International Finance Discussion Papers

Number 323

May 1988

MONEY, INTEREST, AND CAPITAL IN A CASH-IN-ADVANCE ECONOMY

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ABSTRACT

A cash-in-advance constraint on consumption is incorporated into a standard model of consumption and capital accumulation. Monetary policy consists of lump-sum cash transfers. Methods are developed for establishing the existence and uniqueness of an equilibrium, and for explicitly constructing this equilibrium. The model economy's dependence on monetary policy is explored.

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Wilbur John Coleman II*

1. INTRODUCTION

Does monetary policy induce a substitution between consumption and capital? Savings-based models[1] of money demand exhibit a substitution: higher inflation increases the relative cost of saving via money and hence leads to a substitution from money to capital (and, in equilibrium, out of consumption). In, however, transactions-based models of money demand, inflation may have no real effect: the inflation tax may act like a constant proportional consumption tax, a tax which acts like a lump-sum tax. Clearly money holdings in a cash-in-advance economy contain a significant transactions-based component, but does it also contain a savings-based component; can, for example, the cash-in-advance constraint be slack in a deterministic economy? Monetary policy, though, generally consists of more than a deterministic money growth rate, and a varying monetary growth rate should have an effect similar to a varying consumption tax. In this paper I set up a general equilibrium model to address these issues.

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*I wish to thank Robert E. Lucas, Jr., Lars P. Hansen, and Robert M. Townsend for their many helpful comments, and Heidi Lyss for the excellent graphics. The author is a staff economist in the International Finance Division. This paper represents the views of the author and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or other members of its staff.

The framework of this paper stems from Lucas and Stokey [19], and Townsend [26]. Townsend endogenizes the demand for money by carefully spelling out a trading environment for capital, home produced goods, and market produced goods. Lucas and Stokey develop an endowment economy consisting of cash and credit goods in which, depending upon the timing of information and monetary shocks, one of many joint distributions describes the relationship among the economy's variables. Insofar as Townsend includes capital, and Lucas and Stokey do not, the model developed here is closer to his. But in terms of monetary policy, exploring its interactions with the real economy, and the recursive methods used to solve the model, the model developed here is much closer to Lucas and Stokey. In some sense, this paper can be viewed as extending Lucas and Stokey's recursive methodology to Townsend's model. With this setup I can pose some important questions which Lucas and Stokey could not, and I can answer these questions at a level of detail which Townsend could not.

Overview of the Model

The model developed here is based on the infinite horizon Planned Growth (PG) economy of, say, Brock and Mirman [2]. Think of a market based extension where capital is either entirely carried over from the previous period or purchased along well-established lines (fixed suppliers), but consumption is purchased in a decentralized market. Thus while consumption requires cash, cash is not required to carry over existing capital or to accumulate new capital. Agents begin a period with money and a value of output, and purchase consumption, capital, and end-of-period money. Consider a monetary policy which consists of beginning-of-period lump-sum cash transfers. Using this Monetary Growth

(MG) economy, I will try to address the consumption-capital substitution questions.[2]

A surprising result is that the cash-in-advance constraint *can* be slack in a deterministic MG economy. This is possible because the return on capital--the real interest rate--must be greater than or equal to the return on money--minus inflation. Suppose the cash-in-advance constraint is always binding. This means that the rate at which cash is spent is equal to the rate at which output is consumed, which is equal to the inflation rate (fix the money supply). Nothing guarantees that minus this inflation rate is not greater than the real interest rate. In situations where money's return is greater than capital's, inflation must rise. This can only happen if cash is spent at a rate faster than the rate at which output is consumed, which can only occur if excess cash is held. Hence a savings motive to holding money can occur.

Much more straightforword results are obtained in addressing the variable inflation tax issue. Clearly if money supply shocks are unpredictable, then so is the consequent inflation tax, thus this monetary policy is neutral. But if money varies predictably, relatively high expected inflation increases the cost of consuming in these episodes, thus leading to a substitution out of consumption and real balances and into capital. A wide variety of comovements between real and nominal variables exists, where the one selected depends upon the correlation between money supply shocks and production shocks.

The Solution Methodology

The Monetary Growth economy could quickly lead to a dead end. This economy should not be Pareto Optimal, hence a central planner cannot, via Debreu [8], be invoked to solve the model. Competitive equilibrium conditions can be obtained by other means, but the central planner's approach is amenable to explicitly constructing the solution (the value function is usually the fixed point of a contraction mapping). Can the solution be constructed by other means? I spend a fair amount of time, in this paper, doing just that. First, to make matters simple, this alternate approach is developed in a similar setting, the underlying PG economy, where the central planner's method is at hand. Clearly success here is a prerequisite to success in richer models. This approach is then extended to the MG economy.

In the Planned Growth economy, my approach is to obtain convergence by iterating some fixed point equation, call it A, which can be motivated without the use of a central planner. Most of the problems arise because in general A will not be a contraction, and its domain is likely to be infinite dimensional (e.g. a space of consumption functions). The difficulty in establishing an equilibrium is thus finding a *continuous* A under which some *compact* subset is *invariant*. Once this function and compact subset are constructed, Schauder's fixed point theorem guarantees the fixed point's existence. A further difficulty arises since Schauder's fixed point theorem (versus, say, Banach's) does not guarantee the fixed point's uniqueness nor does it provide a method for the fixed point's construction. For the PG economy, however, an A is found which is monotone and concave: a unique fixed point, obtained by iterating A, thus exists.

Although the Monetary Growth economy's fixed point equation is similar to the Planned Growth economy's A, I cannot prove any general existence or uniqueness theorems except for the special case of log utility.[3] Since this special case does not duplicate the underlying PG's equilibrium, it is worth presenting here. The fixed point equation is similar enough, though, so that the algorithm based on it exhibits essentially all the desirable properties which A does (for the examples I tried). Another way to view this paper is one in which the concept of a particular algorithm is developed, proven to work, and shown to work for the core PG economy, and an extension to the MG economy is shown to work as well.

Outline of the Paper

The PG and MG models--and algorithms to solve them--are developed in the following two sections. The specific questions posed in this Introduction are then addressed, via some simulations, in Section 4. Section 5 concludes this paper.

2. THE PLANNED GROWTH MODEL

Problem Statement

The Planned Growth problem is this: for any discount rate $\beta \in (0,1)$, utility function $u \in U$, and production function $f \in F$, find a time stationary consumption function $c \in C_f(K)$ which maximizes, for any initial capital stock $x_0 \in K$, the quantity

$$\sum_{t=0}^{\infty} \beta^t u[c(x_t)],$$

subject to:

$$c(x_t) + x_{t+1} = f(x_t).$$

U is the set of u's such that

$$u: \mathbb{R}_+ \to \mathbb{R},$$

u is twice continuously differentiable,

$$u'(c) > 0, \quad \lim_{c \to 0} u'(c) = \infty, \quad \lim_{c \to \infty} u'(c) = 0,$$

$$u''(c) < 0, \quad \lim_{c \to 0} u''(c) = -\infty, \quad \lim_{c \to \infty} u''(c) = 0.$$

F is the set of f's such that

$$f:\mathbb{R}_+ \to \mathbb{R}_+, \quad f(0) = 0,$$

f is twice continuously differentiable,

$$f'(x) > 0, \quad f''(x) < 0, \quad \beta f'(0) > 1, \quad f'(0) < \infty,$$

$$f(\bar{x}) = \bar{x} \text{ for some } \bar{x} > 0.$$

The set of maintainable capital stocks is

$$K = [0,\bar{x}].$$

The feasible set of c's consists of

$$C_f(K) = \left\{c: \begin{array}{l} c:K \to K \text{ is continuous,} \\ 0 \leq c(x) \leq f(x). \end{array}\right.$$

Equip $C_f(K)$ with the sup norm. As is well known, the solution to the Planned Growth problem is a $c \in C_f(K)$ such that

$$u'[c(x)] = \beta u'\{c[f(x) - c(x)]\}f'[f(x) - c(x)] \quad \text{for} \quad x \in K. \tag{2.1}$$

The task at hand is to find such a c.

Existence

My attack on (2.1) is to construct a continuous self-map A defined on a convex, compact subset $\tilde{C}_f(K) \subset C_f(K)$. Define, first,

$$\tilde{C}_f(K) = \left\{c: \begin{array}{l} c:K \to K \text{ is continuous,} \\ 0 \leq c(x) \leq f(x), \\ 0 \leq c(y) - c(x) \leq f(y) - f(x) \text{ for } y \geq x. \end{array}\right.$$

The third condition defining $\bar{C}_f(K)$ is equivalent to requiring that both c and $f - c$ are increasing functions. Clearly $\bar{C}_f(K)$ is convex, and the following proposition establishes its compactness.

Proposition 2.1. $\bar{C}_f(K)$ is compact.[4]

Proof. I first show that $\bar{C}_f(K)$ is equicontinuous. This is true if for any $\epsilon > 0$ there exists a $\delta > 0$ such that the following is true: $|c(y) - c(x)| < \epsilon$ holds for any $c \in \bar{C}_f(K)$ and any two points $x,y \in K$ satisfying $|y - x| < \delta$.[5] Choose $\delta = \epsilon/f'(0) > 0$. Using properties of $\bar{C}_f(K)$ and F,

$$|c(y) - c(x)| \leq |f(y) - f(x)| \leq f'(0)\delta = \epsilon.$$

$\bar{C}_f(K)$ is thus equicontinuous. $\bar{C}_f(K)$ is also norm-bounded so by the Arzela-Ascoli theorem[6] it is relatively compact; $\bar{C}_f(K)$ is compact since it is closed. ∎

Define the fixed point equation A by

$$u'[(Ac)(x)] = \beta u'\{c[f(x) - (Ac)(x)]\}f'[f(x) - (Ac)(x)] \qquad (2.2)$$

Proposition 2.2. A unique $A(c)$, $A:\bar{C}_f(K) \to \bar{C}_f(K)$, exists which satisfies (2.2).

Proof. Define $A(c)$ pointwise as the y for which

$$z = \beta u'\{c[f(x) - y]\}f'[f(x) - y] - u'(y)$$

equals zero. Clearly (unless $y = 0$ is the root) z is negative for y close to 0, positive for y close to f(x), and strictly increases as y increases . This proves the existence of a unique A(c). Since z increases with y and decreases with x, A(c) is increasing in x; by (2.2) f - A(c) is increasing in x. Hence $A:\bar{C}_f(K) \subset \bar{C}_f(K)$. ∎

Proposition 2.3. A is continuous and monotone.

Proof. Since $\bar{C}_f(K)$ is equicontinuous and K is compact, continuity follows from the pointwise convergence of $A(c_n) \to A(c)$ as $c_n \to c$,[7] which follows from the continuity of the composing functions. Monotonicity requires $\hat{c} \leq \tilde{c}$ to imply $A(\hat{c}) \leq A(\tilde{c})$. Suppose $(A\hat{c})(x) > (A\tilde{c})(x)$ for this particular value of x, then

$$u'[(A\hat{c})(x)] < \beta u'\{\tilde{c}[f(x) - (A\hat{c})(x)]\}f'[f(x) - (A\hat{c})(x)].$$

This implies

$$\hat{c}[f(x) - (A\hat{c})(x)] > \tilde{c}[f(x) - (A\hat{c})(x)],$$

which contradicts $\hat{c} \leq \tilde{c}$. ∎

The existence of A's fixed point can now be established. To do so I will make use of a Schauder fixed point theorem which states that a continuous self-map of a non-empty convex and compact subset of a normed space has at least one fixed point.[8]

Theorem 2.4. There exists a $c \in \bar{C}_f(K)$ such that

$$c = A(c).$$

Proof. By Proposition 2.1 $\bar{C}_f(K)$ is a convex and compact subset of the normed space $C_f(K)$. By Proposition 2.2 A maps into itself; Proposition 2.3 establishes A's continuity. Hence, via Schauder, a fixed point exists. ∎

One shortcoming of the set $\bar{C}_f(K)$, however, is that

$$0 \in \bar{C}_f(K), \quad 0 = A(0),$$

as is obvious from (2.2), so any existence theorem for a fixed point in $\bar{C}_f(K)$ does not guarantee the existence of a non-zero fixed point. In the Appendix, though, the existence of a fixed point in the set $\bar{C}_f(K) - 0$ is established.

The following property of the solution(s) c will be needed.

Proposition 2.5. A nonzero fixed point $c = A(c)$, $c \in \bar{C}_f(K) - 0$, must satisfy $c(x) > 0$ for $x > 0$.

Proof. Since c is an increasing function, if $c(x_0) = 0$ then $c(x) = 0$ for $x \leq x_0$. The solution, though, cannot be of this form. Choose x_0 such that $c(x_0) = 0$ and $c(x) > 0$ for $x > x_0$. At x_0, the solution must satisfy

$$u'[c(x_0)] = \beta u'\{c[f(x_0) - c(x_0)]\}f'[f(x_0) - c(x_0)].$$

But, since $c[f(x_0)] > 0$ $(f(x_0) > x_0)$, the right hand side is bounded while the left is not. ∎

Uniqueness

I will prove the uniqueness of a positive solution by developing and exploiting the concavity of A.[9] Unfortunately, I need an additional (sufficient, but not necessary) restiction on U to guarantee A's concavity.

Define the set U' as

$$U' = U \cap \{u: \; u'(xy) = u'(x)u'(y)\}.$$

For example, $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma > 0$, is in U'. Consider, now, the following definition of concavity.

Definition. Call the monotone function $A:\bar{C}_f(K) \to \bar{C}_f(K)$ *f-concave* if the following two properties hold for any arbitrarily small $x_0 > 0$. (1) For each $c \in \bar{C}_f(K)$ such that $c(x) > 0$ for $x > 0$, an α exists such that

$$\alpha(c,x_0)f(x) \leq (Ac)(x) \leq f(x) \quad \text{for} \quad x \geq x_0, \quad \alpha > 0. \tag{2.3}$$

(2) For any $0 < t < 1$, an η exists such that

$$(Atc)(x) \geq \eta(t,c,x_0)t(Ac)(x) \quad \text{for} \quad x \geq x_0, \quad \eta > 1. \tag{2.4}$$

Proposition 2.6. For any $u \in U'$, A is f-concave.

Proof. Define $\alpha(c,x_0)$ by

$$\alpha(c,x_0) = \min_{x_0 \leq x \leq \bar{x}} \frac{(Ac)(x)}{f(x)} > 0.$$

This α satisfies (2.3). For condition (2.4) note that $(Atc)(x) < (Ac)(x)$ for $x \geq x_0$, hence

$$u'[(Atc)(x)] < \beta u'\{tc[f(x) - (Ac)(x)]\}f'[f(x) - (Ac)(x)] \quad \text{for} \quad x \geq x_0.$$

Since $u \in U'$, the right hand side above is just $u'[t(Ac)(x)]$, hence an η which satisfies (2.4) is

$$\eta(t,c,x_0) = \min_{x_0 \leq x \leq \bar{x}} \frac{(Atc)(x)}{t(Ac)(x)} > 1. \qquad \blacksquare$$

Theorem 2.7. The fixed point of A is unique in $\bar{C}_f(K) - 0$ if, as in Proposition 2.6, A is f-concave.

Proof. Assume there exist two nonzero solutions c_1 and c_2. By Proposition 2.5 $c_1(x) > 0$ and $c_2(x) > 0$ for $x > 0$. Suppose, for now,

$$c_1(x) = c_2(x) \quad \text{for} \quad 0 \leq x \leq x_0, \quad x_0 > 0,$$

where x_0 can be arbitrarily small. Assume, without loss of generality,

$$c_1(x) < c_2(x) \quad \text{for some} \quad x > x_0. \tag{2.5}$$

Since A is f-concave,

$$\begin{aligned} c_1(x) &= (Ac_1)(x) \quad \text{for all} \quad x, \\ &\geq \alpha(c_1,x_0)f(x) \quad \text{for} \quad x \geq x_0. \end{aligned}$$

Since c_1 and c_2 are equal for $x \le x_0$, and since $c_2(x) \le f(x)$ for all x,

$$c_1(x) \ \ge\ \alpha(c_1,x_0)c_2(x) \quad \text{for all } x. \tag{2.6}$$

Because of conditions (2.5) and (2.6), there exists a t_0 $(\alpha(c_1,x_0) \le t_0 < 1)$ such that

$$c_1(x) \ge t_0c_2(x) \quad \text{for all } x,$$

and, for any $t > t_0$,

$$c_1(x) < tc_2(x) \quad \text{for some } x \ge x_0. \tag{2.7}$$

Combine these results and use the monotoneity of A to obtain, for every $x \ge x_0$,

$$\begin{aligned} c_1(x) &= (Ac_1)(x) \\ &\ge (At_0c_2)(x) \\ &\ge \eta(t_0,c_2,x_0)t_0(Ac_2)(x) \\ &\ge \eta(t_0,c_2,x_0)t_0c_2(x). \end{aligned}$$

Since $\eta(t_0,c_2,x_0)t_0 > t_0$, (2.7) contradicts this last inequality. Thus if c_1 and c_2 agree on $[0,x_0]$ for an arbitrarily small x_0, then they agree on $[0,\bar{x}]$. In the limit, then, as $x_0 \to 0$, if c_1 and c_2 agree at 0 (which they must) then they agree on $[0,\bar{x}]$. ∎

See Krasnosel'skiĭ and Zabreĭko [15] for a version of the concavity definition and Theorem 2.7 where $\alpha(c,x_0)$ does not depend on x_0.

Constructing the Solution

The result of this section is stated in the following theorem.

Theorem 2.8.[10] The sequence $\{c_n\}$ defined by

$$c_{n+1} = A(c_n), \quad c_0 \in \bar{C}_f(K) - 0 \quad \text{given},$$

converges to the unique nonzero fixed point, say c^*, if A is f-concave.

Proof. Since $c_0 \in \bar{C}_f(K) - 0$, there exists a nonzero $\underline{c}$ such that

$$0 \leq \underline{c} \leq c_0 \leq f,$$

and because A is monotone,

$$A^n(\underline{c}) \leq A^n(c_0) \leq A^n(f).$$

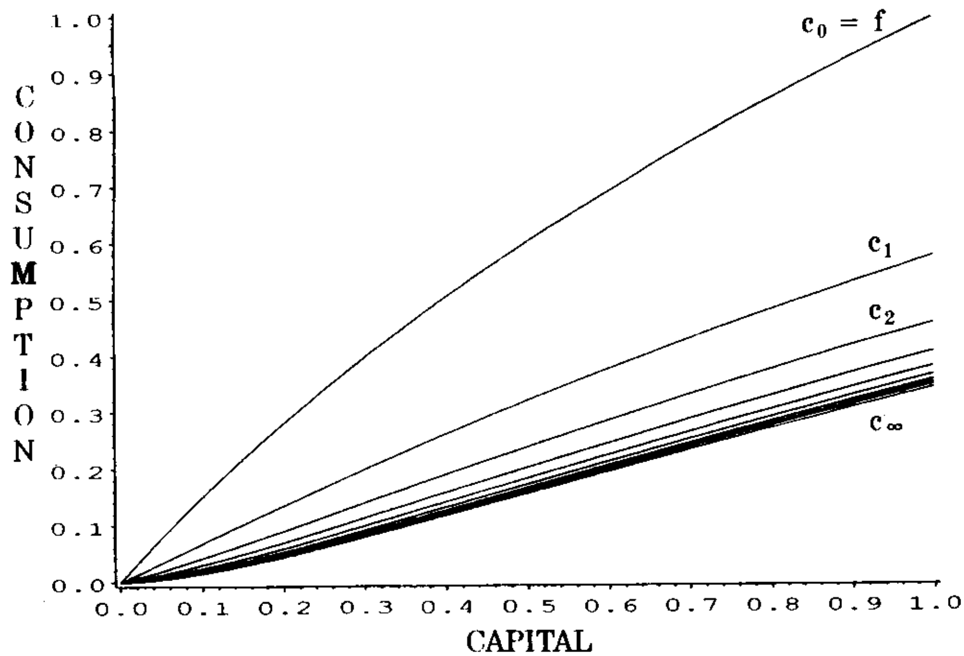
Since $\bar{C}_f(K)$ is compact, both $A^n(\underline{c})$ and $A^n(f)$ converge and by Theorem 2.7 they converge to the unique solution c^* (it can be easily shown that $A^n(\underline{c})$ does not converge to zero; use, e.g., a $\underline{c}$ of the form defined in the Appendix). Thus, in the limit,

$$c^* \leq \lim A^n(c) \leq c^*.$$

Convergence here is pointwise but, as in Proposition 2.2, since $\bar{C}_f(K)$ is equicontinuous and K is compact,

$$\| \lim A^n(c) - c^* \| = 0. \qquad \blacksquare$$

Figure 1 displays an actual converging sequence of consumption functions based on the foregoing algorithm.[11] As this figure shows, convergence is obtained fairly rapidly and smoothly. These results are not specific to the particular values of the parameters.



$\beta = .95,\ u'(c) = c^{-.5},\ f(x) = \{[1 + 1.6x]^{.1} - 1\}/.1$

FIG. 1. PG CONVERGENCE

3. THE MONETARY GROWTH MODEL

Problem Statement

The Monetary Growth economy is comprised of a single representative consumer whose ex post utility from a consumption sequence $\{c_t\}_{t=0}^{\infty}$ is

$$\sum_{t=0}^{\infty} \beta^t u(c_t).$$

Consider utility functions in the set

$$U \cap \{u: \ cu'(c) \leq B < \infty, \ \text{for} \ c \in K, \ K \ \text{redefined below}\}.$$

Treat this consumer as an expected utility maximizer. To complete the description of the consumer's problem I will specify the distribution with respect to which expectations are taken and specify what the consumer is choosing to maximize expected utility.

The economy's exogenous variables are summarized in a sequence of shocks

$$\{s_t\}_{t=0}^{\infty}, \quad s_t \in S, \quad S \ \text{is a finite set.}$$

At time 0, the shock $s_0 \in S$ is known. The joint distribution of these shocks is determined by the Markov probabilities

$$\Pr\{s_{t+1} = s' | s_t = s\} = \pi(s'|s), \quad t \geq 0.$$

Aggregate capital at time t is denoted as X_t, which is determined recursively by

$$X_{t+1} = g(X_t, s_t), \quad t \geq 0, \quad X_0 \in K \text{ given}, \quad g: K \times S \to K.$$

To the representative consumer, the aggregate investment function g is a known and fixed function. Denote the representative consumer's capital stock as x_t, which produces, at time t, an output of

$$f(x_t, s_t).$$

Consider production functions in the set $F(S)$, where

$F(S)$ is the set of f's such that

$$f: \mathbb{R}_+ \times S \to \mathbb{R}_+, \quad f(0,s) = 0 \text{ for every } s \in S,$$

$$f(\cdot, s) \text{ is twice continuously differentiable},$$

$$0 < f'(x,s) < \infty, \quad f''(x,s) < 0 \text{ for every } x \in \mathbb{R}_+, \ s \in S,$$

$$\text{an } \bar{x} > 0 \text{ exists such that } x > \bar{x} \text{ implies } f(x,s) < x \text{ for every } s \in S, \text{ and } f(\bar{x},s) = \bar{x} \text{ for some } s \in S.$$

Define $K = [0, \bar{x}]$ as before, but with this new value of $\bar{x}$. As in section 2, $f(K \times S) \subset K$.

The aggregate money supply at time t is denoted as M_t, where the sequence of money, $\{M_t\}$, is determined recursively by

$$M_{t+1} = h(s_{t+1})M_t, \quad M_0 \in (0,\infty) \text{ known}, \quad h:S \rightarrow (0,\infty).$$

Refer to h as the monetary policy function; this function is known and fixed. Consider only $h \in H$, where

$$H = \{h: \; h(s) > 0, \; \beta \sum_{s' \in S} \frac{\pi(s'|s)}{h(s')} < 1 \; \text{ for any } \; s \in S\}.$$[12]

Denote the representative consumer's time t money holdings, after any monetary transfer, as m_t, and the consumer's time t demand for money to be carried into $t + 1$ as m'_{t+1}. At $t + 1$, the single representative consumer receives the lump-sum monetary transfer

$$[h(s_{t+1}) - 1]M_t.$$

Post-transfer money, m_{t+1}, then is

$$m_{t+1} = m'_{t+1} + [h(s_{t+1}) - 1]M_t.$$

Refer to m'_{t+1} as pre-transfer money.

The price of consumption (and capital) at time t in terms of time t money is denoted as P_t, which is determined by

$$P_t = M_t p(X_t, s_t), \quad p:K \times S \rightarrow (0,\infty].$$

The function p is time-stationary and known. The discount on a bond is denoted as q_t, where

$$q_t = q(X_t, s_t), \quad q: K \times S \to [0,1].$$

Denote the representative consumer's time t demand for a bond maturing at $t + 1$ as b_{t+1}. This bond is thus purchased with $q_t b_{t+1}$ dollars and pays b_{t+1} dollars, both payments made with contemporaneous money.

With output $f(x_t, s_t)$, post-transfer money holdings m_t, and bonds b_t, the consumer must choose current consumption c_t, capital stock x_{t+1}, pre-transfer money holdings m'_{t+1}, and bonds b_{t+1}. This choice must obey the budget constraint

$$p(X_t, s_t)(c_t + x_{t+1}) + \frac{m'_{t+1}}{M_t} + q_t \frac{b_{t+1}}{M_t} = p(X_t, s_t) f(x_t, s_t) + \frac{m_t}{M_t} + \frac{b_t}{M_t}, \tag{3.1}$$

and the cash-in-advance constraint

$$p(X_t, s_t) c_t \leq \frac{m_t}{M_t}. \tag{3.2}$$

From a functional perspective, then, the representative consumer chooses four functions such that

$$c_t = C(x_t, X_t, \frac{m_t}{M_t}, \frac{b_t}{M_t}, s_t), \tag{3.3}$$

$$x_{t+1} = G(x_t, X_t, \frac{m_t}{M_t}, \frac{b_t}{M_t}, s_t), \tag{3.4}$$

$$\frac{m'_{t+1}}{M_t} = L(x_t, X_t, \frac{m_t}{M_t}, \frac{b_t}{M_t}, s_t), \tag{3.5}$$

$$\frac{b_{t+1}}{M_t} = B(x_t, X_t, \frac{m_t}{M_t}, \frac{b_t}{M_t}, s_t). \tag{3.6}$$

For a fixed C, G, L and B, expected utility is a well-defined quantity.

Define $v(x_0, X_0, \frac{m_0}{M_0}, \frac{b_0}{M_0}, s_0)$ as the maximized objective function. The value function v satisfies (using the shorter expressions C_0, C_1, etc., to denote values of functions evaluated at time 0 or time 1 variables respectively)

$$v(x_0, X_0, \frac{m_0}{M_0}, \frac{b_0}{M_0}, s_0) = \tag{3.7}$$

$$\max_{C,G,L,B} \{u(C_0) + \beta \sum_{s_1 \in S} v\left[G_0, X_1, \frac{L_0 M_0 + (h_1 - 1)M_0}{h_1 M_0}, \frac{B_0 M_0}{h_1 M_0}, s_1\right] \pi(s_1 | s_0)\}$$

where the maximization is subject to (3.1)-(3.2), using the functions in (3.3)-(3.6).

A *stationary* equilibrium for this economy is a v, C, G, L, B, g, p and q such that, for any $x \in K$ and $s \in S$,

v satisfies (3.7)

C, G, L and B maximize the right hand side of (3.7) subject to (3.1)-(3.2)

$$G(x,x,1,0,s) = g(x,s), \tag{3.8}$$

$$1 = L(x,x,1,0,s), \tag{3.9}$$

$$0 = B(x,x,1,0,s), \tag{3.10}$$

$$C(x,x,1,0,s) + G(x,x,1,0,s) = f(x,s), \tag{3.11}$$

$$p(x,s)C(x,x,1,0,s) \leq 1. \tag{3.12}$$

Define $c(x,s) \equiv C(x,x,1,0,s)$. Equation (3.8) equates the single representative consumer's capital with the economy's, and (3.9) equates money demand to money supply. Equation (3.10) requires the equilibrium number of bonds to equal 0.[13] Equation (3.11) requires the consumer to lie on his budget constraint and (3.12) requires the cash-in-advance constraint to hold.

Let $\bar{\lambda}(x_t,X_t,\frac{m_t}{M_t},\frac{b_t}{M_t},s_t)$ and $\bar{\varphi}(x_t,X_t,\frac{m_t}{M_t},\frac{b_t}{M_t},s_t)$ be the multipliers associated with (3.1) and (3.2), respectively, in the maximation problem in (3.7). The first order conditions for this maximization problem are

$$0 = u'(c_0) - (\bar{\lambda}_0 + \bar{\varphi}_0)p_0,$$

$$0 = \beta\Sigma v_1\left[G_0,X_1,\frac{L_0 + h_1 - 1}{h_1},\frac{B_0}{h_1},s_1\right]\pi(s_1|s_0) - \bar{\lambda}_0 p_0,$$

$$0 = \beta\Sigma v_3\left[G_0,X_1,\frac{L_0 + h_1 - 1}{h_1},\frac{B_0}{h_1},s_1\right]\frac{\pi(s_1|s_0)}{h_1} - \bar{\lambda}_0,$$

$$0 = \beta\Sigma v_4\left[G_0, X_1, \frac{L_0 + h_1 - 1}{h_1}, \frac{B_0}{h_1}, s_1\right]\frac{\pi(s_1|s_0)}{h_1} - \bar{\lambda}_0 q_0.$$

From (3.7) obtain, at the optimum,

$$v_1(x_0, X_0, \frac{m_0}{M_0}, \frac{b_0}{M_0}, s_0) = \bar{\lambda}_0 p_0 f'(x_0, s_0),$$

$$v_3(x_0, X_0, \frac{m_0}{M_0}, \frac{b_0}{M_0}, s_0) = \bar{\lambda}_0 + \bar{\varphi}_0,$$

$$v_4(x_0, X_0, \frac{m_0}{M_0}, \frac{b_0}{M_0}, s_0) = \bar{\lambda}_0.$$

Define

$$\lambda(x,s) \equiv \bar{\lambda}(x,x,1,0,s), \quad \varphi(x,s) \equiv \bar{\varphi}(x,x,1,0,s).$$

Impose the market equilibrium conditions and combine the above equations to arrive at six equations in the six unknown functions c, g, p, q, λ, and φ mapping $K\times S \rightarrow K$ (c and g) or $K\times S \rightarrow \mathbb{R}_+$ (p, q, λ, and φ)

$$c(x,s) + g(x,s) = f(x,s), \qquad (3.13)$$

$$p(x,s)c(x,s) \leq 1 \text{ with equality if } \varphi(x,s) > 0, \qquad (3.14)$$

$$u'[c(x,s)] = [\lambda(x,s) + \varphi(x,s)]p(x,s), \qquad (3.15)$$

$$\lambda(x,s)p(x,s) = \beta\Sigma\lambda[g(x,s),s']p[g(x,s),s']f'[g(x,s),s']\pi(s'|s), \quad (3.16)$$

$$\lambda(x,s) = \beta\Sigma\{\lambda[g(x,s),s'] + \varphi[g(x,s),s']\}\frac{\pi(s'|s)}{h(s')}, \quad (3.17)$$

$$\lambda(x,s)q(x,s) = \beta\Sigma\lambda[g(x,s),s']\frac{\pi(s'|s)}{h(s')}. \quad (3.18)$$

Existence and Uniqueness

I address existence and uniqueness questions only as they pertain to (3.13)-(3.18). These six equations embody two choices between today and tomorrow (equation (3.16) and (3.17)), so these six equations in six unknowns should collapse into two nontrivial fixed point equations in two unknowns. These unknowns will turn out to be the two functions λ and c.

To this end, begin with equations (3.14) and (3.15).

Proposition 3.1. Equations (3.14) and (3.15) can be written as

$$\varphi(x,s) = \max\{c(x,s)u'[c(x,s)] - \lambda(x,s),\ 0\}, \quad (3.19)$$

$$p(x,s) = \min\{\frac{1}{c(x,s)}, \frac{u'[c(x,s)]}{\lambda(x,s)}\}. \quad (3.20)$$

Proof. Obtain, from (3.19),

$$\lambda(x,s) + \varphi(x,s) = \max\{c(x,s)u'[c(x,s)],\ \lambda(x,s)\}.$$

To show that (3.15) holds, write

$$[\lambda(x,s) + \varphi(x,s)]p(x,s)$$
$$= \max\{c(x,s)u'[c(x,s)],\ \lambda(x,s)\}\min\{\frac{1}{c(x,s)},\ \frac{u'[c(x,s)]}{\lambda(x,s)}\}$$

$$= \begin{cases} u'[c(x,s)] & \text{if} \quad c(x,s)u'[c(x,s)] \geq \lambda(x,s) \\ u'[c(x,s)] & \text{if} \quad c(x,s)u'[c(x,s)] \leq \lambda(x,s). \end{cases}$$

To show that (3.14) holds, derive, from (3.20),

$$p(x,s)c(x,s) = \min\{1,\ \frac{c(x,s)u'[c(x,s)]}{\lambda(x,s)}\} \leq 1,$$

and derive, from (3.19),

$$\varphi(x,s) > 0 \quad \text{if} \quad \frac{c(x,s)u'[c(x,s)]}{\lambda(x,s)} > 1. \qquad \blacksquare$$

To formulate the equation in λ, use (3.19) in (3.17) to obtain

$$\lambda(x,s) = \beta\Sigma\max\{\lambda[f(x,s) - c(x,s),s'], \tag{3.21}$$
$$c[f(x,s) - c(x,s),s']u'[c(f(x,s) - c(x,s),s')]\}\frac{\pi(s'|s)}{h(s')}.$$

To formulate the equation in c, use (3.20) in (3.16) to obtain

$$\min\{\frac{\lambda(x,s)}{c(x,s)},\ u'[c(x,s)]\} \tag{3.22}$$

$$= \beta\Sigma\min\{\frac{\lambda[f(x,s) - c(x,s),t]}{c[f(x,s) - c(x,s),t]},\ u'[c(f(x,s) - c(x,s),s')]\}$$
$$f'[f(x,s) - c(x,s),s']\pi(s'|s).$$

Consider fixed points λ and c in sets $\Gamma_B(K\times S)$ and $\bar{C}_f(K\times S)$, respectively. Let $C(K\times S)$ be the space of continuous functions taking $K\times S \to \mathbb{R}$, equipped with the sup norm, and define $\Gamma_B(K\times S) \subset C(K\times S)$ as

$$\Gamma_B(K\times S) = \left\{ \lambda: \begin{array}{l} \lambda: K\times S \to \mathbb{R}_+, \quad \lambda(\cdot,s) \text{ is continuous,} \\ 0 \leq \lambda(x,s) \leq B. \end{array} \right.$$

Let $C_f(K\times S)$ be the obvious extension of $C_f(K)$ to stochastic production and define, also, the subset $\bar{C}_f(K\times S) \subset C_f(K\times S)$ as

$$\bar{C}_f(K\times S) = \left\{ c: \begin{array}{l} c:K\times S \to K, \quad c(\cdot,s) \text{ is continuous,} \\ 0 \leq c(x,s) \leq f(x,s), \\ 0 \leq c(y,s) - c(x,s) \leq f(y,s) - f(x,s) \quad \text{for} \quad y \geq x. \end{array} \right.$$

Clearly $\Gamma_B(K\times S)$ is a complete metric space and $\bar{C}_f(K\times S)$, as the similar set $\bar{C}_f(K)$ in section 2, is a compact subset of a normed space.

The Fixed Point Equation in λ

The result of this subsection is stated as

Theorem 3.2. For any $c \in \bar{C}_f(K\times S)$, there exists a unique $\lambda \in \Gamma_B(K\times S)$ which satisfies (3.21); this defines a continuous function $\lambda = \Psi(c)$.

Proof. Define, for a fixed $c \in \bar{C}_f(K\times S)$, the function $T:\Gamma_B(K\times S) \to T(\Gamma_B(K\times S))$ as the right hand side of (3.21). I first show that $T(\Gamma_B(K\times S)) \subset \Gamma_B(K\times S)$. Clearly $T(\lambda)$ is continuous for continuous λ. The upper bound B holds for $T(\lambda)$ since

$$(T\lambda)(x,s) \leq \beta\Sigma\max\{B,\ B\}\frac{\pi(s'|s)}{h(s')} \leq B.$$

T is a contraction. To prove this, choose a λ_1 and λ_2, both in $\Gamma_B(K\times S)$. Then (writing $g = f(x,s) - c(x,s)$ to shorten the equations),

$$\|T(\lambda_1) - T(\lambda_2)\|$$

$$= \max_{x,s} \Big|\beta\Sigma[\max\{\lambda_1(g,s'),\ c(g,s')u'(c(g,s'))\} - \max\{\lambda_2(g,s'),\ c(g,s')u'(c(g,s'))\}]\frac{\pi(s'|s)}{h(s')}\Big|$$

$$\leq \max_{s} [\beta\Sigma\frac{\pi(s'|s)}{h(s')}] \max_{x,s'} \big|\max\{\lambda_1(x,s'),\ c(x,s')u'[c(x,s')]\} - \max\{\lambda_2(x,s'),\ c(x,s')u'[c(x,s')]\}\big|.$$

The following four cases arise for this last inequality.

case i: $\lambda_1(x,s') \geq c(x,s')u'[c(x,s')]$, $\lambda_2(x,s') \geq c(x,s')u'[c(x,s')]$.

case ii: $\lambda_1(x,s') < c(x,s')u'[c(x,s')]$, $\lambda_2(x,s') < c(x,s')u'[c(x,s')]$.

case iii: $\lambda_1(x,s') \geq c(x,s')u'[c(x,s')]$, $\lambda_2(x,s') < c(x,s')u'[c(x,s')]$.

case iv: $\lambda_1(x,s') < c(x,s')u'[c(x,s')]$, $\lambda_2(x,s') \geq c(x,s')u'[c(x,s')]$.

For cases (i) and (ii), clearly

$$\big|\max\{\lambda_1(x,s'),\ c(x,s')u'[c(x,s')]\} - \max\{\lambda_2(x,s'),\ c(x,s')u'[c(x,s')]\}\big| \leq |\lambda_1(x,s') - \lambda_2(x,s')|.$$

For case (iii),

$$|\max\{\lambda_1(x,s'),\ c(x,s')u'[c(x,s')]\} - \max\{\lambda_2(x,s'),\ c(x,s')u'[c(x,s')]\}|$$
$$= |\lambda_1(x,s') - c(x,s')u'[c(x,s')]|$$
$$\leq |\lambda_1(x,s') - \lambda_2(x,s')|.$$

This last inequality follows since

$$0 \leq \lambda_2(x,s') \leq c(x,s')u'[c(x,s')] \leq \lambda_1(x,s')$$

is true by hypothesis. Case (iv) is similar to case (iii).

Hence,

$$\|T(\lambda_1) - T(\lambda_2)\| \leq \max_s \left[\beta\Sigma\frac{\pi(s'|s)}{h(s')}\right]\|\lambda_1 - \lambda_2\|.$$

Since $0 < \beta\Sigma\frac{\pi(t|s)}{h(t)} < 1$ for every s, T is a contraction. $\Gamma_B(K\times S)$ is a complete metric space, so by Banach's fixed point theorem there exists a unique $\lambda \in \Gamma_B(K\times S)$ which solves (3.21). Since T is a contraction which is continuous in c, the dependence $\lambda = \Psi(c)$ is continuous. ∎

The Fixed Point Equation in c

In general I would like to prove the existence of a c to (3.22) where $\Psi(c)$ replaces λ. This I am unable to do. Here I only consider the existence and uniqueness of a fixed point c for the special case of log utility. With $u(c) = \log(c)$, (3.22) simplifies to

$$\frac{\lambda(s)}{c(x,s)} = \beta\Sigma\frac{\lambda(t)}{c[f(x,s)-c(x,s),t]}f'[f(x,s)-c(x,s),s']\pi(s'|s), \quad (3.23)$$

where

$$\lambda(s) \;=\; \beta\Sigma\frac{\pi(s'|s)}{h(s')}.$$

Note that λ is independent of the consumption function and capital, and the notation on x is suppressed. Equation (3.23) is quite similar to the first order condition from the stochastic PG model. The existence proof for a c which solves (3.23) is roughly the same as section 2's PG existence proof, so I will not spell the proof out here. Also, if a positive solution $c(x,s) > 0$ for $x > 0$ exists, the proof of this solution's uniqueness in the set $\bar{C}_f(K \times S) - 0$ is similar to section 2's proof.

Constructing the Solution

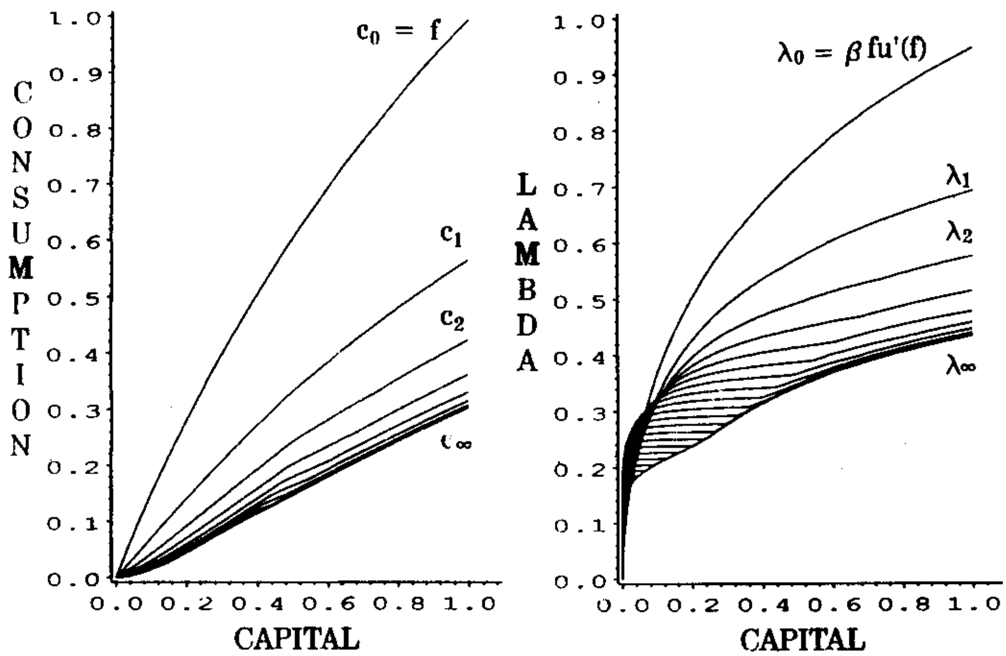
To explicitly construct the solution, I will rely more on a joint determination of the solution.[14] Beginning with some λ_0 and c_0, recursively update to a λ_1 and c_1 such that

$$\lambda_1(x,s) \;=\; \beta\Sigma\max\{\lambda_0[f(x,s) - c_1(x,s),s'],$$
$$c_0[f(x,s) - c_1(x,s),s']u'[c_0(f(x,s) - c_1(x,s),s')]\}\frac{\pi(s'|s)}{h(s')},$$

$$\min\{\frac{\lambda_1(x,s)}{c_1(x,s)},\; u'[c_1(x,s)]\}$$

$$= \;\beta\Sigma\min\{\frac{\lambda_0[f(x,s) - c_1(x,s),t]}{c_0[f(x,s) - c_1(x,s),t]},\; u'[c_0(f(x,s) - c_1(x,s),s')]\}$$
$$f'[f(x,s) - c_1(x,s),s']\pi(s'|s).$$

Figure 2 displays a particular sequence $\{\lambda_n, c_n\}$ computed according to the above algorithm. In general, this figure exhibits the same rapid and smooth convergence as did Figure 1. It is striking how similar the consumption sequence in Figure 2 is to that in Figure 1. Note also that the λ sequence is non-monotone towards the origin.



$\beta = .95$, $u'(c) = c^{-.5}$, $f(x) = \{[1 + 1.6x]^{.1} - 1\}/.1$

FIG. 2. MG CONVERGENCE

Welfare and Optimal Monetary Growth

For any solution to the MG economy, the discounted expected utility obtained, starting from (x,s), is $v(x,s)$, where v solves

$$v(x,s) = u[c(x,s)] + \beta\Sigma v[g(x,s),s']\pi(s'|s).$$

Since c and g depend on h, v also depends on h. This section considers an optimal h, one which maximizes v. It should come as no surprize that optimality is obtained when $h = \beta$.

Clearly v is at its maximum when c and g solve the underlying stochastic PG model. This solution is obtained when c and g statisfy

$$c(x,s) + g(x,s) = f(x,s),$$

$$u'[c(x,s)] = \beta\Sigma u'\{c[g(x,s),s']\}f'[g(x,s),s']\pi(s'|s).$$

But these two equations, along with the following ones, solve the MG model:

$$h = \beta,$$ [15]

$$\varphi(x,s) = 0, \quad \lambda(x,s) = \xi \quad \text{for any} \quad \xi > B,$$

$$p(x,s) = u'[c(x,s)]/\xi, \quad q(x,s) = 1.$$

Equation (3.13) is obviously satisfied. For (3.14),

$$p(x,s)c(x,s) = c(x,s)u'[c(x,s)]/\xi < 1,$$

which is consistent with $\varphi(x,s) = 0$. (3.15) holds since, per above, the right hand side is

$$[\lambda(x,s) + \varphi(x,s)]p(x,s)$$
$$= \xi u'[c(x,s)]/\xi = u'[c(x,s)].$$

(3.16) holds since $\lambda(x,s)p(x,s) = u'[c(x,s)]$ and (c,g) solve the underlying stochastic PG model. (3.17) holds since

$$\beta\Sigma\{\lambda[g(x,s),s'] + \varphi[g(x,s),s']\}\frac{\pi(s'|s)}{h(s')}$$
$$= \frac{\beta}{\beta}\Sigma\xi\pi(s'|s) = \xi.$$

Finally, $q(x,s) = 1$ (zero nominal interest rate) clearly solves (3.18).

The above analysis also makes clear that, for an arbitrary monetary policy, the MG economy may not be Pareto Optimal. Consumption and investment in the monetary economy may differ from its underlying real economy. Subject to the MG technology, a central planner can improve on welfare by letting c and g solve the PG model, and setting

$$p(x,s) = 1/c(x,s).$$

This divergence, of course, is the reason for this section.

4. MONETARY POLICY AND THE ECONOMY

Monetary policy's effect on consumption and capital can best be understood indirectly through its effect on nominal interest rates: if monetary policy does not affect relative nominal interest rates, then it should have no effect on real variables. Clearly a varying monetary growth rate can alter relative interest rates, thus leading to a non-neutrality. I will explicitly look at this effect later in this section. A somewhat more interesting question, developed a bit in the introduction, is whether or not a constant increase in the rate of monetary growth can alter relative rates. This effect, it turns out, hinges on the possibility of zero nominal interest rates (a slack cash-in-advance constraint).

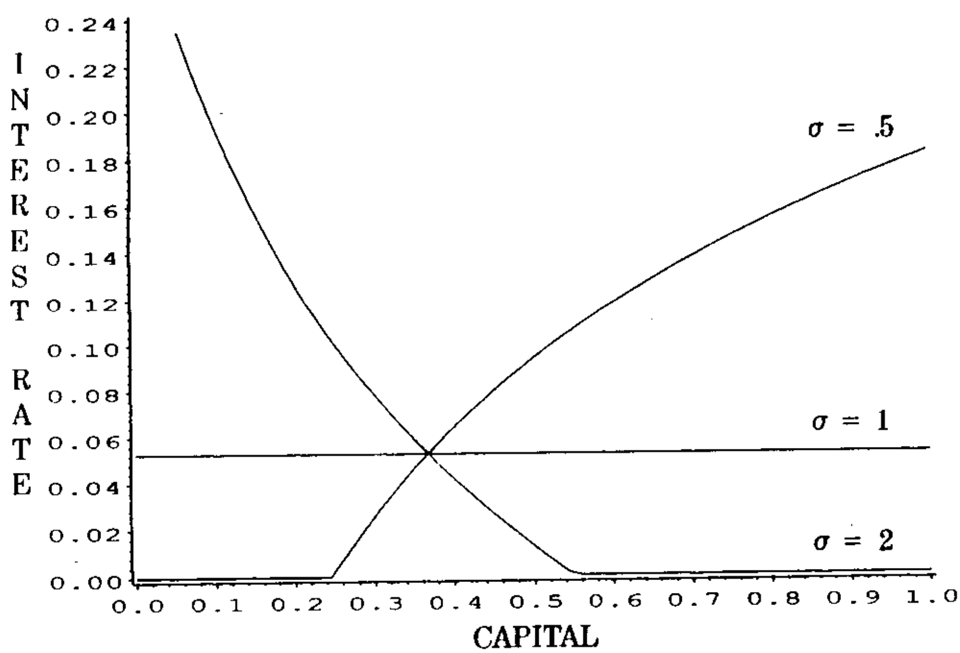
I expect zero nominal interest rates in the Monetary Growth economy whenever rates are negative in the corresponding economy where consumption is subject to an equality cash-in-advance constraint. Call this latter economy the Constrained Monetary Growth economy. In the deterministic constrained economy, nominal interest rates (r_t^c) are defined, in sequence notation, as

$$1 + r_t^c = \frac{c_t}{\beta c_{t+1}} f'(x_{t+1})$$

for some optimal consumption sequence. With log utility, the interaction between real rates[16] (marginal productivity) and inflation ($p_{t+1}/p_t = c_t/c_{t+1}$) is large enough to produce a constant positive ($h > \beta$) nominal rate

$$1 + r_t^c = h/\beta.$$

This rate is also the deterministic MG economy's rate at the stationary state, for any utility function.[17] Consider now a more concave utility function ($cu'(c)$ decreasing). Consumption should then grow slower close to the origin (where small changes in consumption lead to larger changes in marginal rates of substitution) and hence interest

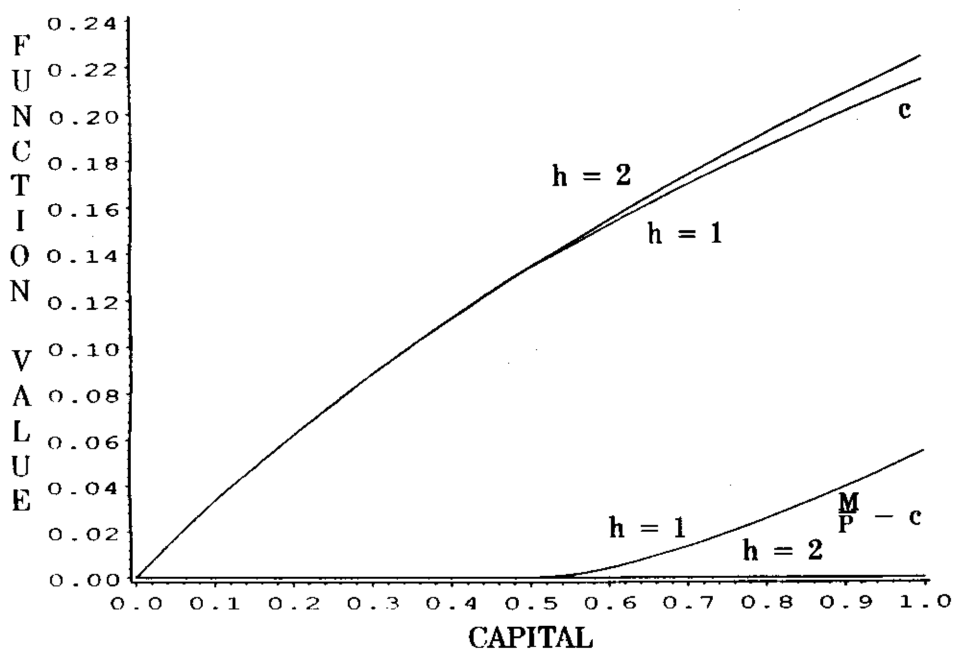


$\beta = .95$, $u'(c) = c^{-\sigma}$, $f(x) = \{[1 + 1.6x]^{.1} - 1\}/.1$, $h = 1$

FIG. 3. MG NOMINAL INTEREST RATES

rates should be relatively higher close to the origin. If interest rates approach negative values, then this will occur to the right of the stationary state for more concave than log utility functions and towards the origin for less concave ones. This pattern for the MG economy is exhibited in Figure 3. A surprising aspect of this figure is the large set of capital values for which interest rates equal zero.

Consider raising the monetary growth rate. As exhibited in Figure 4, higher rates of monetary growth predictably lead to a spending of excess cash. But also exhibited in Figure 4 is a substitution out of investment and into consumption, the *opposite* change of what I expected.



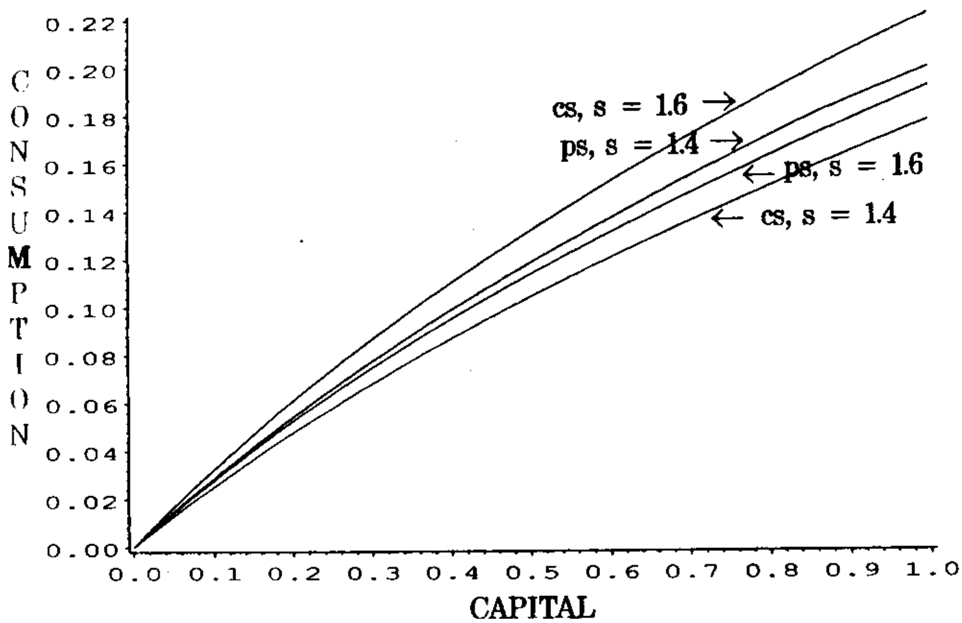
$\beta = .95,\ u'(c) = c^{-2},\ f(x) = \{[1 + 1.6x]^{.1} - 1\}/.1$

FIG. 4. MG CONSUMPTION, EXCESS CASH, AND MONETARY GROWTH

The reason for this substitution is that with no excess cash balances, money is spent at the rate at which output is consumed, and thus in the region where previously excess cash balances were held, where money was spent at a faster rate, relative inflation rates have dropped. This makes consumption relatively cheaper. In a nutshell, intertemporal substitution determines the location of the slack cash-in-advance region, inflation determines the size of this region, and relative inflation rates determine the substitution between consumption and capital.

A variable monetary policy has a much more straightforward (and probably more relevant) effect. First, if monetary policy is stochastic, but the expected monetary growth rate is a constant, then this uncertainty has no real effect. This result can be easily proven by manipulating the first order conditions using this type of monetary policy. This fact dispells the notion that monetary policy has an effect by making the cash-in-advance constraint ex post binding or not binding depending upon a negative or positive monetary shock. Second, if expected monetary growth varies, then so will nominal interest rates. This leads to a real inflation tax effect by lowering consumption in relatively high nominal interest rate states and conversely. Lower consumption means higher investment, lower real interest rates, and lower real balances, etc.

When production is also stochastic, various correlations between monetary shocks and these real shocks will result in quite different effects of monetary policy on the real economy. Consider, for example, a pro-shock monetary policy as one with a positive correlation. Consumption will be (suboptimally) smoothed since when output is relatively high, nominal interest rates are relatively high and hence consumption will be lower than with, say, a constant monetary policy. Conversely, consumption will become more variable with a counter-shock monetary policy. These effects are exhibited in Figure 5. Note that even the ordering of consumption is changed with the pro-shock monetary policy.



β = .95, $u'(c) = c^{-2}$, $f(x,s) = \{[1 + sx]^{1} - 1\}/1$,
$\pi(1.4|1.4) = \pi(1.6|1.6) = .7$, cs: h = (2, 1), ps: h = (1, 2)

FIG. 5. MG CONSUMPTION WITH CORRELATED MONEY AND PRODUCTION SHOCKS

5. CONCLUDING REMARKS

This paper has been an attempt to integrate a transactions theory of money into a general equilibrium theory of capital, with the result being a computable model economy capable of picturing a rich dependence of the real economy on monetary policy. This model economy was able to address a particular consumption-capital substitution question, and appears well-suited for handling other questions. This exercise has, I think, shed light on the workings of a theoretical cash-in-advance economy as well as on an effect of monetary policy in an actual economy.

Equally as significant was the derivation of an algorithm capable of constructing solutions to these types of models. We currently are somewhat short on algorithms which can provide solutions in terms of decision rules, and it's getting well beyond the stage where explicit solutions based on simplifying assumptions are not much more than a check on algebra.

APPENDIX

Proposition A.1. A nonzero fixed point of A exists.

Proof. Since A is monotone, a sufficient condition for the existence of a nonzero fixed point is the existence of a $\underline{c} \in \bar{C}_f(K)$, not identically zero, such that

$$u'[\underline{c}(x)] \geq \beta u'\{\underline{c}[f(x) - \underline{c}(x)]\}f'[f(x) - \underline{c}(x)] \quad \text{for} \quad x \in K. \qquad \text{(A.1)}$$

Define $\hat{x}$ such that $f'[f(\hat{x})] = 1$. Let $\alpha = f'(\bar{x})$. Define

$$\underline{c}(x) = \begin{cases} 0 & 0 \leq x \leq \hat{x}, \\ \alpha(x - \hat{x}) & \hat{x} \leq x \leq f(\hat{x}), \\ \alpha[f(\hat{x}) - \hat{x}] & f(\hat{x}) \leq x \leq \bar{x}. \end{cases}$$

Note that, for $x \geq \hat{x}$,

$$\beta f'[f(x) - \underline{c}(x)] \leq \beta f'[f(\hat{x})] = 1,$$

and

$$\underline{c}[f(x) - \underline{c}(x)] = \alpha[f(\hat{x}) - \hat{x}].$$

A sufficient condition for (A.1) is thus $\underline{c}(x) \leq \underline{c}[f(x) - \underline{c}(x)]$, which is clearly true. ∎

ENDNOTES

1. See Tobin [25].

2. Other studies include, for example, Tobin [25] and Fischer [9] who approach money from a portfolio perspective, and Sidrauski [22] who approaches money from a consumption good perspective. Stockman [24] develops a cash-in-advance model of money and capital, but focuses mainly on properties of stationary states.

3. For models without capital, Grandmount and Younes [11, 12], Lucas [17], and Lucas and Stokey [19] prove the existence of an equilibrium where money serves as a medium of exchange. Townsend [26] has a general proof of existence for cash-in-advance models with capital, but his proof is somewhat non-constructive.

4. A compact subset of a metric space is any set for which the Bolzano-Weierstrauss theorem holds: every bounded sequence contains a convergent subsequence. This statement is valid for any finite-dimensional normed spaced but is generally invalid for an infinite-dimensional metric space. See Heuser [13, Section 2.10].

5. See Rudin [20, Definition 7.22].

6. The Arzela-Ascoli theorem states that a subset of continuous

functions defined on a compact set is relatively compact if it is bounded and equicontinuous. See Rudin [20, Theorem 7.25].

7. See Rudin [20, Exercise 7.16].

8. See Heuser [13, Theorem 106.3].

9. Beals and Koopmans [1], via a central planner, established uniqueness by relying on the strict quasi-concavity of the maximand.

10. Convergence can also be proven by exploiting A's concavity. This result is basically spelled out in Krasnosel'skiĭ and Zabreĭko [15], but you need to employ the same type of extension used in Theorem 2.7. This extension is rather lengthy. My use of monotonicity in Theorem 2.8 is taken from Lucas and Stokey's [19] Theorem 3.

11. Since $c_0 = f$, this is the optimal consumption function sequence for the finite time horizon Planned Growth problem where the time horizon goes to infinity (for zero investment in the final state).

12. To motivate the restriction embedded in H, I'll have to get a bit ahead of the story. In the deterministic MG economy, the stationary state x^* is determined by $1 = \beta f'(x^*)$, and inflation, since consumption is constant, is $h - 1$. The nominal interest rate is thus $h/\beta - 1$, which, if money is not to strictly dominate capital, must not become negative. Essentially, then, the restriction in H ensures that a stationary state exists.

13. Other solutions exist. For example, a consumer could choose to continually roll over debt and thereby obtain an arbitrarily large expected utility. I could have explicity ruled this out by a variety of methods, one of which is bounding the amount of debt.

14. This is likely how a fixed point theorem will be proven. The problem I had is retaining the property that λ_1/c_1 be a decreasing function (in x).

15. Actually, any monetary policy with this as the constant conditional expectation (precisely stated, when $E(1/h) = 1/\beta$) will do. See the discussion in section 4.

16. Define real interest rates as the return on a consumption bond. This rate is then equal to

$$\frac{\lambda(x,s)p(x,s)}{\beta\Sigma\lambda[g(x,s),s']p[g(x,s),s']\pi(s'|s)} - 1,$$

and expected inflation is

$$\frac{\Sigma h(s')p[g(x,s),s']\pi(s'|s)}{p(x,s)} - 1.$$

Note that the nominal interest rate differs from the real rate by an expected money growth term and an expected inflation term relative to monetary growth.

17. In fact, the deterministic stationary state is independent of money's growth rate. This is not true if money and investment are both subject to a cash-in-advance constraint, as in Stockman [25], for which the stationary state is determined by $h/\beta = \beta f'(x^*)$.

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