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EXOGENEITY AND CAUSAL ORDERING IN MACROECONOMIC MODELS

(See p. 6.)

Christopher A. Sims Draft 9/27/75

The aim of this paper is to clarify the variety of operational meanings which results of exogeneity tests like that I carried out in "Money, Income, and Causality" [ ] may have. After the empirical work for that paper was largely complete, and after I had arrived at a reasonable understanding of what the operational implications of the paper were, I reread Granger's 1969 paper and realized that, in Granger's framework, the results could be summarized as "Money causes income; income does not cause money". Since this way of putting it seemed to me concise, illuminating, provocative, and only slightly misleading, I adopted Granger's terminology in presenting the results. "Cause" being the shaggy old word it is, this terminology, for some readers at least, proved much more provocative and misleading than illuminating.[1]

It might seem that the best course in this paper, then, would be to avoid causal language, retreating to more jargonistic language which might be less subject to multiple interpretations. "Exogenous" for example, at least in the form "strictly exogenous", appears to have a precise and unique meaning in econometric theory, while "causal ordering" has at least two meanings (Wold's and Granger's); and in some applications the notion of exogeneity could replace Granger's

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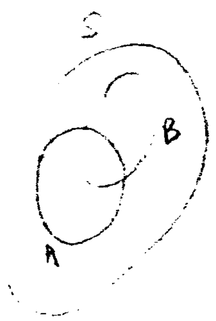
[1] Paul Davidson and Sidney Weintraub, e.g., in their paper "Money as Cause and Effect" ignore my paper's empirical results, while reacting vigorously to its use of causal language: "The identification of correlation with causation violates careful econometric usage and evades the profound cognitive issues. Apparently this confusion is being perpetuated by students of the Monetarists' school (e.g. Sims 1972)." (p. 1117, f.n.2).

notion of causal priority. However the econometric usage of "exogeneity" appears on examination no closer to the dictionary definition of that word than Granger's usage of "causal priority" is to the common meaning of that phrase. Furthermore, in interpreting the results of exogeneity tests, it is probably true that the difficult issues will generally involve determining which of the numerous possible meanings of "causal priority" are reasonably treated as equivalent to Granger's meaning in the case at hand.

This paper begins, therefore, with yet another attempt to be precise about what "causal ordering" means, this time exploring interrelations among a vareity of meanings the word has been given, rather than attempting to give it any new, narrower, technical meaning. With this foundation, the paper then goes on to discuss operational interpretations of Granger and Wold causal orderings in econometric applications of various types.

1. Causal Orderings as Recursions

Using the central idea of Simon (1952) and generalizing it slightly, consider a space S of "outcomes" and two sets of restrictions on it characterized by the subsets A and B of S. A and B together produce the "result" $A \cap B$. Now consider two spaces, X and Y, together with associated functions, P_X, P_Y, mapping S into X and Y respectively. The following definition covers many existing uses of "causal ordering" as special cases.



$P_X : S \to X$

$P_Y : S \to Y$

Definition: The ordered pair (A,B) of restrictions on S determines a causal ordering from X to Y (equivalently, makes X causally prior to Y) if and only if $P_X(A \cap B)=P_X(A)$ and $P_Y(A)=Y$.

Paraphrasing, (A,B) makes X causally prior to Y if and only if A restricts X without restricting Y, while the addition of B restricts Y without further restricting X.

It should be clear that in this definition the causal ordering is a characteristic of the system (A,B) and the output space Y, not of the result $A \cap B$. Given a system (A,B) which does not make X causally prior to Y, we can always define $B'=A \cap B$, $A'=P_X^{-1}(P_X(A \cap B))$, and $Y'=P_Y(A)$, and (A',B') will be construction make X causally prior to Y', yet (A',B') has the same result as (A,B).

An example of a causal ordering is a pair of linear equations in two unknowns, one of which involves only one unknown. The space S is Euclidean 2-space, X and Y are two copies of the real line, P_X projects a point in S into its first co-ordinate, and P_Y projects a point in S into its first co-ordinate, and P_Y projects a point in S into its second co-ordinate. The set A is the line determined by the first equation, which involves only X (a vertical line, if X is the horizontal axis), and B is the line determined by the second equation.

Another example is a two-simultaneous-equation econometric model in Wold causal chain form. Here S is the space of joint distributions of the endogenous variables conditional on the predetermined variables; X is the space of marginal distributions (again conditional on predetermined variables) for the first endogenous variable; Y is the space of marginal distributions for the second endogenous variable; P_X, P_Y project joint distributions in S into corresponding marginals; A is the equation involving only one endogenous variable and hence determining its marginal distribution; and B is the other equation, which specifies the conditional distribution of the second endogenous variable given the first.

A third example is the triangular autoregressive representation of a bivariate covariance-stationary process (x(t),y(t)), in which x is causally prior to y in Granger's (1969) sense. Here the space S is joint autocovariance functions of the processes x(t) and y(t), Y is the space of autocovariance functions for y(t) alone, and X is the space of autocovariance functions for x(t). The restrictions A are the first equation of the joint autoregressive representation, which is a univariate autoregressive representation for X. The set B is determined by the second equation of the joint autoregressive representation.

Though here and in the remainder of the paper we will deal only with pairs (X,Y) of inputs and outputs, it should be clear that two-element orderings like those considered here can be extended to n-element orderings in a natural way.

2. Characterizing a relation as "causal" or "structural".

In this scme it is natural to think of A as a particular input, and of B as specifying the way inputs generate output. In the form

E.g., (A,B,C) orders X, Y, Z in that order if (A,B) makes X causally prior to Y and (A B, C) makes XxY causally prior to Z when the mapping P_{XY}: $P_{XY}(s) = (P_X(s), P_Y(s))$ is used to map S into XxY.

of exogeneity test I used in my 1972 AER article it is natural to think of "causal priority of x" as a characteristic of the distributed lag regression of y on x, that is as a characteristic of B, not of B and A, jointly. There is a nearly precisely corresponding usage in engineering and physics, where operators mapping "input" functions into "output" functions are characterized as "causal" or "non-causal" (or as "realizable" or "non-realizable") again without reference to any particular input. These characterizations of B, the input-output connection, as causal or non-causal arise from intuitive notions of what causal systems "in nature" must be like. The reason it seems plausible to define a causal ordering as we have is that in a system (A,B) with a causal ordering from X to Y it is natural to contemplate varying the input A, holding B fixed, and obtaining outputs $P_Y(A \cap B)$ determined by A. Of course we can always undertake this experiment as a mathematical exercise, but the practical significance of the experiment depends critically on whether there is in nature a mechanism corresponding to the set of constraints B, which would remain fixed while we varied A. Thus even though there will be many systems (A,B) which generate the same result A B and imply different causal orderings, not all of them are of equal practical interest. The most interesting systems are those in which the input-output relation B is one which would in fact remain fixed if we varied A.[1]

The idea that causal orderings are at least implicitly linked to the possibility of varying the causally prior input has appeared before, in Simon [ ], e.g. Here as throughout I am providing references only to papers I have encountered in unsystematic reading in this area.

We will call such a B "structural", and a precise definition of this form follows.

If variation of A is to be even formally possible, B must have a form which "accepts" variation in A. To be precise, we will say:

Definition: The set $B \subset S$ accepts X as input to Y if for any $A \subset S$ which constrains only X [i.e., any A such that $P_X^{-1} P_X(A)=A$], (A,B) makes X causally prior to Y.

Though similar notions are sometimes given the name "realizable", or "causal" in physical science literature, the formally explicit notion closest to that defined below seems to be Hurwicz's [5] definition of "structural", so I use that word.

Definition: The set B is structural for inputs X if (1) B accepts X as input and (2) ~~when any~~ [if set $C \subset X$ is "true" (or is "implemented") ⇒ ~~then~~ $P_Y(P_X^{-1}(C) \cap B)$ is true].

In Hurwicz's [5] formulation, the "inputs" considered in defining structural relations are mappings from one space of equation systems into another. These mappings correspond to interventions of the form, e.g., "~~fix~~ decrease the coefficient on x_i in the j'th equation ~~at~~ by α". The role of B is played by some set of equation systems to which the transform is applied. The input is interpreted as implemented when, e.g., a certain excise tax rate is fixed at α, in which case a "structural" equation system must be one in which the coefficient on x_i in the j'th equation system does in fact vary with the excise tax rate.

Since the property of being "structural" is a property of the way we interpret the system as applying to the real world, not of the system's form, there is no way of proving that a system is structural by examining the system's form. We can test whether a system is structural by using it to predict the effects of an intervention, making the intervention, and observing the result. In this way we may prove the system is not structural, but there can be no guarantee that other interventions, or even the same intervention repeated, will be predicted well by the system just because one or several test interventions are predicted well.

Nonetheless it may be possible to specify enough properties which we know a structural system ought to have to allow us to distinguish potentially structural from surely non-structural systems. The use of such restrictions to distinguish non-structural systems is called "identification" in economics, "realizability theory" in some physical science applications.

In any application where inputs and outputs are dated, it is sensible to assume that a structural relation for inputs must not be one which determines past outputs from future inputs. This notion has not received much prominence in writings on econometric methodology, because the hypothetical inputs with respect to which simultaneous equation models are identified are paradigmatically one-time transformations of a system which is assumed not to have been subject to intervention during the period of observation. We consider possible alterations of the supply or demand curve, e.g., in a system which has had stable supply and demand curves in the sample period. In this context there is no time-stream of inputs and outputs. Furthermore, econometricians might resist the idea that dating of variables can be used to formulate general restrictions on structural relations. Indeed it is possible for example that a structural relation between two endogenous variables in an econometric model could involve a two-sided distributed lag. The claim that the conditional expectation of y_t given the past and future of x_t has non-zero partial derivative with respect to future x's precludes that conditional expectation from being a structural relation only if variations in x are the inputs with respect to which the structure is claimed to be identified. In economics, there is [more often than not] no "variable" in the system whose time paths are the identifying interventions.

The condition that future inputs should not determine past outputs is called "causality" in some physical applications. For example, an operator mapping input functions of time into output functions of time is termed "causal" if it determines y_t (the value of the output function at t) from x_s (values of the input function) at $s \leq t$. In the formal framework of this paper, we can define this causality property as follows.

Consider a family of spaces X_t^+, X_t^-, Y_t^-, Y_t^+, t ranging over the real line, where X_t^+ is to be thought of as <u>future</u> inputs and Y_t^- as <u>past</u> outputs. The corresponding functions P_{Xt+} and P_{Yt-} map S into X_t^+ and Y_t^-, respectively.

<u>Definition</u>: The subset B of S is <u>causal</u> if and only if:

i) For any t, B accepts X_t^- as input; and

ii) $t > r$, $P_{xt+}^{-1}(P_{xt+}(A)) = A$, and (A,B) makes X_t^+ causally prior to Y_r^- imply $P_{Yr-}(A \wedge B) = Y_{r-}$.

Paraphrasing, if we attempt to feed into B an input which specifies characteristics of future inputs only, the result will contain no information about past outputs.

Note that being causal in this sense is only a necessary condition for an input-output mechanism to be structural. The mistake of treating this causality condition as sufficient for a relation to be structural is exactly the old post hoc ergo propter hoc fallacy. But, as we have already seen, causality of a relation is in this respect no different from any other identifying restriction. No characteristic of a relation's internal structure can guarantee that the relation is structural, because being structural is a characteristic of the way we connect the relation to reality, not a property of the relation by itself.

We have discussed three terms: "causal ordering", "causal", and "structural". The first two refer to properties of the logical structure of a model which may be plausible requirements if we are to contemplate treating the model as structural relative to variation in x as the identifying interventions. The controversy surrounding "causality" in economics tends to arise in situations where a model with a causal order from x to y or a model containing a relation which is causal for x as input fits some historical data, and it is then asserted that the model can be used accurately to project the effects of varying the path of x. That is, the fit of the causal model to the data is used to buttress a claim that the model is structural relative to variations in the path of x as identifying interventions.

In the remainder of the paper we discuss the justification for and dangers in interpreting fitted models displaying Wold or Granger causal orderings as structural when the mechanism determining the causally prior variable is to be varied.

3. Interpreting Wold and Granger Causal Orderings

Consider now a dynamic, stochastic, linear, econometric model,

$$(1)\quad \begin{cases} a_{11}*y_1 + a_{12}*y_2 = u_1 \\ a_{22}*y_2 + a_{22}*y_2 = u_2 \,. \end{cases}$$

The "*" indicates convolution, being read $a_{11}*y_1(t) = \sum_{s=-\infty}^{\infty} a_{11}(s)y_1(t-s)$. We will take the system to have been normalized with $a_{11}(0) = 1$ and $a_{22}(0) = 1$, and we will assume $a_{ij}(s) = 0$, all $s<0$, $i=1,2$, $j=1,2$. This latter condition is natural because we would like the system to accept as input arbitrary initial conditions -- values for $y_i(t)$, $t \le 0$ and $u_i(t)$ $t \le 0$. It is also natural in many applications to require that (1) be causal when u_i, $i=1,2$ are jointly covariance stationary, (1) takes realizations of covariance stationary processes u as input and produces realizations of covariance-stationary processes y as output, and past and future input and output are defined by, e.g., setting U_{t-} as the values of $u_i(s)$, $i=1,2$, $s \le t$. This amounts to the standard condition that the coefficients $a_{ij}(s)$ form a stable operator.[1]

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[1] It bears repeating that the reason for imposing stability of the operator applied to y in (1) is not simply that we know that the real-world y is not explosive. Systems of the form (1) with the $a_{ij}(s)$ operator "unstable" may fit covariance stationary pairs of u, y processes. The "instability" of the $a_{ij}(s)$ operator implies non-stationarity of y only if we impose the additional requirement that the system be causal with u as input and y as output. (For example, if the system is causal from stationary u's to stationary y's when we reverse the sign of the time index, then the left-hand-side coefficients will generally form an "unstable" operator, if instability is defined in the conventional way in terms of the absolute values of the roots of the characteristic polynomial.)

The system (1) displays a Wold causal ordering[1] if $a_{21}(0)=0$, u_1 and u_2 are serially uncorrelated, and u_1 and u_2 are mutually uncorrelated. The system displays a Granger causal ordering[2] if $a_{21}(s)=0$, all s, and $u_1(t)$ and $u_2(s)$ are mutually uncorrelated for all t, s. (Note that the possibility that $u_i(t)$, $u_i(s)$ are correlated is left open.)

Each ordering implies a convenient statistical property for the first equation of (1). The Wold ordering implies that $y_2(t)$ is uncorrelated with $u_1(s)$ for $s \geq t$, i.e. that y_2 is <u>predetermined</u> in the first equation. The Granger ordering implies that $y_2(s)$ and $u_1(t)$ are uncorrelated for all t, s, i.e. that y_2 is <u>exogenous</u>[3] in the first equation. The conditions that y_2 be predetermined or exogenous in the first equation are not equivalent to Wold and Granger causal orderings, because the second equation of (1) need not exist or take the form given in (1) in order for y_2 to be predetermined or exogenous in the first equation.[4] In fact by appropriate definitions of input and output spaces,

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1/ See [ ] for a forceful presentation of the argument that structural models are likely to take a form with a Wold ordering.

2/ See [ ] for a presentation of this notion of causal ordering. Granger's original definition is not confined to linear covariance-stationary systems.

3/ Some writers use the term "strictly exogenous" where we use "exogenous" to sharpen the distinction from "predetermined".

4/ The potential advantages and disadvantages of testing exogeneity without estimating the second equation of (1) are discussed in [ ].

it is possible to make the conditions that y_2 be predetermined or exogenous in the first equation of (1) equivalent to the condition that the first equation be causal in the sense defined earlier in this paper. Nonetheless in practice it can be helpful in organizing thought on the subject to think of exogeneity or predeterminedness as restrictions on a two-equation system. If there is a second equation of the form given in (1), and if u_1, u_2 are covariance-stationary, then Wold and Granger orderings are equivalent to y_2's being predetermined and exogenous, respectively, in the first equation.

The problem of whether and how to interpret Wold or Granger orderings as structural can arise in two guises. One of these is relatively familiar. Standard simultaneous equation models in most applications are implemented subject to numerous maintained hypotheses of exogeneity and predeterminedness. These maintained hypotheses are, at least in principle, generated by considering what we know about causal orderings in the real world phenomena being modeled, and translating those real world orderings into Wold or Granger orderings. This familiar problem of the criteria for making assumptions that variables are exogenous or predetermined was treated by Koopmans [ ] some twenty years ago, and published work on the subject has advanced little since then. This paper unfortunately will have little to say in this area, but some remarks on the subject may appear in a later draft.

The other guise of the problem arises when a model displaying a causal ordering has been shown to fit some sample of data and we must decide whether to interpret this historically observed Wold or Granger ordering as structural. That is, we must decide whether to treat the first equation of (1) as fixed while we generate arbitrary time paths for y_2. The question can always be recast as, "Is the true structure applying to arbitrary variations in the time path of y_2 possibly not (1), yet nonetheless consistent with (1)'s fitting the historical data?"

One reason this form of the issue has not received much attention until recently is that with the Wold ordering, which entered the econometric literature earlier, the issue never arises in pure form. If y_1 and y_2 are jointly covariance stationary and have an autoregressive representation, then there is <u>always</u> a system of the form (1) displaying a Wold ordering.[/] Thus if we have no other identifying restrictions on (1), the demonstration that a system like (1) with a Wold ordering will fit the data is no evidence at all that the ordering is structural. The structural system could be practically any system of the form (1) and still imply that a Wold-ordered system would fit the historical data. This does not mean that a Wold ordering is untestable, only that a Wold ordering can be tested only in conjunction with other identifying restrictions on the system. Thus debate over whether the estimated causal ordering is structural is likely to be diverted into dispute over whether the other identifying restrictions are valid in the particular application under consideration.

The Granger ordering, on the other hand, is all by itself a restriction on the class of jointly covariance-stationary processes y_1, y_2 which could satisfy (1). That this created the possibility of testing the

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[/] See [ ].

null hypothesis that y_2 is exogenous in the first equation without any other identifying restrictions as maintained hypotheses was pointed out early as 1963 by Hannan [ ], but the first applications of the idea in economics of which I am aware were those by myself [ ] and Sargent [ ]. When a Granger ordering can be shown to fit some historical data, we can no longer say that any structure could have produced the empirical result, and the problem arises of determining what structural systems, if any, which do not imply causal orderings from y_2 to y_1 might have produced the observed good fit of a model with a Granger ordering.

Lucas [ ] and others have put forth a very general argument that linear dynamic models like (1) are probably never structural. The form of the distributed lags in (1) is determined in part by the use people make of historical data on variables in the model in projecting the future. Any "intervention" which changes some relation in the system is likely to change the optimal procedure for using historical data in making projections, and hence to change the form of all distributed lags in the system, assuming people adjust their expectation-formation rules at least part way toward the optimal form. There is no doubt that this argument is in principle correct, and it is at least possible that the argument will prove to be of such substantial quantitative importance that econometric methodology will be back at the drawing boards for years before it again has models that appear to be able to relate data structurally to questions of policy.

Before simply proceeding to the next topic, let me indicate some reasons for hope that treating linear dynamic models as structural could turn out sometimes not to be a serious error. Though expectational elements probably enter nearly every economic dynamic behavioral relation, it is not clear how often it is likely to turn out that the form of such relations is strongly sensitive to the expectational rule. It is certainly possible to construct dynamic optimization models in which, because of strong elements of "physical inertia" or "costs of adjustment" the lag distributions describing optimal behavior depend only weakly on the expectation-formation rule. Also, it is certainly true that there are contexts in which it is legitimate to consider analysis of optimal choice for the time path of y_2 by policy-makers without supposing that this choice must inherently involve altering the historical form of the stochastic process determining y_2. That is, a policy choice which appears to the decision maker as the problem of projecting the effects of several alternative time paths for y_2 using the first equation of (1), then picking the "best" of these, might result in a joint covariance structure for y_1 and y_2 consistent with (1). The precise sense in which this can be true and how much comfort is to be gained from this possibility is a deep and controversial question which would divert us from the main point of this paper.

4. Generating Spurious Granger Orderings

Suppose we had a structural model in which one equation took the form:

$$a_{22}*y_2(t) = b(y_1(t)-\hat{y}_1(t)) + u_2(t) \quad , \tag{2}$$

where $\hat{y}_1(t)$ is the minimum variance linear forecast of $y_1(t)$ based on values of $y_1(s)$, $y_2(s)$ for $s<t$ and u_2 is serially uncorrelated

and uncorrelated with past values of y_2 or y_1. Then if y_1, y_2 are jointly covariance-stationary and have an autoregressive representation, $\hat{y}_1(t)$ will be a linear combination of past values of y_2 and y_1, allowing us to write

3) $$y_1(t) - \hat{y}_1(t) = a_{11}*y_1(t) + a_{12}*y_2(t) = u_1(t).$$

Clearly $u_1(t)$ is serially uncorrelated and uncorrelated with past values of y_1, y_2. Now we can rewrite the right-hand-side of (2) as $b\ u_1(t) + u_2(t)$. By construction, u_1 *and* u_2 are correlated, if at all, only contemporaneously. Thus there is some constant α such that if we set $v_2 = bu_1 + u_2$, $v_1(t) = u_1(t) - \alpha v_2(t)$ is serially uncorrelated and uncorrelated with $v_2(s)$ for all s. Thus, taking linear combinations of (2) and (3) we arrive at

4) $$a_{11}*y_1(t) + (a_{12} - \alpha a_{22})*y_2(t) = v_1(t)$$
$$a_{22}*y_2(t) = v_2(t),$$

which has the form of (1) and implies a Granger causal ordering.

Neither equation of (4) is likely to be structural, though the first equation would be consistently estimated by least squares regression and would pass a test for exogeneity of y_2.

The reason this example is interesting is that there are behavioral theories which lead to equations of the form (2), and which therefore, when they are true, imply that spurious causal orderings are likely to be observed. For example, suppose policy-makers know that a structural relation between y_1 and y_2 exists, of the form

5) $$b_{21}*y_1 + b_{22}*y_2 = u_2 .$$

Suppose further that policy-makers can set y_1 at any value they wish each period, subject to a "disturbance" u_1 and to the fact that when they choose $y_1(t)$ they know $y_1(s)$, $y_2(s)$ only for $s \leq t-1$. In particular then they can use any rule of the form

6) $$b_{11}*y_1 + b_{12}*y_2 = u_1$$

in forming y_1, subject to the condition that $b_{12}(0) = 0$, with $b_{11}(0)$ normalized at 1.0.

Let us take the policy-makers to be minimizing a quadratic objective function of the form $\mathrm{Var}[(g*y_2(t))^2]$. Now if u_1, u_2 form a linearly regular covariance-stationary process, they will have a joint moving average representation of the form

7) $$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = H * \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} ,$$

where $v_1=u_1-\hat{u}_1$, $v_2=u_2-\hat{u}_2$ are one-step-ahead forecast errors. The vector process v_1, v_2 will be serially uncorrelated.[1] Assuming that the optimal choice of (6) implies that the b_{ij} coefficients in (5) and (6) form a stable operator[2], (5), (6), and (7) will jointly imply that we can write

8) $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = B^{-1}*H* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} ,$$

which in turn implies

9) $$g*y_2 = f_1*v_1 + f_2*v_2 \quad .$$

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[1] This is a version of the Wold decomposition of the process. See, e.g., Rozanov [ ].

[2] This rules out some, but not all interesting cases. See [ ].

From the normalization rules $g(0) = b_{11}(0) = b_{22}(0) = 1$, the fact that by construction $H(0) = I$ in (7), and the fact that $b_{12}(0) = 0$, we can be sure that in (9) $f_2(0) = 1$, $f_1(0) = -b_{21}(0)$. Except for the requirement that the coefficient of $v_1(t)$ be one and that that of $v_2(t)$ be zero, however, equation (6) allows $y_1(t)$ to be chosen to be an arbitrary linear combination of current and past $v_1(s)$, $v_2(s)$.[1] Then if b_{21} has a one-sided inverse under convolution (i.e., is "stable"), we can also make $b_{21}*y_1$ equal to an arbitrary linear combination of current and past v_1, v_2 except for the restriction that the coefficient on current v_1 be $b_{21}(0)$ and that on current v_2 be zero. This in turn, through (5) implies that y_2 can be taken to be an arbitrary linear combination of current and past v_1, v_2 except that the coefficients on contemporary v_1, v_2 must be $-b_{21}(0)$, 1, respectively. Finally, if g is invertible, the coefficients on right-hand side of (9) can be chosen aribitarily except for the previously listed restrictions on the 0-order coefficients.

Now the variance of $g*y_2$ is $\sum_{s=0}^{\infty}[f_1^2(s)\,\sigma_{11} + 2f_1(s)f_2(s)\sigma_{12} + f_2^2(s)\sigma_{22}]$, where σ_{ij} is the covariance of v_i with v_j. Since the summand is non-negative for all s, the minimum clearly occurs with $f_1(s)=f_2(s)=0$, all $s \neq 0$.

We have arrived at the conclusion that the following equation will hold:

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[1] To be strictly true, this result would require that v_1 and v_2 be expressible as linear combinations of current and past u_1, u_2, i.e. that u_1, u_2 have an autoregressive representation. However even when the autoregressive representation does not exist, it will be possible to make y_1 approximate an arbitrary linear combination of past v_1, v_2 arbitrarily well by appropriate choice of coefficients in (6).

10) $$g{*}y_2 = v_2 - b_{21}(0)v_1.$$

But this is precisely the form of equation given in (2), which we have already shown to imply that y_1 and y_2 will fit a model of the form (1) displaying a Granger causal ordering. Neither of the equations of the model displaying the ordering will be (5) or (6), so the ordering is certainly not structural. (One of the equations of the ordered model will be (10), however, so if we recognized the situation we could at least identify g.) Furthermore, it is the <u>policy</u> variable which will appear second in the causal ordering.[1]

Though this discussion has been framed in terms of an abstract "policy-maker", it might apply to a "representative decision-maker" as well. For example, employers setting a wage one period in advance, attempting thereby to achieve a target level of employment, might generate exactly such a structure, with spurious causal ordering from employment to wages.

Since the foregoing discussion has introduced assumptions here and there along the way, it may be worthwhile to summarize formally what has been demonstrated.

<u>Theorem 1</u> Suppose: i) (5) holds with $b_{22}(0) = 1$, b_{22} and b_{21} both possessing one-sided inverses under convolution; ii) u_1, u_2 form a covariance-stationary process with an autoregressive representation; and iii) the coefficients of (6) are chosen so as to minimize $Var[g{*}y_2]$, where g has a one-sided inverse under convolution. Then the resulting

[1] The likelihood that optimal control might generate spurious causal orderings was first pointed out to me by Milton Friedman in private correspondence.

autocovariance structure for y_1, y_2 admits a Granger ordering from y_2 to y_1, with neither equation of the ordered system in general represented by (5) or (6).

The assumptions of the theorem are in fact quite restrictive, and should not be read as implying that "optimal control generates spurious causal orderings". Perhaps most restrictive is the requirement that the objective function involve y_2 alone -- the objective cannot be to keep y_1 close to y_2, for example. Also restrictive is the requirement that the "information delay" be one period. If the delay is more than one period, no spurious ordering is generated.[1] Note the special nature of the disturbance in the policy rule (6): u_1 must be influences on the policy variable which the policy-maker cannot eliminate, but which he does anticipate and attempt to counteract; u_1 cannot represent the effect of other policy-objectives or of imperfections in the process of policy-optimization. If u_1 is identically zero, the system becomes singular; the policy equation (6) can be estimated without error; no problem of spurious ordering arises. Finally it is by no means usual for us to have any good a priori reason to assume b_{21} to be invertible. Quite often, in fact, there will be a very small contemporaneous effect of y_1 on y_2 (small $b_{21}(0)$), which is likely to lead to b_{21}'s being non-invertible.

As I showed in [ ], the Granger ordering is equivalent in a covariance-stationary system to the requirement that the two-sided distributed lag regression of y_1 on y_2 puts zero-coefficients on

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[1] Though I have not studied the case carefully, I believe that if the optimal control problem is solved by policy-makers at a smaller time unit than applies to the fitted data, an approximate spurious ordering is likely to arise.

future values of y_2. The condition thus appears related to analysis of "leads and lags", and it naturally occurs to people that leads and lags between two series can be generated by their common dependence, with different lags, on some third series, even where there is no structural causal ordering between the two original series.

Consider the system

11) $$y_1 = c_1*z + v_1$$

$$y_2 = c_2*z + v_2 \; .$$

Even if v_1 and v_2 are independent of each other and of z (so that (11) becomes what Sargent and I have called an "unobservable index" model -- see [ ]), there are no conditions on c_1 and c_2 alone which guarantee that y_1 and y_2 will satisfy a system like (1) with a Granger ordering. However, certain joint conditions on the c_i and the covariance properties of z and the v_i will imply a spurious Granger ordering.—/

The two-sided distributed lag regression of y_1 on y_2 has coefficients given by $R_{12}*R_{22}^{-1}$, where $R_{12}(s) = Cov(y_1(t), y_2(t-s))$, $R_{22}(s) = Cov(y_2(t), y_2(t-s))$, and R_{22}^{-1} is the bounded inverse of R_{22} under convolution—/. Using (11) and a convenient assumption that v_1, v_2, and z are mutually orthogonal, we can write $g = R_{12}*R_{22}^{-1}$ $= c_1*R_z*c_2'*(c_2*R_z*c_2' + R_{v2})^{-1}$, where $R_z(s) = Cov(z(t), z(t-s))$

Private conversation with Gary Skoog and unpublished work by John Geweke have been helpful to me on this topic.

The inverse Fourier transform of the inverse of the spectral density of y_2, where this latter "inverse" is taken frequency-by-frequency and is the inverse under ordinary multiplication.

is the autocovariance function of z and R_{v2} is the autocovariance function of v_2. The "'" notation is defined by $f'(s) = f(-s)$. This yields fairly directly a sufficient condition for a spurious Granger ordering:

<u>Theorem 2</u>: If in (11) z, v_1, and v_2 are mutually orthogonal, if c_2 is invertible under convolution, and if $c_2 * R_z * c_2' = \lambda R_{v2}$, where λ is a constant, then y_1 and y_2 can be written in the form (1) with a Granger ordering from y_2 to y_1.

Proof: The result follows when we rewrite the expression for g in the preceding paragraph as $g = c_1 * c_2^{-1} * c_2 * R_z * c_2' * (c_2 * R_z * c_2' + R_{v2})$ and substitute $\lambda^{-1} c_2 * R_z * c_2'$ for R_{v2}.

Theorem 2 is more interesting for the unlikeliness of its assumptions than for its positive result. The only likely example of a real-world case where its assumptions are plausible is where one is not dealing with time series at all, so that c_2, R_z, and R_{v2} all vanish for values of their arguments other than zero — i.e., all variables are serially uncorrelated. In this case Granger orderings hold in both directions, y_1 to y_2 and y_2 to y_1, so it is easy to avoid the error of treating one of those orderings as structural.

Theorem 2 does not give necessary conditions for generating spurious orderings from (11), however. Another kind of sufficient condition for a spurious ordering is:

<u>Theorem 3</u>: Suppose y_2 has a univariate representation as a finite-order autoregression of order p, that z is a finite-order moving average process of order q, that c_2 is zero for $s>r$, and that

z, v_1, and v_2 in (11) are mutually orthogonal. Then if $c_1(s)$ vanishes for $s \leq p+q+r$, y_1, y_2 can be represented as displaying a Granger ordering from y_2 to y_1.

Proof: Evident from inspection of $g = c_1^* R_z * C_2' * R_{22}^{-1}$.

This theorem's assumptions are even more artificial than Theoren 2's. To round out these results it is worthwhile to state one necessary condition for a spurious ordering of this type:

Theorem 4: If in (11) z, v_1, and v_2 are mutually orthogonal, and if it is possible to write a system of the form (1) with y_2 causally prior to y_1 in Granger's sense, then c_1 in (11) has no bounded one-sided inverse under convolution.

Proof: The proof is quite technical, and is reserved for an appendix to a later draft.

The conclusion from this latter set of three theorems is that common dependence of y_1 and y_2 on a third variable with different lags does not, in any natural way, tend to generate spurious causal orderings. In fact, one might expect instead that testing for a Granger ordering would be a useful way to distinguish lead-lag relations which do not imply a structural causal ordering from those which do.

The cases we have discussed to this point are, as far as I know, the leading examples of structural models which might generate spurious Granger orderings. They show clearly that it is a mistake to act as if a structural causal ordering is automatically implied when a Granger ordering fits the historical data. On the other hand, it seems clear that there will be a large class of applications where, if a Granger ordering fits the data, the most plausible structure consistent with that empirical result will be one in which the ordering is structural, not spurious.

What about Wold orderings? As already pointed out, they can be tested usefully only where additional identifying restrictions are available (perhaps including some assumptions of strict exogeneity!) Hence discussions of whether a Wold ordering is structural cannot abstract from the particular application to the extent possible with Granger orderings. Would orderings are also likely to be less frequently useful, since they depend on an assertion that residuals in the structural equations are known to be serially uncorrelated, an assertion which we will rarely be able to make with confidence in time series. Outside of time series, where all data are serially uncorrelated, Wold and Granger orderings reduce to the same thing, and the need for additional restrictions to make the ordering testable is common to both types. There is no reason in principle, however, why in those applications when an identified model points strongly to a system with a Wold ordering, we should not contemplate treating the ordering as structural with roughly the same mixture of confidence and apprehension as is appropriate for a Granger ordering.

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