A NOTE ON THE TRUNCATED

NORMAL DISTRIBUTION

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In the application of probability models to economic problems, many instances arise where a particular random variable is theoretically constrained at one or both tails. Even though a normal distribution may be involved in establishing the "significance" of a particular coefficient in a regression, it may be improper to use the normal distribution to generate observations of that coefficient in a Monte Carlo experiment because extreme values may cause the coefficient to change sign. One way, of course, of avoiding this problem is to sample from one of the many approximations to the normal such as those presented by Chew [1] and Hoyt [4] where the extreme values may be set consistent with the theoretical model.

Another way of avoiding the extreme value problem is to sample from a normal distribution but to discard the observations which are outside certain limits. This procedure defines a truncated normal variate[1/] with two desirable properties: (1) the sampling distribution is consistent with the normal probability model in the sense that the density functions are proportional, and (2) the sample is especially easy to generate since normal random number generators are standard subroutines at most computer installations. The purpose of this note is to present the computation formulae for the first two moments of the truncated normal distribution together with a table of selected values.[2/] Cohen [2] has previously published a formula for the variance of the truncated normal distribution corresponding to (5) below. However, there is an error in

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[1/] Cf. Cramér [3]. Kendall and Stuart [5] discuss the problem of inferring the parent population given the truncated sample.

[2/] By utilizing the off diagonal elements of the table, this procedure also provides a convenient method of generating skewed distributions over finite intervals.

his second equation. The term inside the second set of brackets on the right hand side of that equation should read as equation (3) below.[3/]

Let X be a standard normal variate with mean $\mu_X = 0$ and variance $\alpha_X^2 = 1$. If in sampling from the X population we discard values of X which are less than c and greater than d, then we have the truncated normal variate Y defined on [c,d] with density function

(1) $$g(y) = \alpha f(y)$$

where $$f(y) = \frac{}{\sqrt{2\Pi}} e^{-y^2/2} = ke^{-y^2/2}$$

is the density function of X, and $\alpha = \frac{1}{F(d)-F(c)}$

where $$F(x) = \int_{-}^{x} f(u)du.$$

The mean of Y is given by

(2) $$\mu_Y = \alpha K\int_c^d ye^{-y^2/2} dy.$$

Letting $u = \frac{y^2}{2}$, we have

$$\mu_Y = \alpha K\int_{c^2/2}^{d^2/2} e^{-u} du$$

or

(3) $$\mu_Y = \alpha[f(c^2/2) - f(d^2/2)].$$

The variance of Y is

(4) $$\alpha_Y^2 = \alpha\int_c^d Ky^2 e^{-y^2/2} dy - \mu_Y^2$$

$$= \alpha K\int_c^d yd(-e^{-y^2/2}) - \mu_Y^2$$

[3/] Cohen's equation (in his notation) is:

$$(\sigma_c)^2 = \left[1 + \frac{k_1 f(k_1)-k_2 f(k_2)}{\int_{k_1}^{k_2} f(u)du}\right] - \left[\frac{f(k_1)-f(k_2)}{\int_{k_1}^{k_2} f(u)du}\right]^2$$

$$= \alpha[-Kye^{-y^2/2} \Big|_c^d + \int_c^d Ke^{-y^2/2}\, dy] - \mu_Y^2$$

$$= \alpha[cf(c) - df(d) + F(d) - F(c)] - \mu_Y^2 \tag{5}$$

Note that as $d \to +\infty$ formulae (3) and (5) approach the formulae given by Cramér [3] for the standard normal variable. The mean and variance of the truncated normal may be computed from formulae (3) and (5) with the aid of only the standard normal tables. Although the ordinate tables are not customarily found in textbook appendices, they are readily available in such standard references as Owen [6]. With the aid of a computer, it is equally as easy to compute these moments directly from (2) and (4) using numerical methods such as Simpson's Rule. This was the procedure used in preparing the table below, where the computations should be exact except possibly for rounding of the last digit.

REFERENCES

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[4] Hoyt, John P. "A Simple Approximation to the Standard Normal Probability Density Function," The American Statistician, 22, No. 2 (April 1968), 25-26.

[5] Kendall, Maurice G. and Stuart, Allen. The Advanced Theory of Statistics, Vol. II, New York: Hafner Publishing Company, 1961, pp. 522-7.

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(Means) and Variances of the
Truncated Normal Distribution

Right Limits (d)

Left Limits (c)	0	.5	1	1.5	2	2.5	3
0	(0) 0	(0.24479) 0.02065	(0.45992) 0.07963	(0.62194) 0.16471	(0.72286) 0.25129	(0.77241) 0.31463	(0.79108) 0.34744
- .5	-(0.24479) 0.02065	(0.00000) 0.08057	(0.20663) 0.17278	(0.35625) 0.28024	(0.44575) 0.37660	(0.48816) 0.44082	(0.50367) 0.47188
-1	-(0.45992) 0.07963	-(0.20663) 0.17278	(0.00000) 0.29116	(0.14519) 0.41570	(0.22966) 0.51982	(0.26876) 0.58559	(0.28278) 0.61614
-1.5	-(0.62194) 0.16471	-(0.35625) 0.28024	-(0.14519) 0.41570	(0.00000) 0.55152	(0.08296) 0.66116	(0.12081) 0.72854	(0.13423) 0.75918
-2	-(0.72286) 0.25129	-(0.44575) 0.37660	-(0.22966) 0.51982	-(0.08296) 0.66116	(0.00000) 0.77382	(0.03755) 0.84229	(0.05078) 0.87315
-2.5	-(0.77241) 0.31463	-(0.48816) 0.44082	-(0.26876) 0.58559	-(0.12081) 0.72854	-(0.03755) 0.84229	-(0.00000) 0.91124	(0.01320) 0.94222
-3	-(0.79108) 0.34744	-(0.50367) 0.47188	-(0.28278) 0.61614	-(0.13423) 0.75918	-(0.05078) 0.87315	-(0.01320) 0.94222	-(0.00000) 0.97324