Historical copy

Staff Report -- WP-17

OPTIMAL OPEN MARKET STRATEGY: THE USE OF INFORMATION VARIABLES

By

John H. Kareken, Thomas Muench, and Neil Wallace*

3/72

* Staff members of the Department of Economics, University of Minnesota, and Consultants to the Federal Reserve Bank of Minneapolis. All views expressed herein are the sole responsibility of the authors and should not be interpreted as representing those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

OPTIMAL OPEN MARKET STRATEGY:
THE USE OF INFORMATION VARIABLES

I. Introduction

Undeniably, the Federal Open Market Committee, which decides and executes open market policy for the Federal Reserve System, is much of the time in the dark about what has been going on in the economy. In saying this we do not mean to be at all derisive. It is simply that the Committee decides policy each day. And yet most days it does not know the values for the preceding day of all the various economic variables--or, indeed, even the obviously important ones. Thus, only on the first day of the statement week does the Committee know the value of the money stock for the day before. And on how many days of, for example, the calendar quarter does it know, to a reasonable approximation, the value of GNP for the day before? Certainly not many.

We can imagine it not mattering in the slightest that the FOMC is, as we have said, much of the time in the dark. The consensus judgment of economists is surely, though, that the past influences the present. And information lags being what they are, the presumptions must be that most days there are initial conditions which the Committee does not know. So it would seem not entirely idle to investigate how a central bank, or the responsible decision unit thereof, should conduct open market operations if much of the time it is in the dark or, in other words, if most days there are some unknown initial conditions. That is what we have done, although based on certain simplifying assumptions. And in sections II and III of this paper we present an open market operations strategy for the central bank which much of the time has to contend with unknown initial conditions. Then, in section IV, we compare this strategy with one which we believe is not unlike that of the FOMC.

The strategy of sections II-III is optimal, but as we have indicated, based on certain simplifying assumptions. There are three assumptions which, because they are so blatantly unrealistic, we particularly regret having had to make. The first is that the central bank has but one target variable. The second is that the economic structure, although stochastic, is linear. And the third is that the central bank knows with certainty the constants and coefficients of this structure. These assumptions, taken together, imply that the central bank has to optimize only over the current day, not a succession of days extending far (perhaps indefinitely far) into the future.[1/] That is why they are so convenient. Still, we should be happier if we had been able to manage with different assumptions--among them, that the central bank is uncertain about the constants and coefficients of the structure constraining it.

What we think of as our information assumptions seem to us, however, to be quite realistic. In the economic structure of this paper, which holds

---

1/ For us, the "day" is the decision period, an arbitrary but brief stretch of time. There is also the "week," made up of several days, and the quarter, made up of several weeks.

over days, there are four variables: a nominal rate of interest; two monetary aggregates; the central bank's portfolio of assets and the stock of deposits; and nominal GNP, which we take as the central bank's target variable. And, as we assume, both the interest rate and its asset portfolio are observed by the central bank essentially without lapse of time. At the beginning of each day, it finds out what the values of the interest rate and its asset portfolio were for the day before. (It obviously has to know the value of either the interest rate or its portfolio, since for us it must have used one or the other of these variables as its instrument variable.) But neither the deposit stock nor GNP is always observed without lapse of time. The central bank gets observations on the daily deposit stock only at the beginning of each week and observations on daily GNP only at the beginning of each quarter. Consequently, only on the first day of the quarter can it determine initial conditions. On all the other days, it cannot. On all the days of the quarter except the first, it must therefore guess initial conditions.

And the optimizing central bank, in guessing initial conditions for the current day, uses all such observations as have become available since it last decided open market policy. It uses the most recent observation on the interest rate or its portfolio. On the first day of the week, it also uses the newly received observations on the daily money stock. (And of course on the first day of the quarter it uses the newly received observations on daily GNP.) Precisely how it uses these observations--or how optimal open market policy depends on these observations--is what we explain in some detail in sections II and III.

We refer to all those variables which the central bank uses in guessing initial conditions, or alternatively in deciding open market policy, as information variables. This is the natural phrase, since observations on these variables constitute information about initial conditions. It would be misleading to refer to any one of them as an intermediate target variable, although this is an oft-heard phrase. Unless the economic structure is very special, the optimizing central bank does not use an intermediate target variable. It does not, that is, start out by calculating a target value or sequence of values for the deposit stock or for any other variable, except of course its true or ultimate target variables (for us, nominal GNP), and then strive over several days or weeks for equality between target and actual values. It does in effect calculate sequences of values--expected values, however, not target values. Indeed, it does this each day, but only so that subsequently it can compare certain of these values with actual values. This it has to do in deciding open market policy, for comparisons of actual and previously expected values are what determine its guesses about initial conditions. But once having observed discrepancies, it has to calculate new sequences of values--expected values, as we think of them. If there are discrepancies, then initial conditions are not what they were expected to be. And expected values are therefore not what they were expected to be.

Admittedly, there is little point in fussing about descriptive phrases. And it is perhaps acceptable to go on using the customary one--intermediate target variable. Those who do should, though, keep in mind that the optimizing central bank likely changes its target value or values each day. It has to make a change, at least on days when it observes discrepancies between actual and previously calculated target (for us, expected) values. Once discrepancies have been observed, the value of, say, the deposit stock consistent with the desired value of GNP, or any other ultimate target variable, is not what it was. It cannot be, for initial conditions are not as expected.

II. The Optimal Open Market Strategy

In this section, we first set out those simplifying assumptions to which we have referred and then show how, based on these assumptions, the central bank should proceed day by day in the conduct of open market operations. As it turns out, what the central bank should do each day depends on certain expectations. It is in section III, however, that we provide the formulae for these expectations.

We begin by giving some market equilibrium conditions. The first, for the goods market, is

$$(1)\quad Y(t) = \alpha_0 + \alpha_1 R(t) + a_1(t),\ \alpha_1 < 0,$$

where $Y(t)$ is nominal GNP for day t, $R(t)$ is the nominal rate of interest, also for day t, and $a_1(t)$ is a day t random disturbance. The second equilibrium condition, for the money market, is

$$(2)\quad R(t) = \beta_0 + \beta_1 Y(t) + \beta_2 P(t) + \beta_3[a_2(t)-a_3(t)]$$

where $P(t)$ is the central bank's asset portfolio and $a_2(t)$ and $a_3(t)$ are random disturbances, all for day t.

Underlying the second equilibrium condition are two private-sector structural (demand) equations. One is

$$(3)\quad M^d(t) = \alpha_2 + \alpha_3 Y(t) + \alpha_4 R(t) + a_2(t),\ \ \alpha_3 > 0,\ \alpha_4 < 0,$$

where $M^d(t)$ is the private sector's desired stock of deposits for day t or, since by assumption there is no coin or currency, its desired stock of money. The other, already having been transformed, is

$$(4)\quad M(t) = \alpha_5 + \alpha_6 R(t) + \alpha_7 B^d(t) + a_3(t),\ \alpha_6, \alpha_7 > 0,$$

where M(t) is the banking sector's day t stock of deposit liabilities and $B^d(t)$ is its desired stock of reserves for day t. The second equilibrium condition follows then from $M^d(t) = M(t)$ and $P(t) = B^d(t)$.2/ And, as is easily established,

$$\beta_0 = \frac{\alpha_2 - \alpha_5}{\alpha_6 - \alpha_4}$$

$$\beta_1 = \frac{\alpha_3}{\alpha_6 - \alpha_4} > 0$$

$$\beta_2 = \frac{-\alpha_7}{\alpha_6 - \alpha_4} < 0$$

and

$$\beta_3 = \frac{1}{\alpha_6 - \alpha_4} > 0 .$$

The central bank is, by assumption, subjectively certain about the α_i. It is not, however, about the $a_i(t)$, which as noted above are random variables. All the central bank knows about the $a_i(t)$ are the stochastic processes which generate them. These are

$$(5)\quad a_i(t) = \rho_i a_i(t-1) + u_i(t),\ |\rho_i| < 1,$$

2/ In equating P(t) and $B^d(t)$, we ignore a number of so-called determinants of bank reserves (for example, float). Also there is the assumption that $B^r(t)$, required reserves for day t, is determined by M(t). We might better have assumed that $B^r(t)$ is determined by M(t-j), where j > 0.

where the $u_i(t)$, random variables, are normally distributed, with

$$E(t)[u_i(t)] = 0$$

and

$$E(t)[u_i(t)u_j(t+h)] = \begin{cases} V(t)[u_i(t)] = Vu_i \text{ for } i=j \text{ and } h=0 \\ 0 \text{ otherwise}. \end{cases}$$

The symbol E(t)[·], which appears very often on the pages following, stands for the expected value of [·] as of the beginning of day t, or for the conditional expectation of [·], which depends on all observations through day t-1.

The past does then influence the present, although in a particularly simple way. But it is not important that the assumed lag structure is not complex. If it were, there would still be an optimal strategy and its derivation would be essentially the same as that which follows. It is only required that the economic structure be linear.3/

The central bank's loss function is

$$L = \sum_1^Q [Y(t)-\tilde{Y}]^2$$

where Q is the number of days in the quarter and $\tilde{Y}$ is the desired or target value of Y(t).4/ It minimizes expected loss, but subject to equations (1),

---

3/ Incidentally, it is easy enough to do away with the influence of the past. One has only to assume, for all i, that $\rho_i = 0$.

4/ We might have assumed the central bank's loss function to be

$$L = [\sum_1^Q Y(t)-\sum_1^Q \tilde{Y}]^2$$

where $\sum_1^Q Y(t)$ is the familiar quarterly GNP. It seemed unreasonable, though, to force the central bank only to "average" the Y(t) or, in effect, to make it not care about within-quarter fluctuations of Y(t). The assumption of a constant target value, $\tilde{Y}$, is of no consequence. We could easily have managed with an arbitrary path of target values, $\tilde{Y}(t)$.

(2) and (5) and of course the information at hand, which as already explained is most of the time incomplete.

To be more precise, what the central bank minimizes at the beginning of day 1 is E(1)L. And since, for $j>0$, $E(t-j)[\cdot] = E(t-j)\{E(t)[\cdot]\}$,

$$E(1)L = E(1)\{[Y(1)-\tilde{Y}]^2 + E(2)\{[Y(2)-\tilde{Y}]^2 + \cdots$$
$$+ E(Q-1)\{[Y(Q-1)-\tilde{Y}]^2 + E(Q)\{[Y(Q)-\tilde{Y}]^2\}\}\cdots\}.$$

The central bank minimizes E(1)L by first minimizing $E(Q)\{[Y(Q)-\tilde{Y}]^2\}$. The minimum of $E(Q)\{[Y(Q)-\tilde{Y}]^2\}$, denoted by $\tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}$, is in general a function of all initial conditions. Then the central bank minimizes

$$E(Q-1)\{[Y(Q-1)-\tilde{Y}]^2 + \tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}\}.$$

And so it proceeds, until in the end it has minimized

$$E(1)\{[Y(1)-\tilde{Y}]^2 + \tilde{E}(2)\{[Y(2)-\tilde{Y}]^2 + \cdots + \tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}\}\cdots\} .$$

That the central bank, proceeding in the way described, does indeed minimize E(1)L is established by induction in the following way. Whatever the outcomes for days 1,2,...,Q-1 or the observations at hand at the beginning of day Q,

$$\tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\} \leqq \bar{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}$$

where $\bar{E}(Q)$ is the expected loss associated with any policy whatever for day Q. We must now establish that if (i) $\tilde{E}(t+1)\{\sim\} \leqq \bar{E}(t+1)\{-\}$ where

$$\tilde{E}(t+1)\{\sim\} = \tilde{E}(t+1)\{[Y(t+1)-\tilde{Y}]^2 + \tilde{E}(t+2)\{[Y(t+2)-\tilde{Y}]^2 + \cdots$$
$$+ \tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}\}\cdots\}$$

and

$$\bar{E}(t+1)\{-\} = \bar{E}(t+1)\{[Y(t+1)-\tilde{Y}]^2 + \bar{E}(t+2)\{[Y(t+2)-\tilde{Y}]^2 + \cdots$$
$$+ \bar{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}\}\cdots\},$$

then (ii) $\tilde{E}(t)\{\sim\} \leqq \bar{E}(t)\{-\}$.

By hypothesis, statement (i), and the nature of the expectations operator,

$$\bar{E}(t)\{\tilde{E}(t+1)\{\sim\}\} \leqq \bar{E}(t)\{\bar{E}(t+1)\{-\}\}.$$

Adding $\bar{E}(t)\{[Y(t)-\tilde{Y}]^2\}$ to both sides,

$$\bar{E}(t)\{[Y(t)-\tilde{Y}]^2 + \tilde{E}(t+1)\{\sim\}\} \leqq \bar{E}(t)\{-\}.$$

The desired conclusion, statement (ii), follows, since by definition $\tilde{E}(t)\{\sim\}$ is less than or equal to the left-hand side of this inequality.

The central bank has two possible instrument variables, however, P and R. It therefore does not simply minimize expected loss by choice of a value for a given instrument variable. Rather, it minimizes by choice of a value for R and a value for P and then by comparing its two minimum expected losses.

With R as the instrument variable, the reduced-form equation for Y(t) is equation (1) or, by equations (5),

$$(6) \quad Y(t) = \alpha_0 + \alpha_1 R(t) + \rho_1 a_1(t-1) + u_1(t).$$

By definition, though, $u_i(t-1)$ is the sum of $E(t)[a_i(t-1)]$, the expectation of $a_i(t-1)$ as of the beginning of day t, and $e_i(t)$, the deviation of $a_i(t-1)$ from that expectation; that is,

$$(7) \quad a_i(t-1) = E(t)[a_i(t-1)] + e_i(t).$$

Therefore,

$$(8)\quad Y(t) = \alpha_0 + \alpha_1 R(t) + \rho_1 E(t)[a_1(t-1)] + \rho_1 e_1(t) + u_1(t).$$

And so

$$E(Q)[Y(Q)-\tilde{Y}]^2 = \{\tilde{Y}-\alpha_0-\alpha_1 R(Q)-\rho_1 E(Q)[a_1(Q-1)]\}^2$$

$$+ E(Q)[\rho_1{}^2 e_1(Q)^2 + u_1(Q)^2] .$$

Since the second of the terms on the right-hand side of this equation does not depend on R(Q), the optimal or minimizing value of R(Q) satisfies the equation

$$(9)\quad \tilde{Y} = \alpha_0 + \alpha_1 R(Q) + \rho_1 E(Q)[a_1(Q-1)].$$

Also,

$$\tilde{E}(Q)[Y(Q)-\tilde{Y}]^2 = \rho_1{}^2 E(Q) e_1(Q)^2 + E(Q) u_1(Q)^2$$

$$= \rho_1{}^2 \Pi_{11}(Q) + \Phi_{11}$$

where Φ is the variance-covariance matrix of the $u_i(t)$ and $\Pi(t)$ is the variance-covariance matrix of the $e_i(t)$.[5/]

With P as the instrument variable, the reduced-form equation for Y(t) is

$$(10)\quad Y(t) = \delta_0 + \delta_1 P(t) + \delta_2 a_1(t) + \delta_3[a_2(t)-a_3(t)]$$

or, by equations (5),

$$(11)\quad Y(t) = \delta_0 + \delta_1 P(t) + \Delta_\rho' a(t-1) + \Delta' u(t)$$

5/ That is,

$$\Pi(t) = E(t)[e(t)e(t)']$$

where $e(t)' = [e_1(t), e_2(t), e_3(t)]$

where

$$\Delta' = (\delta_2, \delta_3, -\delta_3)$$

$$\Delta_\rho' = (\rho_1\delta_2, \rho_2\delta_3, -\rho_3\delta_3)$$

$$a(t)' = [a_1(t), a_2(t), a_3(t)]$$

$$u(t)' = [u_1(t), u_2(t), u_3(t)]$$

$$\delta_0 = \frac{\alpha_0 + \alpha_1\beta_0}{1 - \alpha_1\beta_1}$$

$$\delta_1 = \frac{\alpha_1\beta_1}{1 - \alpha_1\beta_1} > 0$$

$$\delta_2 = \frac{1}{1 - \alpha_1\beta_1} > 0$$

and

$$\delta_3 = \frac{\alpha_1\beta_3}{1 - \alpha_1\beta_1} < 0 .$$

Then, by equations (7),

$$E(Q)[Y(Q)-\tilde{Y}]^2 = \{\tilde{Y}-\delta_0-\delta_1 P(Q)-\Delta_\rho' E(Q)[a(Q-1)]\}^2$$

$$+ E(Q)[\Delta' u(Q)u(Q)'\Delta+\Delta_\rho' e(Q)e(Q)'\Delta_\rho] .$$

It follows that the optimal value of P(Q) satisfies the equation

$$(12) \quad \tilde{Y} = \delta_0 + \delta_1 P(Q) + \Delta_\rho' E(Q)[a(Q-1)]$$

and

$$\tilde{E}(Q)[Y(Q)-\tilde{Y}]^2 = \Delta'\Phi\Delta + \Delta_\rho'\Pi(Q)\Delta_\rho .$$

The central bank therefore uses R or P as its instrument variable for day Q, depending on whether

$$(13) \quad \Phi_{11} + \rho_1^2\Pi_{11}(Q) \lesseqgtr \Delta'\Phi\Delta + \Delta_\rho'\Pi(Q)\Delta_\rho .$$

If it uses R, then open market policy is determined by equation (9). And if it uses P, then open market policy is determined by equation (12). It follows that $E(Q-1)\{[Y(Q-1)-\tilde{Y}]^2 + \tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}\}$ is either

$$E(Q-1)\{[Y(Q-1)-\tilde{Y}]^2 + \Phi_{11} + \rho_1{}^2\Pi_{11}(Q)\}$$

or

$$E(Q-1)\{[Y(Q-1)-\tilde{Y}]^2 + \Delta'\Phi\Delta + \Delta_\rho'\Pi(Q)\Delta_\rho\} .$$

But Φ is a constant. And as we show below (section III), so is $\Pi(t)$; that is to say, for all t, it is independent of the sequence of instrument variable choices and values up through day t-1. Consequently, the day Q-1 choices of instrument variable and value can affect only $E(Q-1)[Y(Q-1)-\tilde{Y}]^2$ and thus should be made so as to minimize this term. The minimization problem of day Q-1 is therefore exactly the same as that of day Q. The day Q-1 analogue of equation (9) therefore determines the optimal value of R(Q-1). The day Q-1 analogue of equation (12) determines the optimal value of P(Q-1). And the analogue of inequalities (13) determines whether R or P is the instrument variable for day Q-1.

We might continue through the minimization process, but likely it has already become apparent that, with $\Pi(t)$ independent of past instrument variable choices and values, there is only one term of

$$E(t)\{[Y(t)-\tilde{Y}]^2 + \tilde{E}(t+1)\{[Y(t+1)-\tilde{Y}]^2 + \cdots + \tilde{E}(Q)\{[Y(Q)-\tilde{Y}]^2\}\} \cdots \},$$

namely $E(t)\{[Y(t)-\tilde{Y}]^2\}$, that depends on policy for day t. The optimal value of R(t), denoted by $\tilde{R}(t)$, is therefore the solution of the day t analogue of equation (9),

$$(14) \quad \tilde{R}(t) = \frac{1}{\alpha_1}\{\tilde{Y}-\alpha_0-\rho_1 E(t)[a_1(t-1)]\}$$

and the optimal value of P(t), denoted by $\tilde{P}(t)$, is the solution of the day t analogue of equation (12),

$$(15) \quad \tilde{P}(t) = \frac{1}{\delta_1}\{\tilde{Y}-\delta_0-\Delta_\rho' E(t)[a(t-1)]\}.$$

Finally, the central bank uses R or P as its instrument variable, depending on whether

$$(16) \quad \Phi_{11} + \rho_1^2\Pi_{11}(t) \lesseqgtr \Delta'\Phi\Delta + \Delta_\rho'\Pi(t)\Delta_\rho.$$

III. The Central Bank's Expectations and a Proof of Independence

To decide open market policy for day t, the central bank has to know the $E(t)[a_i(t-1)]$ and $\Pi(t)$. In this section, we therefore derive formulae for these expectations and then, in part using that for $\Pi(t)$, give an inductive proof that $\Pi(t)$ is independent of past policy decisions.

As is well known, if Y(t-1) and $X_1(t-1), X_2(t-1), \ldots, X_N(t-1)$ are normal random variables and the $X_i(t-1)$, the only variables observed at the beginning of day t, are uncorrelated with each other, then the expectation of Y(t-1) with the $X_i(t-1)$ known, or as of the beginning of day t, is

$$(17) \quad E(t)[Y(t-1)] = E(t-1)[Y(t-1)]$$

$$+ \sum_i \eta_i(t)\{X_i(t-1) - E(t-1)[X_i(t-1)]\}.$$

E(t-1)[Y(t-1)] is the expectation of Y(t-1) with the $X_i(t-1)$ unknown, or as of the beginning of day t-1; $E(t-1)[X_i(t-1)]$ is the expectation of $X_i(t-1)$ with Y(t-1) unknown, or as of the beginning of day t-1; and

$$\eta_i(t) = \frac{C(t-1)[Y(t-1), X_i(t-1)]}{V(t-1)[X_i(t-1)]} \equiv \frac{E(t-1)[y(t-1)x_i(t-1)]}{E(t-1)[x_i(t-1)]^2}$$

where

$$y(t-1) = Y(t-1) - E(t-1)[Y(t-1)]$$

and

$$x_i(t-1) = X_i(t-1) - E(t-1)[X_i(t-1)].$$

As is also well known, the residual from the expectation given by equation (17)--that is, $Y(t-1) - E(t)[Y(t-1)]$--is independent of the $X_i(t-1)$. We use this result below in deriving the formulae for $\Pi(t)$.

At the beginning of day t, then, the central bank uses equations like equation (17) to determine the $E(t)[a_i(t-1)]$. Its information variables, the counterparts of the $X_i(t-1)$, are not the same, however, for all t. For one thing, it does not necessarily use the same variable as its instrument variable on days t and t+1. And an instrument variable cannot be an information variable, for by definition its expected value as of the beginning of day t and its actual value for day t are the same.[6/] Also, recall our information assumptions. At the beginning of day t, where $t = 1,2,\ldots,Q$, $P(t-1)$ and $R(t-1)$ are known. But $M(t-1)$ is known only for $t = 1,\omega+1,2\omega+1,\ldots,(s-1)\omega+1$, there being ω days in a week and s weeks in the quarter. And $Y(t-1)$ is known only for $t = 1$. It is not therefore simply a matter of deriving one set of formulae, the counterparts of equation (17), which hold for all t. There are several sets of formulae and to derive them we have to consider the central bank's situation at the beginning of the first day of the quarter, the second or any subsequent day of the first week and the first day of the second or any other week of the quarter.

6/ We show below, however, that the $E(t)[a_i(t-1)]$ and $\Pi(t)$ are the same, whether P was used as the instrument variable for day t-1 or R was.

A. Days of the first week

The first day of the quarter (t=1) is decidedly special. At its beginning, the central bank--knowing P(0), R(0), M(0) and Y(0)--is able actually to calculate the $a_i(0)$. In a manner of speaking, therefore, it uses equations (1), (3) and (4) to determine the $E(1)[a_i(0)] = a_i(0)$. And, of course, $\Pi(1) = 0$.

We now show how the central bank obtains the $E(t)a_i(t-1)$ and $\Pi(t)$, given that it has the $E(t-1)a_i(t-2)$ and $\Pi(t-1)$. At the beginning of day t, where $2 \leqq t \leqq \omega$, the central bank gets observations on P(t-1) and R(t-1). One of these observations provides no information about the $a_i(t-1)$, though, since the central bank must have used either P or R as its instrument variable for day t-1. If it used P, then by equation (17)

$$(18)\quad E(t)[a_i(t-1)] = E(t-1)[a_i(t-1)] + \phi_i(t)r(t-1)$$

where

$$r(t-1) = R(t-1) - E(t-1)[R(t-1)]$$

$$\phi_i(t) = \frac{C(t-1)[a_i(t-1),\ R(t-1)]}{V(t-1)[R(t-1)]}$$

and, by equations (5),

$$(19)\quad E(t-1)[a_i(t-1)] = \rho_i E(t-1)[a_i(t-2)].$$

Now, the reduced-form equation for R(t-1), derived from equations (1) and (2), is

$$(20)\quad R(t-1) = \gamma_0 + \gamma_1 P(t-1) + \Gamma' a(t-1)$$

where

$$\Gamma' = (\gamma_2, \gamma_3, -\gamma_3)$$

$$\gamma_0 = \frac{\beta_0+\alpha_0\beta_1}{1-\alpha_1\beta_1}$$

$$\gamma_1 = \frac{\delta_1}{\alpha_1}$$

$$\gamma_2 = \beta_1\delta_2$$

and

$$\gamma_3 = \frac{\delta_3}{\alpha_1} \ .$$

Therefore, by equation (19),

$$\text{(21)} \quad E(t-1)[R(t-1)] = \gamma_0 + \gamma_1 P(t-1) + \Gamma_\rho' E(t-1)[a(t-2)]$$

where

$$\Gamma_\rho' = (\rho_1\gamma_2, \rho_2\gamma_3, -\rho_3\gamma_3) .$$

And $r(t-1)$ being the difference between $R(t-1)$ and $E(t-1)[R(t-1)]$, it follows from equations (5) and (7) and (20) and (21) that

$$\text{(22)} \quad r(t-1) = \Gamma_\rho' e(t-1) + \Gamma' u(t-1).$$

Also, by equations (5), (7) and (19),

$$\text{(23)} \quad a_i(t-1) - E(t-1)[a_i(t-1)] = \rho_i e_i(t-1) + u_i(t-1),$$

so

$$\text{(24)} \quad C(t-1)[a_i(t-1), R(t-1)] = \Gamma_i V u_i + \rho_i \Gamma_\rho' \Pi_i(t-1)$$

where Γ_i is the ith element of Γ and $\Pi_i(t-1)$ is the ith column of $\Pi(t-1)$,

$$\text{(25)} \quad V(t-1)[R(t-1)] = \Gamma'\Phi\Gamma + \Gamma_\rho'\Pi(t-1)\Gamma_\rho$$

and

$$\text{(26)} \quad \phi_i(t) = \frac{\Gamma_i V u_i + \rho_i \Gamma_\rho' \Pi_i(t-1)}{\Gamma'\Phi\Gamma + \Gamma_\rho'\Pi(t-1)\Gamma_\rho} \quad .$$

Finally, by equations (5), (18) and (23),

$$(27)\quad e_i(t) = \rho_i e_i(t-1) - \phi_i(t)r(t-1) + u_i(t-1).$$

And since the $e_i(t)$ are independent of $r(t-1)$,

$$E(t)[e(t)e(t)'] = E(t-1)[e(t)e(t)'].$$

In consequence,

$$(28)\quad \Pi_{ij}(t) = \rho_i\rho_j\Pi_{ij}(t-1) + E(u_iu_j) - \phi_i(t)\phi_j(t)V(t-1)[R(t-1)],$$

or, by equations (25) and (26),

$$(29)\quad \Pi_{ij}(t) = \rho_i\rho_j\Pi_{ij}(t-1) + E(u_iu_j) - \frac{[\Gamma_iVu_i + \rho_i\Gamma_\rho'\Pi_i(t-1)]\,[\Gamma_jVu_j + \rho_j\Gamma_\rho'\Pi_j(t-1)]}{\Gamma'\Phi\Gamma + \Gamma_\rho'\Pi(t-1)\Gamma_\rho}.$$

If P was the instrument variable for day t-1, then the appropriate expectations for deciding open market policy at the beginning of day t are those of equations (18) and (29). These, however, are also the appropriate expectations if R was the instrument variable for day t-1. If it was, then by equation (17)

$$(30)\quad E(t)[a_i(t-1)] = E(t-1)[a_i(t-1)] + \Theta_i(t)p(t-1).$$

But there is a reduced-form equation for P(t-1), obtainable from equation (20),

$$(31)\quad P(t-1) = -\frac{1}{\gamma_1}[\gamma_0 - R(t-1) + \Gamma'a(t-1)],$$

so

$$(32)\quad p(t-1) = -\frac{1}{\gamma_1}[\Gamma_\rho'e(t-1) + \Gamma'u(t-1)].$$

In words, p(t-1) is simply a multiple of the discrepancy, r(t-1), which would have been observed had P been used as the instrument variable [see equation (22)]. And since the $\{a_i(t-1) - E(t-1)[a_i(t-1)]\}$ are independent of the instrument variable choice and value of day t-1 [see equation (23)], it follows that $\Theta_i = -\gamma_1\phi_i$ and, from this equality, that the $E(t)[a_i(t-1)]$ and $\Pi(t)$ are the same whether P was used as the instrument variable for day t-1 or R was.7/

Thus, assuming known unique values of the $E(t-1)[a_i(t-2)]$ and of $\Pi(t-1)$, values assumed to be independent of policy decisions up through day t-2, we have now found unique values of the $E(t)[a_i(t-1)]$ and of $\Pi(t)$. These depend on those prior expectations, on certain functions of day t-1 outcomes, and on the parameters of the structure (including the Vu_i), but not on the day t-1 policy decision. For $\Pi(t)$, this is clear from inspection of equations (29); for the $E(t)[a_i(t-1)]$, one must inspect equations (18), (19), (22) and (26). Since we showed at the outset of this section that the $E(1)[a_i(0)]$ and $\Pi(1)$ are independent of policy up through day 0, it follows that the $E(t)[a_i(t-1)]$ and $\Pi(t)$ are independent of the entire sequence of policy decisions up through day t-1. That this is true for $\Pi(t)$, which is our main concern, can be seen in a slightly different and perhaps more revealing way. Because all of the parameters of the economic structure, including the Vu_i, are known at the beginning of day 1, the central bank, using equations (29), is at that time able to calculate

---

7/ This result is easily checked. One has only to redo our derivations, starting, however, with equations (30) and (31). It is perhaps not surprising that the $E(t)[a_i(t-1)]$ and $\Pi(t)$ are the same, no matter which variable, P or R, was chosen as instrument variable at the beginning of day t-1. These variables are linearly related [equation (20)],and the disturbance in the relationship is independent of which variable was chosen as the instrument variable.

all $\Pi(t)$.[8/]

Strictly speaking, the proof and the formulae we have provided are only for $1 \leqq t \leqq \omega$. It is easy enough, however, to extend the proof for $t > \omega$. Moreover, as will be apparent, the formulae found here apply for second and subsequent days of all the remaining weeks. Only the formulae for the first day of the second and following weeks are missing.

B. The first day of the second week

At the beginning of the first day of the second week, day $\omega+1$, the central bank gets observations on the daily deposit stock for the first week. And knowing $M(1), M(2), \ldots, M(\omega)$, it revises its expectations of the $a_i(t-1)$ for $2 \leqq t \leqq \omega$. It puts itself back at the beginning of day 2 and, using observations on $M(1)$ and $P(1)$ or $R(1)$, calculates revised expectations of the $a_i(1)$, denoted by $E^*(2)[a_i(1)]$. Then it puts itself at the beginning of day 3 and, using observations on $M(2)$ and $P(2)$ or $R(2)$, calculates the revised expectations $E^*(3)[a_i(2)]$. And in the end, having calculated the $E^*(t)[a_i(t-1)]$ for $2 \leqq t \leqq \omega$, it comes to expectations of the $a_i(\omega)$, which are in part what it needs to determine open market policy for day $\omega+1$.

If P was the instrument variable for day t-1, then for day t the central bank's information variables are R and M. To use equations similar to equation (17) to determine the $E^*(t)[a_i(t-1)]$, it has to have information variables which are uncorrelated. Obviously, $R(t-1)$ and $M(t-1)$ are

---

8/ Note that there is no circularity of reasoning, since in this section we have not assumed optimality of policy, nor, in fact, anything about policy, except that each day either P or R is used as the instrument variable.

correlated. But there are transforms,

$$(33)\quad Z_1(t-1) = m^*(t-1) - \alpha_6 r^*(t-1)$$

and

$$(34)\quad Z_2(t-1) = \gamma_3 Z_1(t-1) + r^*(t-1)$$

which are uncorrelated.

By equations (4) and (20), the reduced-form equation for M(t-1) is

$$(35)\quad M(t-1) = \alpha_5 + \alpha_6\gamma_0 + (\alpha_7+\alpha_6\gamma_1)P(t-1) + \alpha_6\Gamma' a(t-1) + a_3(t-1);$$

therefore,

$$(36)\quad m^*(t-1) \equiv M(t-1) - E^*(t-1)[M(t-1)]$$

$$= \alpha_6\Gamma'\{a(t-1) - E^*(t-1)[a(t-1)]\}$$

$$+ a_3(t-1) - E^*(t-1)[a_3(t-1)] .$$

But since

$$(37)\quad a_i(t-1) - E^*(t-1)[a_i(t-1)] = \rho_i e^*(t-1) + u_i(t-1),$$

it follows that

$$(38)\quad m^*(t-1) = \alpha_6[\Gamma' e^*(t-1) + \Gamma' u(t-1)] + \rho_3 e_3^*(t-1) + u_3(t-1).$$

And by the analogue of a previous argument [see the derivation of equation (22)],

$$(39)\quad r^*(t-1) = \Gamma_\rho' e^*(t-1) + \Gamma' u(t-1),$$

so

$$(40)\quad Z_1(t-1) = \rho_3 e_3^*(t-1) + u_3(t-1)$$

and

$$(41)\quad Z_2(t-1) = \Gamma_\rho^{*\prime} e^*(t-1) + \Gamma^{*\prime} u(t-1)$$

where

$$\Gamma^{*\prime} = (\gamma_2, \gamma_3, 0)$$

and

$$\Gamma_\rho^{*\prime} = (\rho_1\gamma_2, \rho_2\gamma_3, 0).$$ 9/

Now, since $Z_1(t-1)$ and $Z_2(t-1)$ are uncorrelated,

$$\text{(42)} \quad E^*(t)[a_i(t-1)] = E^*(t-1)[a_i(t-1)] + \sum_{k=1}^{2} \phi_{ik}^*(t) Z_k(t-1)$$

where

$$\phi_{ik}^*(t) = \frac{C^*(t-1)[a_i(t-1),\ Z_k(t-1)]}{V^*(t-1)[Z_k(t-1)]} .$$

As is readily verified,

$$\phi_{i1}^*(t) = \begin{cases} 0 \text{ for } i = 1,2 \\ 1 \text{ for } i = 3 \end{cases}$$

and

$$\phi_{i2}^*(t) = \begin{cases} \dfrac{\rho_i \Gamma_\rho^{*\prime} \Pi_i^*(t-1) + \Gamma_i^{*\prime} V u_i}{\Gamma_\rho^{*\prime} \Pi^*(t-1) \Gamma_\rho^* + \Gamma^{*\prime} \Phi \Gamma^*} & \text{for } i = 1,2 \\ 0 \text{ for } i = 3 \end{cases}$$

9/ That $Z_1(t-1)$ and $Z_2(t-1)$ are uncorrelated is established by the following argument. Because the $a_i(0)$ are known, $e_3^*(1) = 0$. Therefore, by equation (40), $Z_1(1) = u_3(1)$. And since $Z_1(1)$ is known at the beginning of day 2, $u_3(1)$ is too. Consequently,

$$E^*(2)[a_3(1)] = E^*(2)[\rho_3 a_3(0) + u_3(1)] = a_3(1)$$

and $e_3^*(2) = 0$, so $u_3(2)$ is known as of the beginning of day 3. More generally, $Z_1(t) = u_3(t)$ and $e_3^*(t) = 0$ for $t \leqq \omega$, so as alleged, $Z_1(t-1)$ and $Z_2(t-1)$ are uncorrelated. It is a consequence of the particular economic structure of section II, however, that $e_3^*(t) = 0$ and that $u_3(t)$ is known as of the beginning of day t+1. In general, it is not possible to identify any of the day t disturbances at the beginning of day t+1. But this does not matter, since for any linear economic structure there exist uncorrelated transforms of all the natural information variables that contain all the information.

where $\Pi_i^*(t-1)$ is the ith column of $\Pi^*(t-1)$, the variance-covariance matrix of the $e_i^*(t-1)$ and Γ_i^* is the ith element of Γ^*.

And by equations (37) and (42),

$$(43)\quad e_i^*(t) = \rho_i e_i^*(t-1) - \sum_{k=1}^{2} \phi_{ik}^*(t) Z_k(t-1) + u_i(t-1),$$

so

$$(44)\quad \Pi_{ij}^*(t) = \begin{cases} \rho_i \rho_j \Pi_{ij}^*(t-1) + E(u_i u_j) - \phi_{i2}^*(t)\phi_{j2}^*(t)V^*(t-1)Z_2(t-1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i,j \neq 3 \\ 0 \text{ for } i = 3 \text{ or } j = 3 \ . \end{cases}$$

There is the possibility, however, of the central bank having used R as its instrument variable for day t-1. If it did, then P and M are its information variables. The uncorrelated transforms are

$$(45)\quad Z_3(t-1) = m^*(t-1) + \alpha_6 \gamma_1 p^*(t-1)$$

and

$$(46)\quad Z_4(t-1) = \frac{\gamma_3}{\gamma_1} Z_3(t-1) - p^*(t-1)$$

where $p^*(t-1)$ is the analogue of $p(t-1)$. But $Z_3(t-1) = Z_1(t-1)$ and $Z_4(t-1) = \frac{1}{\gamma_1} Z_2(t-1)$, so the $E^*(t)[a_i(t-1)]$ are the same, whether P was chosen as the instrument variable at the beginning of day t-1 or R was.

The central bank, at the beginning of day ω+1, uses equations (42) and (44)--but, as already indicated, repeatedly--to get the necessary expectations. Its situation at the beginning of day 2ω+1, or any other first day of the week, is exactly the same, however, as its situation at the beginning of day ω+1, so we have now completed our task of providing all the required expectations formulae.

C. A generalization

In this subsection, we provide the expectations formulae for the central bank which is constrained by a complex (but still linear) economic structure. These formulae are easy generalizations of those derived above. Setting them out, we nevertheless make an important point: that in deciding open market policy, the optimizing central bank "looks at everything." This point has been made by others, but perhaps not fully justified.

We assume here that there are K interest rates, the kth of which is $R_k(t)$, and N monetary aggregates, the nth of which is $M_n(t)$. At the beginning of day t where $t = 1,2,\ldots,Q$ the central bank observes $P(t-1)$ and $R_k(t)$, $k = 1,2,\ldots,K$. And at the beginning of day t where $t = 1,\omega+1,\ldots,(s-1)\omega+1$ it observes $M_n(t-1), M_n(t-2),\ldots,M_n(t-\omega)$, $n = 1,2,\ldots,N$.

It is enough to give the formulae which apply if P was the instrument variable for day t-1. Thus,

$$(47)\quad E(t)[a_i(t-1)] = E(t-1)[a_i(t-1)] + \sum_{k=1}^{K} \Psi_{ik}(t)Z_k(t-1)$$

for $i = 1,2,\ldots,I \geqq K+N+1$ and $t = 2,3,\ldots,\omega,\omega+2,\ldots,Q$ where $Z_k(t-1)$, $k = 1,2,\ldots,K$ are independent linear transforms of the $r_k(t-1)$ and

$$\Psi_{ik}(t) = \frac{C(t-1)[a_i(t-1),\ Z_k(t-1)]}{V(t-1)[Z_k(t-1)]} \quad .$$

Also,

$$(48)\quad \Pi_{ij}(t) = \rho_i\rho_j\Pi_{ij}(t-1) + E(u_iu_j) - \sum_{k=1}^{K} \Psi_{ik}(t)\Psi_{jk}(t)V(t-1)[Z_k(t-1)] .$$

And, finally, for $t = \omega+1, 2\omega+1,\ldots,(s-1)\omega+1$,

$$(49)\quad E(t)[a_i(t-1)] = E^*(t-1)[a_i(t-1)] + \sum_{k=1}^{K+N} \Psi^*_{ik}(t)Z^*_k(t-1)$$

where $Z_k^*(t-1)$ are independent linear transforms of the $r_k^*(t-1)$ and the $m^*(t-1)$ and

$$\Psi_{ik}^* = \frac{C^*(t-1)[a_i(t-1),\ Z_k^*(t-1)]}{V^*(t-1)[Z_k^*(t-1)]} ;$$

and

$$(50)\quad \Pi_{ij}^*(t) = \rho_i \rho_j \Pi_{ij}^*(t-1) + E(u_i u_j) - \sum_{k=1}^{K+N} \Psi_{ik}^*(t) \Psi_{jk}^*(t) V^*(t-1)[Z_k^*(t-1)].$$

In deciding open market policy at the beginning of day t, the optimizing central bank does then use all of its most recent observations, those on interest rates and, if there are any, those on monetary aggregates. So much is clear from equations (47) and (49). Nor does it respond to or "weight" its most recent observations in an arbitrary way. The "weights" are determined by the economic structure.

IV. An Alternate Strategy

We have but one remaining task: to evaluate an alternative strategy for open market operations. This alternative strategy is, we believe, essentially like the present strategy of the FOMC. In our judgment, it is the strategy the FOMC would currently be using if the simplifying assumptions of section II were satisfied and if it did not care at all how much interest rates change from day to day or week to week. That of course is why we bother evaluating it and why in this section we represent the FOMC as actually using it.

We require a description of our alternative (the FOMC's) strategy. At the beginning of day 1, the first day of the quarter, the FOMC selects a target value for daily GNP, denoted as above by $\tilde{Y}$. Then, using the equalities $E(1)[Y(t)] = \tilde{Y}$ for $t = 1,2,\ldots,Q$, it finds the implied sequence of values for M, which we denote by $E(1)[M(t)]$. The $E(1)[M(t)]$, which are the expected values as of the beginning of day 1, might better be referred to, however, as target values. For the FOMC, once having determined the sequence $E(1)[M(t)]$, then takes as its intermediate objective that

$$(51)\quad E(T)[M(T+h)] = E(1)[M(T+h)],$$

be satisfied for all $T = 1,\omega+1,2\omega+1,\ldots,(s-1)\omega+1$ and $h = 0,1,\ldots,\omega-1$. $E(T)[M(T+h)]$ is the expected value of $M(T+h)$ as of the beginning of day T, the first day of some week.

The FOMC goes about ensuring that equations (51) are satisfied in the following way. At the beginning of day T, it calculates a sequence of target values for what might be called its operating variable, $R(t)$, a sequence which satisfies equations (51) and which we denote by $\bar{E}(T)[R(T+h)]$ where $h = 0,1,\ldots,\omega-1$. And having calculated this sequence, the FOMC then uses it straight through the week. At the beginning of day T+h, it calculates the desired value of its portfolio, $\bar{P}(T+h)$, which by definition satisfies the FOMC's proximate objective:

$$(52)\quad E(T+h)[R(T+h)] = \bar{E}(T)[R(T+h)].$$

Thus, the desired value of $P(T+h)$, or open market policy for day T+h, is such as to make the expected value of $R(T+h)$ as of the beginning of day T+h equal to the target value of $R(T+h)$, which is the expected value as of the beginning of day T. And this latter value is such as to make the expected values of $M(T+h)$, as of the beginning of day T and as of the beginning of day 1, equal.

We can now ask, though, how nearly optimal $\bar{P}(t)$ is. How does it compare with $\tilde{P}(t)$, our optimal portfolio value? By equations (3), (1) and (5),

$$(53)\quad M(t) = \varepsilon_0 + \varepsilon_1 R(t) + A'_\rho a(t-1) + A'u(t)$$

where

$$\varepsilon_0 = \alpha_2 + \alpha_0\alpha_3$$

$$\varepsilon_1 = \alpha_4 + \alpha_1\alpha_3$$

$$A'_\rho = (\rho_1\alpha_3,\ \rho_2,\ 0)$$

and

$$A' = (\alpha_3,\ 1,\ 0).$$

Therefore,

$$(54)\quad E(t)[M(t)] = \varepsilon_0 + \varepsilon_1 E(t)[R(t)] + A'_\rho E(t)[a(t-1)]$$

and so

$$(55)\quad E(T)[M(T+h)] - E(1)[M(T+h)] = \varepsilon_1\{E(T)[R(T+h)] - E(1)[R(T+h)]\}$$

$$+ A'_\rho\{E(T)[a(T+h-1)] - E(1)[a(T+h-1)]\}\ .$$

But using equation (51), equation (55) can be solved for the target path of $R(t)$:

$$(56)\quad \bar{E}(T)[R(T+h)] = E(1)[R(T+h)] - \frac{A'_\rho}{\varepsilon_1}\{E(T)[a(T+h-1) - E(1)[a(T+h-1)]\}\ .$$

Then, by equation (52),

$$(57)\quad E(T+h)[R(T+h)] = E(1)[R(T+h)] - \frac{A'_\rho}{\varepsilon_1}\{E(T)[a(T+h-1) - E(1)[a(T+h-1)]\}.$$

Now, by equation (20),

$$(58)\quad E(T+h)[R(T+h)] = \gamma_0 + \gamma_1 P(T+h) + \Gamma'_\rho E(T+h)[a(T+h-1)]$$

and

$$(59)\quad E(1)[R(T+h)] = \gamma_0 + \gamma_1 E(1)[P(T+h)] + \Gamma_\rho' E(1)[a(T+h-1)] .$$

By an obvious extension of equation (15), however,

$$(60)\quad E(1)[P(T+h)] = \frac{1}{\delta_1}(\tilde{Y}-\delta_0) - \frac{1}{\delta_1}\Delta_\rho' E(1)[a(T+h-1)],$$

so, in place of equation (59), we have

$$(61)\quad E(1)[R(T+h)] = \gamma_0 + \frac{\gamma_1}{\delta_1}(\tilde{Y}-\delta_0) - (\frac{\gamma_1}{\delta_1}\Delta_\rho' - \Gamma_\rho')E(1)[a(T+h-1)].$$

And by substitution from equations (58) and (61) into equation (57),

$$(62)\quad \bar{P}(T+h) = \frac{1}{\delta_1}(\tilde{Y}-\delta_0) - \frac{1}{\delta_1}\Delta_\rho' E(1)[a(T+h-1)]$$

$$- \frac{1}{\gamma_1}\Gamma_\rho'\{E(T+h)[a(T+h-1)] - E(1)[a(T+h-1)]\}$$

$$- \frac{1}{\gamma_1 \varepsilon_1} A_\rho'\{E(T)[a(T+h-1)] - E(1)[a(T+h-1)]\}.$$

Consequently, by subtraction using equation (15),

$$(63)\quad \bar{P}(T+h) - \tilde{P}(T+h) = (\frac{1}{\delta_1}\Delta_\rho' - \frac{1}{\gamma_1}\Gamma_\rho')\{E(T+h)[a(T+h-1)] - E(1)[a(T+h-1)]\}$$

$$- \frac{1}{\gamma_1 \varepsilon_1} A_\rho'\{E(T)[a(T+h-1)] - E(1)[a(T+h-1)]\}.$$

Although $(\frac{1}{\delta_1}\Delta_\rho' - \frac{1}{\gamma_1}\Gamma_\rho') = (\frac{\rho_1}{\delta_1},0,0)$, it is nonetheless true that $\bar{P}(T+h) - \tilde{P}(T+h)$ vanishes identically for $T=1$, $\omega+1$, $2\omega+1,\ldots,(s-1)\omega+1$ and $h=0,1,\ldots,\omega-1$ if and only if $\rho_1=\rho_2=0$. For other than the first day of the quarter, FOMC strategy is optimal if and only if $\rho_1=\rho_2=0$.[10/] But $\rho_1=\rho_2=0$ means that the disturbances of

---

10/ Strictly speaking, this proposition holds only if there is no choice of instrument variables. The FOMC uses its portfolio as its instrument variable. It might do better, however, using some interest rate(s). See below (the concluding paragraph of this section).

equations (1) and (3) are not serially correlated or, more generally, that the past influences neither the private sector's desired expenditure for newly produced goods and services nor its desired stock of demand deposits. And this, it would seem, is not what most economists believe.

After day 1, the FOMC in effect uses M as its ultimate target variable. It starts off the quarter with a sequence of expected values, E(1)[M(1)], E(1)[M(2)],...,E(1)[M(Q)], which throughout it uses in calculating or determining desired values for R, its proximate target variable, and P, its instrument variable. This is why it is appropriate to refer to the sequence E(1)[M(1)], E(1)[M(2)],...,E(1)[M(Q)] as a sequence of target values. There are economists (we might call them "monetarists") who would have the FOMC do in part exactly what it does: namely, use M as its ultimate target variable or, more specifically, stay with initial expected values of M. It would therefore seem worthwhile to get optimality conditions for the use of M as the ultimate target variable, on the assumption, however, that the FOMC determines its target value for R(t) not at the beginning of day T, where $t=1,\omega+1,\ldots,(s-1)\omega+1$, but at the beginning of day t, where $t=1,2,\ldots,Q$. Then the desired portfolio for day t, denoted by $\hat{P}(t)$, is determined from

$$(64)\quad E(t)[M(t)] = E(1)[M(t)].$$

And by equation (35),

$$(65)\quad \hat{P}(t) - \tilde{P}(t) = \left(\frac{1}{\delta_1}\Delta_\rho' - \frac{1}{\gamma_1}\Gamma_\rho' - \frac{1}{\gamma_1\varepsilon_1}A_\rho'\right)\{E(t)[a(t-1)] - E(1)[a(t-1)]\}.$$ 11/

But since

11/ It will be noted that if h=0 in equation (63), then it and this equation are the same. What this means is that, given M as the ultimate target variable, FOMC strategy for the first day of the week is optimal.

$$\left(\frac{1}{\delta_1}\Delta'_\rho - \frac{1}{\gamma_1}\Gamma'_\rho - \frac{1}{\gamma_1\varepsilon_1}A'_\rho\right) = \left[\rho_1\left(1 - \frac{\alpha_1\alpha_3}{\alpha_4+\alpha_1\alpha_3}\right), - \rho_2\frac{\alpha_1}{\alpha_4+\alpha_1\alpha_3}, 0\right],$$

it is immediate that $\hat{P}(t) = \tilde{P}(t)$ if and only if $\alpha_4,\rho_2=0$ or $\rho_1,\rho_2=0$.

Those who would have the FOMC use M as its ultimate target variable evidently believe then that the past influences neither the public sector's desired expenditures nor its desired stock of deposits or, what is more likely, <u>that neither the past nor the rate of interest influences the private sector's desired stock of deposits</u>. But believing the latter--that neither the past nor the rate of interest influences the private sector's desired stock of deposits--it is possible to criticize the FOMC for choosing a sequence of target values for R at the beginning of the week. For, by equation (63), if $\alpha_4,\rho_2=0$ and $\rho_1\neq 0$, then $\bar{P}(t)\neq\tilde{P}(t)$.

To this point, we have simply accepted the FOMC's use of P, its portfolio, as its instrument variable. It happens, though, that even if neither the past nor the rate of interest influences the private sector's desired stock of deposits, the FOMC may still be well-advised to use R rather than P as its instrument variable (or, more generally, some interest rate or rates rather than, for example, the so-called monetary base). Some monetarists have denied that it could be; but as inspection of inequalities (16) will reveal, $\alpha_4,\rho_2=0$ (or $\rho_1,\rho_2=0$) does not imply that P is the better instrument variable.

V. Conclusion

We end as we began, by calling attention to those simplifying assumptions of ours. So far as we know, what we have provided is not the optimal strategy for open market operations, but rather an optimal strategy. And we should be clear about our alternative strategy, to which we have referred, with some justification, as the FOMC strategy. On our assumptions, it is suboptimal. But on other assumptions it may not be.

Yet, we find it difficult to imagine the use of an intermediate target variable, or several of them, being optimal. Using an intermediate target variable, the central bank presumably does not calculate a target value at the beginning of every decision period. Instead, at the beginning of some decision period it calculates a sequence of values for several periods ahead. Moreover, whatever happens, it keeps to this sequence. And that, as we have shown, is what is wrong with using an intermediate target variable.

Nor is it possible, in defense of the use of an intermediate target variable, to plead lack of knowledge, whether of recent behavior of information variables or of economic structure. Certainly it may be objected that the typical central bank does not know that version of the structure which determines daily GNP or, say, the daily values of interest rates. But then what version does it know, however imperfectly? Perhaps the one which determines quarterly GNP and quarterly averages for interest rates and the various monetary aggregates. If so, then all it ought to bother calculating is the quarterly (average) value for its instrument variable. Calculating the corresponding value for any other variable, one designated as an intermediate target variable, does no good. Knowing only that version of the structure which determines quarterly values, or only how quarterly observations are related, the central bank is unable to use daily or weekly or monthly observations in adjusting its instrument variable. In effect, it gets no information from the beginning of the quarter to the end. Thus, whether or not the central bank knows all the various versions of the economic structure, the use of an intermediate target variable, or several of them, is inappropriate.

APPENDIX: DEPOSIT STOCK OBSERVATION ERRORS

One of our information assumptions was that at the beginning of the first day of each week the central bank gets error-free observations on the deposit stock for all the days of the week just past. We might, however, have made a more realistic assumption: that it gets only estimates for the days of the week just past and, at the same time, error-free observations for the days of the week before that. Actually, it would not have made all that much difference if we had; our optimal strategy would not have been much different. But since the FOMC initially gets estimates of the daily deposit stock, it is perhaps worth explaining how the introduction of observation errors changes the optimal open market strategy.

To anticipate, the central bank's expectations change with the introduction of deposit stock observation errors. It gets its expected values of, say, initial conditions exactly as it would if there were no observation errors, by using observations on all information variables and such estimates of the deposit stock as are available. But, again, its expectations are not the same as they would be if there were no errors of observation. Nor is the sequence of optimal values for the instrument variable(s); as will be recalled, expectations of initial conditions partly determine optimal policy. The central bank proceeds, however, exactly as it would if there were no observation errors; in determining its optimal instrument variable and the optimal value thereof, it uses formulae entirely analogous to those given in the text.

The estimate of the deposit stock for day t is

$$\hat{M}(t) = M(t) + v(t)$$

where v(t) is normally distributed ,

$$E(t)[v(t)] = 0$$

$$E(t)[v(t)v(t+h)] = \begin{cases} V(t)[v(t)] = Vv \text{ for } h=0 \\ 0 \text{ otherwise} \end{cases}$$

and

$$E(t)[v(t)u_i(t)] = 0 \quad \text{for all } i.$$

The central bank starts off the first day of the quarter with full information, except about the daily deposit stock. For the last week of the quarter just past, it has only estimates of the daily deposit stock. Not knowing $M(0)$, it therefore goes back, as it were, to the beginning of day $-(\omega-1)$, the first day of the last week of the quarter just past, and calculates the $a_i(-\omega)$. (Since it has full information, determining these initial conditions is a simple exercise in arithmetic.) Then, using the calculated $a_i(-\omega)$, observations of $Y(-\omega+1)$ and $P(-\omega+1)$ or $R(-\omega+1)$ and the estimate $\hat{M}(-\omega+1)$, it calculates the revised expectations $E^*(-\omega+2)[a_i(-\omega+1)]$ and $\hat{\Pi}(-\omega+2)$. There is a generalization of equation (17) which does not require that the X's be uncorrelated and a particular version of this generalization is what the central bank uses. It proceeds day by day through the last week of the quarter just past and ends with the needed expectations $E^*(1)[a_i(0)]$ and $\hat{\Pi}(1)$ where as can be shown

$$\hat{\Pi}(1) = \tau(1)Vv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad 0<\tau(1)<1.$$

Having come up with the $E^*(1)[a_i(0)]$ and $\hat{\Pi}(1)$, the central bank is able to determine open market policy for the first day of the current quarter. Using $\hat{\Pi}(1)$, it determines which is the better instrument variable, P or R.[12/]

12/ It might be expected that by increasing Vv, R can be made the optimal instrument variable for day 1. And, indeed, if $\rho_2 \neq \rho_3$, that is so. If $\rho_2 = \rho_3$, however, then inequalities (16) apply unchanged.

And using the $E^*(1)[a_i(0)]$, it determines the optimal value of its chosen instrument variable.

At the beginning of day t where $t=2,3,\ldots\omega$, the central bank uses equations which are just like equations (18) and (29), except that $\hat{\Pi}(t)$ appears in place of $\Pi(t)$.

At the beginning of the first day of the second week, the central bank observes the daily deposit stock for the days of the last week of the previous quarter and gets estimates for the days of the first week of the current quarter. It therefore starts off the second week knowing all the $a_i(0)$. So it is able, using the known $a_i(0)$ and the estimates $\hat{M}(1)$, $\hat{M}(2),\ldots,\hat{M}(\omega)$, to revise its expectations of the $a_i(t)$, $t=1,2,\ldots,\omega$ and thus come to expected values for the $a_i(\omega+1)$.

Because $\hat{M}(t)$ is only an estimate of $M(t)$,

$$\hat{m}^*(t) = \hat{M}(t) - E^*(t)[\hat{M}(t)] = m^*(t) + v(t).$$

It follows that

$$\hat{z}_1(t) \equiv \hat{m}^*(t) - \alpha_6 r^*(t) = Z_1(t) + v(t)$$

and

$$\hat{z}_2(t) \equiv \gamma_3 \hat{z}_1(t) + r^*(t) = Z_2(t) + \gamma_3 v(t).$$

But then $\hat{z}_1(t)$ and $\hat{z}_2(t)$ are correlated, so with observation errors the revised expectations of the $a_i(t-1)$, or of initial conditions for day t, are

$$E^*(t)[a_i(t-1)] = E^*(t-1)[a_i(t-1)] + \sum_{k=1}^{2} \hat{\phi}^*_{ik}(t)\hat{z}_k(t-1) \ \underline{13/}$$

13/ These equations are substitutes for equations (42).

where

$$\begin{bmatrix} \hat{\phi}^*_{i1}(t) \\ \\ \hat{\phi}^*_{i2}(t) \end{bmatrix} = \{E^*(t-1)[\hat{z}(t-1)\hat{z}(t-1)']\}^{-1} \, C^*(t-1)[\hat{z}(t-1),a_i(t-1)]$$

and

$$\hat{z}(t-1)' = [\hat{z}_1(t-1),\hat{z}_2(t-1)].$$

Finally,

$$\hat{\Pi}^*_{ij}(t) = \rho_i\rho_j\hat{\Pi}^*_{ij}(t-1) + E(u_iu_j) - \hat{\phi}^{*\prime}_i(t)E^*(t-1)[\hat{z}(t-1)\hat{z}(t-1)']\hat{\phi}^*_j(t).$$ 14/

14/ These equations substitute for equations (44). Note that $\hat{\Pi}^*(1) = 0$.