_1-1 \leq y < \bar{\theta}_1 X_0+2P_1-1\},$$

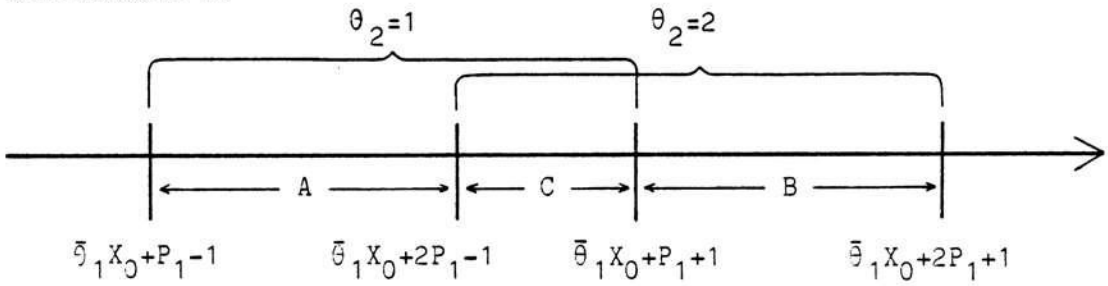
$$B = \{y: \bar{\theta}_1 X_0+P_1+1 < y \leq \bar{\theta}_1 X_0+2P_1+1\}, \text{ and}$$

$$C = \{y: \bar{\theta}_1 X_0+2P_1-1 \leq y \leq \bar{\theta}_1 X_0+P_1+1\},$$

see (Figure 7).

Figure 7

X_1 is uniformly distributed on specified interval conditional on



If we observe $X_1 \varepsilon A$, we know that $\theta_2 = 1$, because points in the set A have a zero probability of occurring when $\theta_2 = 2$. Similarly, if we observe $X_1 \varepsilon B$, we know that $\theta_2 = 2$. If we observe, however, $X_1 \varepsilon C$, there is no information upon which to change our priors, because points in C have an equal probability of occurring whether $\theta_1 = 1$ or $\theta_1 = 2$ is true. Thus, if X_1 falls in C, we would still attach the probabilities p and 1-p to $\theta_2 = 1$ and $\theta_2 = 2$, respectively.

Learning, in this case, amounts to reducing the probability that X_1 falls in C. In order to compute how the probabilities of X_1 falling in particular regions depend on P_1, we first determine the lengths of the subintervals A, B, and C:

$$\ell(A) = (\bar{\theta}_1 X_0 + 2P_1 - 1) - (\bar{\theta}_1 X_0 + P_1 - 1) = P_1,$$

$$\ell(B) = (\bar{\theta}_1 X_0 + 2P_1 + 1) - (\bar{\theta}_1 X_0 + P_1 + 1) = P_1, \text{ and}$$

$$\ell(C) = (\bar{\theta}_1 X_0 + P_1 + 1) - (\bar{\theta}_1 X_0 + 2P_1 - 1) = 2 - P_1.$$

For $0 \leqq P_1 \leqq 2$, we have

$$\text{prob}(X_1 \varepsilon A | \theta_2 = 1) = 1/2\ell(A) = P_1/2,$$

$$\text{prob}(X_1 \varepsilon C | \theta_2 = 1) = 1/2\ell(C) = 1 - P_1/2,$$

$$\text{prob}(X_1 \varepsilon B | \theta_2 = 2) = 1/2\ell(B) = P_1/2, \text{ and}$$

$$\text{prob}(X_1 \varepsilon C | \theta_2 = 2) = 1/2\ell(C) = 1 - P_1/2.$$

Thus,

$$\text{prob}(X_1 \varepsilon A) = \text{prob}(\theta_2 = 1) \cdot \text{prob}(X_1 \varepsilon A | \theta_2 = 1) = pP_1/2,$$

$$\text{prob}(X_1 \varepsilon B) = \text{prob}(\theta_2 = 2) \cdot \text{prob}(X_1 \varepsilon B | \theta_2 = 2) = (1-p)P_1/2, \text{ and}$$

$$\text{prob}(X_1 \varepsilon C) = \text{prob}(\theta_2 = 1) \cdot \text{prob}(X_1 \varepsilon C | \theta_2 = 1) + \text{prob}(\theta_2 = 2) \cdot \text{prob}(X_1 \varepsilon C | \theta_2 = 2)$$

$$= p(1 - P_1/2) + (1-p)(1 - P_1/2) = 1 - P_1/2.$$

When $P_1 = 0$, X_1 will fall in C with probability 1, so that there will be no learning. For $P_1 \geqq 2$, the distributions of X_1 conditional on θ_2 have no overlap, so that X_1 has a zero probability of falling in C. Since optimal policy [when there is inherent uncertainty] $\tilde{P}_1^I$ is between 0 and 1 (to be shown), optimal policy $\tilde{P}_1$ when there is learning must be between 0 and 2. The lower bound comes from $\tilde{P}_1 \geq \tilde{P}_1^I$. The upper bound comes from the fact that as $\tilde{P}_1$ exceeds $\tilde{P}_1^I$, first-

period loss increases while learning increases. However, there is no learning after P_1 reaches 2; that is, $C = \phi$ is the most that can be learned and that occurs at $P_1 = 2$.

Our simple model of estimation uncertainty can be expressed by equations (1) - (3) with

(1) $$\gamma = \frac{V_2}{V_1},\ \hat{X}_1 = \theta_1 X_0 + 1,\ \hat{X}_2 = \theta_1 \hat{X}_1$$

(2a) $$E_0(\theta_1(1),\theta_2(1),\theta_3(1)) = E_0(\theta_1(2),\theta_2(2),\theta_3(2)) = (\bar{\theta}_1, 2-p, 0)$$

$$E_1\theta_1(2) = \bar{\theta}_1,\ E_1\theta_2(2) = \begin{cases} 1 & \text{if } X_1 \varepsilon A \\ 2 & \text{if } X_1 \varepsilon B \\ 2-p & \text{if } X_1 \varepsilon C \end{cases} \qquad E_1\theta_3(2) = 0,$$

where use has been made of--

(i) The mean of a Bernoulli random variable taking on the values α and β with probabilities p and 1-p, respectively, is $p \cdot \alpha + (1-p) \cdot \beta$, and for θ_2, $\beta = 2$ and $\alpha = 1$.

(ii) The mean of a random variable which is uniformly distributed on the interval [a,b] is $\frac{1}{b-a}\int_a^b t\,dt = \frac{a+b}{2}$, and for θ_3, b = 1 and a = -1.

(2b) $$E_0(\theta(1)-\bar{\theta}(1))'(\theta(1)-\bar{\theta}(1)) = E_0(\theta(2)-\bar{\theta}(2))'(\theta(2)-\bar{\theta}(2))$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & 1/3 \end{pmatrix},$$

$$E_0(\theta(1)-\bar{\theta}(1))'(\theta(2)-\bar{\theta}(2)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where use has been made of--

(i) The variance of a Bernoulli random variable taking on the values α and β with probabilities p and 1-p, respectively, is

$$E(\cdot)^2 - (E(\cdot))^2 = p\alpha^2 + (1-p)\beta^2 - (p\alpha+(1-p)\beta)^2$$

$$= p(1-p)\alpha^2 - 2p(1-p)\alpha\beta + (1-p)(1-(1-p))\beta^2$$

$$= p(1-p)(\alpha-\beta)^2,$$

and for θ_2, $\beta = 2$ and $\alpha = 1$.

(ii) The variance of a random variable which is uniformly distributed on the interval [a,b] is

$$E(\cdot)^2 - (E(\cdot))^2 = \frac{1}{b-a}\int_a^b t^2 dt - (\frac{a+b}{2})^2$$

$$= \frac{1}{b-a}(\frac{b^3-a^3}{3}) - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4(a^2+ab+b^2)}{12} - \frac{3(a^2+2ab+b^2)}{12} = \frac{a^2-2ab+b^2}{12} = \frac{1}{12}\cdot(b-a)^2,$$

and for θ_3, $b = 1$ and $a = -1$.

(iii) θ_2 and θ_3 are independently distributed, implying

$$E_0\theta_2(t)\theta_3(t') - E_0\theta_2(t)E_0\theta_3(t') = 0, \text{ for } t,t'=1, 2.$$

(iv) θ_3 is independently distributed over time, implying

$$E_0\theta_3(1)\theta_3(2) - E_0\theta_3(1)E_0\theta_3(2) = 0.$$

(v) θ_2 is a fixed, but unknown, variate, implying

$$E_0\theta_2(1)\theta_2(2) - E_0\theta_2(1)E_0\theta_2(2) =$$

$$p\alpha^2 + (1-p)\beta^2 - (p\alpha+(1-p)\beta)^2 = p(1-p)(\alpha-\beta)^2,$$

since $\text{prob}[\theta_2(2)=\alpha|\theta_2(1)=\alpha] = 1$ and $\text{prob}[\theta_2(2)=\beta|\theta_2(1)=\beta] = 1$, and for θ_2, $\beta = 2$ and $\alpha = 1$.

We also have

$$E_1(\theta(2)-\bar{\theta}(2))'(\theta(2)-\bar{\theta}(2)) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} & \text{if } X_1 \varepsilon A \cup B \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & 1/3 \end{pmatrix} & \text{if } X_1 \varepsilon C \end{cases}$$

and

(3) $\underline{X}_1 = X.$

Since learning is possible only when there is more than one period, we shall assume $\gamma > 0$ and solve for $\tilde{U}_2(I_1)$. Recall $U_2(P_2,I_1) = -\gamma E_1(X_2-\hat{X}_2)^2$, $\tilde{P}_2$ maximizes $U_2(P_2,I_1)$, and $\tilde{U}_2(I_1) \equiv U_2(\tilde{P}_2,I_1)$. We now compute $\tilde{P}_2$ and $\tilde{U}_2(I_1)$ depending on the outcome for X_1.

If $X_1 \varepsilon A$, $\tilde{P}_2$ is found by maximizing with respect to P_2:

$$-E_1(X_2-\hat{X}_2)^2$$

subject to $X_2 = \bar{\theta}_1 X_1 + P_2 + \theta_3(2)$. We know from the certainty equivalence model IA (p. 21) that the solution to this problem is

$$\tilde{P}_2 = \hat{X}_2 - \bar{\theta}_1 X_1$$

and

$$\tilde{U}_2(I_1) = -\gamma\sigma_3^2.$$

If $X_1 \varepsilon B$, $\tilde{P}_2$ is found by maximizing with respect to P_2:

$$-E_1(X_2-\hat{X}_2)^2$$

subject to $X_2 = \bar{\theta}_1 X_1 + 2P_2 + \theta_3(2)$. Again we know from the certainty equivalence model IA (p. 21) that the solution to this problem is

$$\tilde{P}_2 = \frac{\hat{X}_2 - \bar{\theta}_1 X_1}{2}$$

and

$$\tilde{U}_2(I_1) = -\gamma\sigma_3^2.$$

Finally, if $X_1 \varepsilon C$, $\tilde{P}_2$ is found by maximizing with respect to P_2:

$$-E_1(X_2 - \hat{X}_2)^2$$

subject to $X_2 = \bar{\theta}_1 X_1 + \theta_2(2)P_2 + \theta_3(2)$ where $\overline{\theta_2(2)}$ and σ_2^2 are given in (2a) and (2b). We know from the inherent uncertainty model IIA (p. 55) that the solution to this problem is

$$\tilde{P}_2 = \left(\frac{\bar{\theta}_2}{\sigma_2^2 + \bar{\theta}_2^2}\right)(\hat{X}_2 - \bar{\theta}_1 X_1)$$

and

$$\tilde{U}_2(I_1) = -\gamma\left[\sigma_3^2 + \left(\frac{\sigma_2^2}{\sigma_2^2 + \bar{\theta}_2^2}\right)(\hat{X}_2 - \bar{\theta}_1 X_1)^2\right]$$

which in our case is

$$\tilde{P}_2 = \left(\frac{2-p}{4-3p}\right)(\hat{X}_2 - \bar{\theta}_1 X_1)$$

and

$$\tilde{U}_2(I_1) = -\gamma\left[\sigma_3^2 + \left(\frac{p(1-p)}{4-3p}\right)(\hat{X}_2 - \bar{\theta}_1 X_1)^2\right].$$

Define $N(p) = p(1-p)$, $D(p) = 4-3p$, and $\Psi(p) = \frac{N(p)}{D(p)}$. Since

$$\begin{aligned} E_0\tilde{U}_2(I_1) = {} & \text{prob}(X_1 \varepsilon A)E_0[\tilde{U}_2(I_1)|X_1 \varepsilon A] + \\ & \text{prob}(X_1 \varepsilon B)E_0[\tilde{U}_2(I_1)|X_1 \varepsilon B] + \\ & \text{prob}(X_1 \varepsilon C)E_0[\tilde{U}_2(I_1)|X_1 \varepsilon C], \end{aligned}$$

we have

$$\begin{aligned} E_0\tilde{U}_2(I_1) &= -\left(\frac{pP_1}{2}\right)\gamma\sigma_3^2 - \left(\frac{(1-p)P_1}{2}\right)\gamma\sigma_3^2 - \left(\frac{2-P_1}{2}\right)\gamma[\sigma_3^2 + \Psi(p)E_0(\hat{X}_2 - \bar{\theta}_1 X_1)^2] \\ &= -\gamma\sigma_3^2 - \gamma\Psi(p)\left(1 - \frac{P_1}{2}\right)E_0(\hat{X}_2 - \bar{\theta}_1 X_1)^2. \end{aligned}$$

Optimal first-period policy is found by maximizing with respect to P_1

$$-E_0(X_1-\hat{X}_1)^2 + E_0\tilde{U}_2(I_1)$$

$$= -\sigma_2^2P_1^2 - \sigma_3^2 - (\bar{a}_1X_0+\bar{a}_2P_1-\hat{X}_1)^2 - \gamma\sigma_3^2 - \gamma\Psi(p)(1 - \frac{P_1}{2})E_0(\hat{X}_2-\bar{a}_1X_1)^2$$

where use has been made of the inherent uncertainty model IIA (p. 55).

The first-order condition for $\tilde{P}_1$ to be a maximizer is a quadratic expression in $\tilde{P}_1$, which yields the two solutions:

$$\tilde{P}_1 = \frac{2\gamma\bar{a}_1^2\Psi(p)(3-2p)+D(p)\pm\sqrt{\gamma^2\Psi(p)^2\bar{a}_1^4(8-15p+7p^2)+\gamma\Psi(p)D(p)\bar{a}_1^2(6-5p)+D(p)^2}}{(3/2)\gamma\bar{a}_1^2N(p)}.$$

We now want to analyze this solution and compare it to the one under inherent uncertainty. First, we can show that for $\gamma\bar{a}_1^2 < 18$, the larger of the two roots is greater than 2 and can be ruled out (see pp. 67-68). Note that for the larger root

$$\tilde{P}_1^+ > \frac{2\gamma\bar{a}_1^2\Psi(p)(3-2p)+D(p)+\sqrt{[7^{\frac{1}{2}}(1-p)\gamma\bar{a}_1^2\Psi(p)+D(p)]^2}}{\frac{3}{2}\gamma\bar{a}_1^2N(p)}.$$

$$= \frac{2[(6+7^{\frac{1}{2}})-(4+7^{\frac{1}{2}})p]}{3(4-3p)} + \frac{4(4-3p)}{3\gamma\bar{a}_1^2p(1-p)}.$$

The first term is greater than 4/3 for $p < 1$, while the second term is greater than 2/3 for $\gamma\bar{a}_1^2 < 18$. Since normally we would expect both γ and $\bar{a}_1$ to be less than 1, we can restrict our attention to the smaller root.

Second, note that with $\hat{X}_2 = \bar{a}_1\hat{X}_1$ the optimal first-period policy when there is inherent uncertainty is given by

$$\tilde{P}_1^I = (\frac{\bar{a}_2}{\sigma_2^2+\bar{a}_2^2})(\hat{X}_1-\bar{a}_1X_0) \text{ (see p. 56)}$$

$$= \frac{2-p}{4-3p} \text{ (recalling } \hat{X}_1 = \bar{a}_1X_0+1).$$

Third, note that the optimal first-period policy when there is estimation uncertainty can be expressed as

$$\tilde{P}_1 = \frac{2\gamma\bar{\theta}_1^2\Psi(p)(3-2p)+D(p)-\sqrt{[\gamma\bar{\theta}_1^2\Psi(p)(3-\frac{5p}{2})+D(p)]^2+\gamma^2\bar{\theta}_1^4\Psi(p)^2(\frac{3p^2}{4}-1)}}{(3/2)\gamma\bar{\theta}_1^2 p(1-p)}.$$

Finally, it is straightforward to show that if we arbitrarily set $\gamma^2\bar{\theta}_1^4\Psi(p)^2(\frac{3p^2}{4}-1)$ to zero, the expression for $\tilde{P}_1$ simplifies to $\tilde{P}_1 = \frac{2-p}{4-3p}$, which is the inherent uncertainty solution. Thus, returning to the actual expression for $\tilde{P}_1$, we see that the term under the radical sign,

$$\gamma^2\bar{\theta}_1^4[\frac{p(1-p)}{4-3p}]^2[\frac{3p^2}{4}-1],$$

represents the adjustment for learning to the inherent uncertainty solution. Since $\frac{3p^2}{4} - 1 < 0$, the rest of the terms are squared and thus positive, and the radical is preceded by a negative sign, it follows that the adjustment for learning adds a positive amount to the inherent uncertainty solution, $\tilde{P}_1 > \tilde{P}_1^I$.

By inspection it is clear that the adjustment for learning increases when γ or $\bar{\theta}_1$ increases. When γ increases, the future is more important to the policymaker, that is, $E_0\tilde{U}_2(I_1)$ has a greater weight in the policymaker's objective function. Thus, learning is more important, since it allows a higher value of $E_0\tilde{U}_2(I_1)$ to be attained.

When $\bar{\theta}_1$ increases, the expected value of $\tilde{P}_2$ conditional on $P_1 = 0$ increases whether $X_1 \varepsilon A$, B, or C:

$$E_0(\tilde{P}_2|P_1=0)\begin{cases} \tilde{X}_2 - \bar{\theta}_1^2 X_0 = \bar{\theta}_1 & X_1 \varepsilon A \\ \dfrac{\tilde{X}_2 - \bar{\theta}_1^2 X_0}{2} = \dfrac{\bar{\theta}_1}{2} & X_1 \varepsilon B \\ (\frac{2-p}{4-3p})(\hat{X}_2 - \bar{\theta}_1^2 X_0) = (\frac{2-p}{4-3p})\bar{\theta}_1 & X_1 \varepsilon C. \end{cases}$$

But a larger value of $\tilde{P}_2$ implies greater variance of the goal variable X_2 if $X_1 \varepsilon C$. This variance can be avoided if X_1--that is, if there is learning. To summarize, a greater value of $\bar{\theta}_1$ implies a greater expected value of $\tilde{P}_2$, which implies more variance in X_2, unless there is certainty about $\theta_2(2)$. Thus, an increase in $\bar{\theta}_1$ increases the value of learning.

We would expect also that the greater the initial uncertainty about θ_2, the greater would be the value of learning. But given our assumed probability distribution of θ_2, it is not possible to increase the variance of θ_2, p(1-p), without also changing the mean of θ_2, 2-p. Thus, we can not determine how $\tilde{P}_1$ changes when σ_2^2 changes because $\bar{\theta}_2$ also changes. Since p(1-p) does appear with a positive sign in the adjustment for learning term, though, it at least suggests that the expected result would obtain with more general probability distributions of θ_2.

Finally, the question arises whether learning by doing can overturn our earlier policy implication that uncertainty requires caution. Earlier we found that optimal policy should respond less to perceived gaps in goal variables from their targets the greater is the uncertainty about policy. Optimal policy with inherent uncertainty must always be less, in absolute value, than the optimal policy when the policy coefficient is known, the certainty equivalence policy. So can the optimal policy under estimation uncertainty ever be greater, in absolute value, than the optimal certainty equivalence policy? The anwser is yes.

The optimal first-period policy when θ_2 is known is

$$\tilde{P}_1^K = \frac{1}{\theta_2}(\hat{X}_1 - \bar{\theta}_1 X_0) = \frac{1}{2-p}(\hat{X}_1 - \bar{\theta}_1 X_0).$$

If we stick to the assumption that $\hat{X}_1 - \theta_1 X_0 = 1$, it appears that the optimal learning policy $\tilde{P}_1$ can never exceed $\tilde{P}_1^K$. Although $\tilde{P}_1$ increases with γ, the limit of $\tilde{P}_1$ as $\gamma \to \infty$ is $\tilde{P}_1^K$.

If, however, we set $\hat{X}_1 = \theta_1 X_0$, then both the certainty equivalence solution $\tilde{P}_1^K$ and the inherent uncertainty solution $\tilde{P}_1^I$ are zero. The optimal solution under estimation uncertainty $\tilde{P}_1$ in this case is positive, because the adjustment for learning to the inherent uncertainty solution is still positive. Since all solutions change continuously with the gap $\hat{X}_1 - \bar{\theta}_1 X_0$, there is a small gap $\hat{X}_1 - \bar{\theta}_1 X_0 > 0$ for which $\tilde{P}_1^I < \tilde{P}_1^K < \tilde{P}_1$. Learning, then, can overturn the implication that uncertainty requires caution.

Footnotes

1/ The survey is not intended to be exhaustive; it includes only a sample of prominent papers on a few key issues.

2/ See, for example, Wallace (1976) and Lucas (1978).

3/ The solution can also be expressed as a feedback rule which states how policy variables should be set each period based on available information.

4/ In the case of $n > 1$ goal variables, the terms V_i and $(X_i - \hat{X}_i)^2$ are 1xn and nx1 matrices, respectively. The quadratic objective function assumed here is not general in that it does not allow cross-product terms either over time or between variables at a point in time.

5/ The variable γ is $\frac{1}{1+r}$, where $r \geqq 0$ is the policymakers' rate of time preference.

6/ In the information lags and multiple goal variable models X_t can be thought of as an n-dimensional vector, and the process becomes

$$X_{it} = \sum_{j=1}^{n} \theta_{ij}(t) X_{jt-1} + \theta_{i,n+1}(t) \cdot P_t + \theta_{i,n+2}(t) \quad i=1,\ldots,n.$$

In more compact notation the process can be written as before:

$$X_t = \theta_1(t) X_{t-1} + \theta_2(t) P_t + \theta_3(t)$$

for all t, where now

$$X_t = \begin{pmatrix} X_{1t} \\ \cdot \\ \cdot \\ \cdot \\ X_{nt} \end{pmatrix}, \quad \theta_1(t) = \begin{pmatrix} \theta_{11}(t) & \cdots & \theta_{1n}(t) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \theta_{n1}(t) & \cdots & \theta_{nn}(t) \end{pmatrix}, \quad \theta_2(t) = \begin{pmatrix} \theta_{1,n+1}(t) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n,n+1}(t) \end{pmatrix},$$

$$\theta_3(t) = \begin{pmatrix} \theta_{1,n+2}(t) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n,n+2}(t) \end{pmatrix}$$

and P_t is a scalar.

7/ Suppose, however, we search initially for the optimal first-period rule $\tilde{f}_1$ over all functions mapping an arbitrary I_0 into P_1. Applying $\tilde{f}_1$ to the known vector I_0 yields optimal first-period policy $\tilde{P}_1$ as a scalar, $\tilde{P}_1 = \tilde{f}_1(I_0)$.

8/ Footnote 7 applies in this case to g_1 and g_2.

9/ Let N denote the vector of realizations of economic variables observed at t=1 but not at t=0 (i.e., N=new information), so that for each I_0 and I_1, N is defined by: $N = I_1 \sim I_0$. Thus, $I_1 = I_0 \cup N$. For arbitrary $\langle g_1, g_2 \rangle \varepsilon G$ we can extend g_2 to $\{I_1\}$ by $\bar{g}_2(I_1) \equiv \bar{g}_2(I_0 \cup N) = g_2(I_0)$ for all N. Then it is clear that each pair of functions $\langle g_1, \bar{g}_2 \rangle$ contained in G is also contained in F. However, as long as the economic process is not deterministic and there are new observations at t=1, there exists a pair $\langle f_1, f_2 \rangle \varepsilon F$ such that $f_2(I_0 \cup N) \neq f_2(I_0 \cup N')$ for $N \neq N'$. The pair $\langle f_1, f_2 \rangle$ is contained in F but not in G.

10/ For one-period problems V_2 is set equal to zero and E_0U is maximized with respect to P_1. X can be considered a vector with n components as in footnote 6 or a scalar (n=1). When n > 1, however, $\sum$ must be defined differently. As in footnote 6,

$$\theta(t) = (\theta_1(t), \theta_2(t), \theta_3(t)) = \begin{pmatrix} \theta_{11}(t) & \cdot & \cdot & \theta_{1n}(t) & \theta_{1,n+1}(t) & \theta_{1,n+2}(t) \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \theta_{n1}(t) & \cdot & \cdot & \theta_{nn}(t) & \theta_{n,n+1}(t) & \theta_{n,n+2}(t) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \theta_{1R}(t) \\ \\ \theta_{nR}(t) \end{pmatrix},$$

where θ_{iR} is the ith row vector of θ. The variance-covariance matrix of θ is defined by

$$\sum(t,t')_s = E_s \begin{pmatrix} (\theta_{1R}(t)-E_s\theta_{1R}(t))' \\ \\ (\theta_{nR}(t)-E_s\theta_{nR}(t))' \end{pmatrix}_{n(n+2)x1} \underset{1xn(n+2)}{(\theta_{1R}(t')-E_s\theta_{1R}(t'),\ldots,\theta_{nR}(t')-E_s\theta_{nR}(t'))}$$

$$\begin{pmatrix} \sum_{11}(t,t')_s & \cdots & \sum_{1n}(t,t')_s \\ \vdots & & \vdots \\ \sum_{n1}(t,t')_s & \cdots & \sum_{nn}(t,t')_s \end{pmatrix},$$

where $\sum_{ij}(t,t')_s$ is the covariance of tth period, i^{th} row coefficients with t'^{th} period j^{th} row coefficients based on information at time s.

11/ Theil (1965), pp. 404-424.

12/ Theil (1965), pp. 508-510. Our two-period model with one goal variable and one policy control variable can be written in the form

$$\begin{pmatrix} X_1 \\ \\ X_2 \end{pmatrix} = \begin{pmatrix} \theta_2(1) & 0 \\ \\ \theta_1(2)\theta_2(1) & \theta_2(2) \end{pmatrix} \begin{pmatrix} P_1 \\ \\ P_2 \end{pmatrix} + \begin{pmatrix} \cdot\ \theta_1(1)X_0+\theta_3(1) \\ \\ \theta_1(2)\theta_1(1)X_0+\theta_1(2)\theta_3(1)+\theta_3(2) \end{pmatrix}$$

$$\equiv \begin{pmatrix} R_{11} & 0 \\ \\ R_{21} & R_{22} \end{pmatrix} P + u.$$

13/ It has implicitly been assumed that $\bar{\theta}_{23} > 0$, but this is not an important restriction. If both $\bar{\theta}_{13}$ and $\bar{\theta}_{23}$ were zero, there would be no effect of policy on either goal variable and one choice of policy would be as good as any other. So we want to assume that at least one of these coefficients is not zero. Since the model is symmetric with respect to X and π, it could just as easily have been assumed that $\bar{\theta}_{13} > 0$.

14/ The economic process may not be stable under both choices, however. In rational expectations-natural rate models attempts to fix a nominal interest rate make the price level indeterminate.

15/ This is essentially the model found in Poole (1970).

16/ For a more complete exposition of these points see Kareken-Miller (1976).

17/ The two-period model described here is a special case of the general model found in Kareken-Muench-Wallace (1973).

18/ This model can be found in Brainard (1967).

19/ Friedman seemed to make this type of argument in his 1953 article.

20/ For the bulk of the paper Fischer and Cooper assume that the lag weights can be expressed as:

$$w_i(t) = (1-\beta)(1-\lambda_{t-i}) \prod_{j=0}^{i-1} \lambda_{t-j}$$

where $\prod_{j=0}^{-1} (\cdot) \equiv 1$ and for all s

$$\lambda_s = \lambda + \varepsilon_s, \; 0 < \lambda < 1$$

$$E\varepsilon_s = 0, \; E\varepsilon_s^2 = \sigma_\varepsilon^2.$$

Under known lags $\sigma_\varepsilon^2 = 0$ and the w's are from a Koyck lag distribution

$$w_i = (1-\beta)(1-\lambda)\lambda^i.$$

The length of lag is defined by the mean lag

$$\frac{\sum_{i=0}^{\infty} w_i(t) \cdot i}{\sum_{i=0}^{\infty} w_i(t)},$$

and the variance of the lag is defined in terms of the parameter σ_ε^2.

The mean lag increases in the known lag case as λ increases. This implies that w_0, the known coefficient on current policy, decreases as the length of lag increases. The model in the known lag case is simply a version of our known coefficients model in IA with

$$\theta_1 = \beta,$$

$$\theta_2 = (1-\beta)(1-\lambda), \text{ and}$$

$$\theta_3(t) = (1-\beta)(1-\lambda) \sum_{i=1}^{\infty} \lambda^i P_{t-i} + U(t).$$

Optimal policy in model IA is given by

$$\tilde{P}_t = \frac{1}{\theta_2}(\hat{X}_t - \theta_1 X_{t-1} - \bar{\theta}_3(t))$$

$$\equiv \frac{1}{(1-\beta)(1-\lambda)}(\hat{X}_t - \beta X_{t-1} - (1-\beta)(1-\lambda) \sum_{i=1}^{\infty} \lambda^i P_{t-i}).$$

The deviation of $\tilde{P}$ from zero needed to close the gap grows as θ_2 declines. Since θ_2 declines with an increase in the mean lag, it is not surprising then that Fischer and Cooper find the longer the lag the more vigorously stabilization policy should be used.

21/ New information is represented by the variable X_0. The response of $\tilde{P}_1$ to new information is then $-\bar{\theta}_1 A - \bar{\theta}_1^2 B$. As long as not both A and B are zero, which will be the case in general with $\sigma_\lambda^2 < \infty$, a feedback rule adjusting P_1 to new information is seen to dominate a nonfeedback rule.

22/ This proposition could be restated in terms of the previous footnote to say that policy should respond less vigorously to new information as the variance of the policy lag increases.

23/ For "learning by doing" models, see Chow [1975], Prescott, and Zellner.

24/ The result that extreme observations improve the precision of OLS estimates can be found in Johnston.

25/ The estimation problem referred to in the example above cannot be solved explicitly for optimal policy--even in the two-period case.

26/ By assuming a positive gap between $\hat{X}_1$ and $\bar{\theta}_1 X_0$, we have assured that the optimal policy $\tilde{P}_1^I$ when θ_2 is inherently uncertain is less than the optimal policy $\tilde{P}_1^K$ when θ_2 is known (see page 51).

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