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"Tobin's q" and the Rate of Investment in General Equilibrium

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This paper is an exercise that was undertaken to practice some of the methods that were taught by R. E. Lucas, Jr., in his Economics 337 class at the University of Chicago in the spring of 1977. The version of the stochastic growth model used here was described by Lucas in his lectures. Propositions 1, 2, and 3 of this paper were stated by Lucas in class and have proofs parallelling those given in Lucas [11]. Propositions 4, 5, and 6, as well as the propositions in Appendixes A and C, are my own responsibility. Robert Litterman performed the calculations. Helpful comments on earlier drafts were made by Lars Hansen, Robert E. Lucas, Jr., Jose Scheinkman, and Robert Townsend. Thanks to the suggestions of Lars Hansen, the proof of proposition 4 is simplified from earlier drafts.

1. Introduction

James Tobin's "q theory" is one of the most prominent current macroeconomic theories about firms' demand schedule for a flow of investment. According to that theory, there is at most times a discrepancy between the price of existing capital goods, say, as reflected in markets for used capital goods, and the price of newly produced capital goods. Tobin calls the ratio of these two prices q. Tobin posits that q is an important argument of firms' demand schedule for investment. "The rate of investment--the speed at which investors wish to increase the capital stock--should be related, if to anything, to q, the value of capital relative to its replacement cost," [22, p. 21]. Such a theory must necessarily stem from a model in which "frictions" are present that prevent the price of existing capital from being driven equal at all times to the price of newly produced capital. For example, in "putty-putty" versions of one-sector growth models, q is always unity. Furthermore, in such models firms have no investment demand schedule, a point emphasized by Tobin [20, 21].

A simple model possessing the friction necessary to permit q to diverge from unity is the one sector growth model with irreversible aggregate investment. In this model, newly produced goods can either be consumed or used to augment the capital stock. But once they are designated as capital, capital goods cannot physically be converted into consumption goods. At the same time, there is a competitive market in existing physical capital. By transacting in this market, individual agents can reverse their past investment decisions, despite the irreversible nature of investment in the aggregate. In this model, there is a relative price which corresponds to q, the value of capital relative to its replacement cost. The irreversibility of investment in the aggregate is the friction that permits q to diverge from unity and which makes it possible for aggregate investment to be positively correlated with q. However, the population regression of aggregate investment on q is in no sense an "investment demand schedule," instead being a mongrel relation that reflects all of the parameters of the model. An econometrician studying such an economy would have no cause to fit such a regression if it is the economy's structure that he is after. Among other things, there is a massive "simultaneity problem." Not only does q, taken as a random process, influence investment decisions, but investment decisions influence q as a random process. But it is not merely a purely econometric simultaneity problem. It is only in a special and qualified sense that even agents who can legitimately view q as exogenous exhibit investment behavior that can be described as a function mainly or solely of q; and this sense does not seem to correspond to the one macroeconomists have had in mind. Indeed, the model in this paper exhibits a feature that probably characterizes virtually any model that possesses the friction necessary to make q diverge from unity: the very same source of friction that makes q diverge from unity also converts agents' decision problem into a nontrivial dynamic one, the solution of which will in general not assume a "myopic" form such as a simple contemporaneous demand schedule relating current investment to current q. Instead, investment decisions will necessarily be functions of agents' views about the future, the current state of which cannot in general be summarized by a single variable such as q.

This paper uses an irreversible investment version of the stochastic one-sector growth model as a vehicle for making some observations about the q theory of investment. We are attracted to the stochastic one-sector growth model because it is perhaps the simplest coherent general equilibrium model available in which one can discuss the mutual determination of investment and q. The one-sector stochastic growth model has been extensively studied (see e.g., Mirman [15], Mirman [16], Brock and Mirman [6], and Mirman and Zilcha [17]), so there is little that is analytically original here. However, because we are discussing a version of the model with irreversible investment, rather than the reversible investment (existing capital can be consumed) version that is extensively discussed in the literature, we have to spend some time discussing the nature of corner solutions in which the constraint that existing capital can't be consumed is binding. The presence of the corner is what permits q to diverge from unity in some states of the world. From a technical point of view, the presence of the corner requires modifications of the proof that the optimum value function associated with the planning problem is differentiable in capital, and of the proof that the stochastic growth model possesses a "stable" configuration of fixed points. It is important to verify the existence of the derivative with respect to capital of the "planners'" optimum value function, since it turns out to be the price of used capital in the competitive market model. It is useful to verify that the stochastic growth model has a "stable" configuration of fixed points because it implies that the endogenous random variables in the model converge in distribution, and that the sample first and second moments converge to population values.

The paper is organized as follows. Section 2 describes an interpretation of the model as a collection of households and firms that interact competitively in markets for output and inputs of capital and labor. Section 3 studies the equilibrium of the competitive model by studying the solution of the planning model that has an identical structure with the competitive model. Section 4 contains some numerical examples of economies, designed to illustrate how the various population moments and regression coefficients depend on the free parameters of the model. Our conclusions are in Section 5. Three appendixes contain various propositions and lemmas needed in the text.

We conclude this section with a heuristic description of the workings of the model. The model implies that capital evolves according to the stochastic difference equation

$$K_{t+1} = b(K_t, \theta_t, \varepsilon_t)$$

where K_t is the economy-wide capital labor ratio, θ_t is a shock to technology and ε_t is a shock to preferences. The function b turns out to be continuous, increasing in K and θ, and decreasing in ε. The function $b(K,\theta,\varepsilon)$ by construction satisfies the restriction that investment is irreversible,

$$K_{t+1} = b(K_t,\theta_t,\varepsilon_t) \geq (1-\delta)K_t$$

where $0 < \delta < 1$ is the depreciation rate of capital. There is a family of $b(K,\theta,\varepsilon)$ curves, as depicted in Figure 1, one curve for each (θ,ε) realization. We have drawn $b(K,\theta_a,\varepsilon_a)$ as the lowest possible $b(K,\theta,\varepsilon)$ curve, where θ_a is the lowest possible realization of θ and ε_a the highest possible realization of ε. We have drawn $b(K,\theta_b,\varepsilon_b)$ as the highest possible $b(K,\theta,\varepsilon)$ curve, where θ_b is the highest value of θ possible and ε_b is the lowest value of ε possible. In between $b(K,\theta_a,\varepsilon_a)$ and $b(K,\theta_b,\varepsilon_b)$ there is a continuum of $b(K,\theta,\varepsilon)$ curves corresponding to different possible realizations of (θ,ε).

The capital-labor ratio K of such a system eventually evolves to within the interval $[K_a,K_b]$, and then stays there forever.[1/] The system evolves stochastically as the capital-labor ratio wanders between K_a and K_b, depending on the sequence of drawings of (θ,ε). For the system depicted in Figure 1, for any $K_t > \tilde{K}$, there is a positive probability that (θ_t,ε_t) will be such that $K_{t+1} = b(K_t,\theta_t,\varepsilon_t) = (1-\delta)K_t$, i.e., the constraint that investment is irreversible becomes binding.[2/] This is true because of the specification that there is a continuum of curves filling the space between $b(K_t,\theta_a,\varepsilon_a)$ and $b(K_t,\theta_b,\varepsilon_b)$. Now it turns out that in such states (θ_t,ε_t) in which the system is on the corner, the price p_{Kt} of used capital relative to the price of new capital (i.e., newly produced output) drops below unity. Roughly speaking, this relative price drops farther below unity, the more binding is the constraint that investment be irreversible. On the other hand, when the irreversibility constraint is not

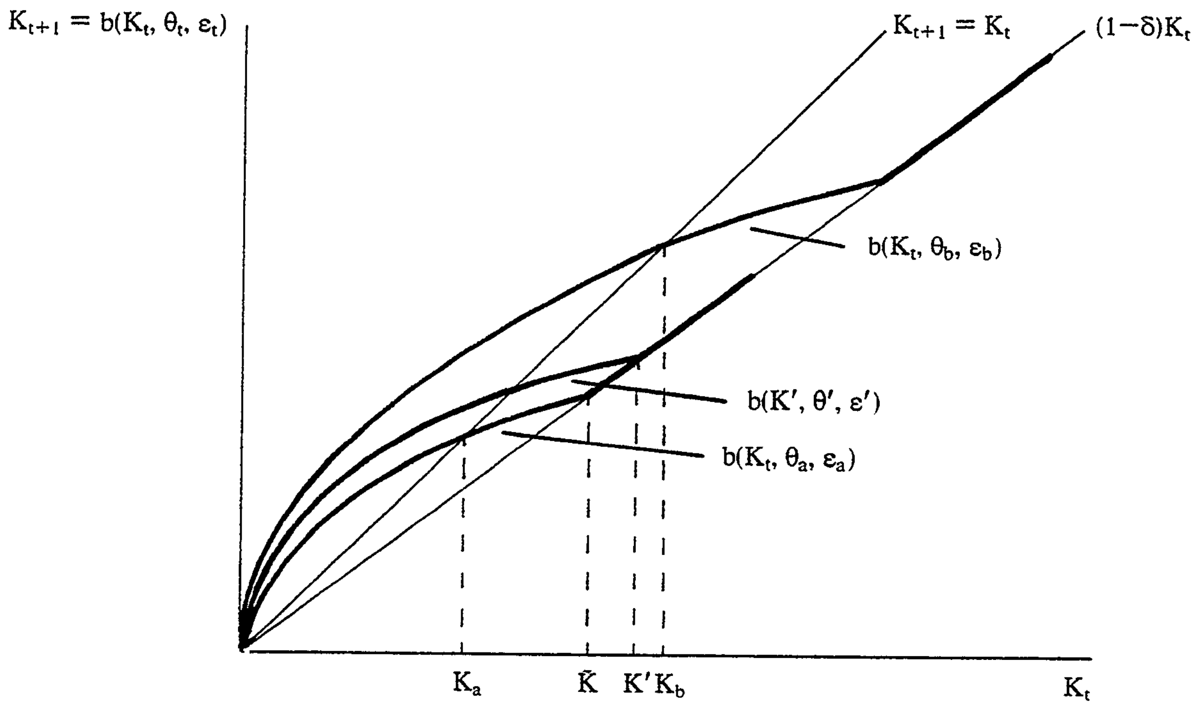


Figure 1

The Random Motion of K_t Over Time

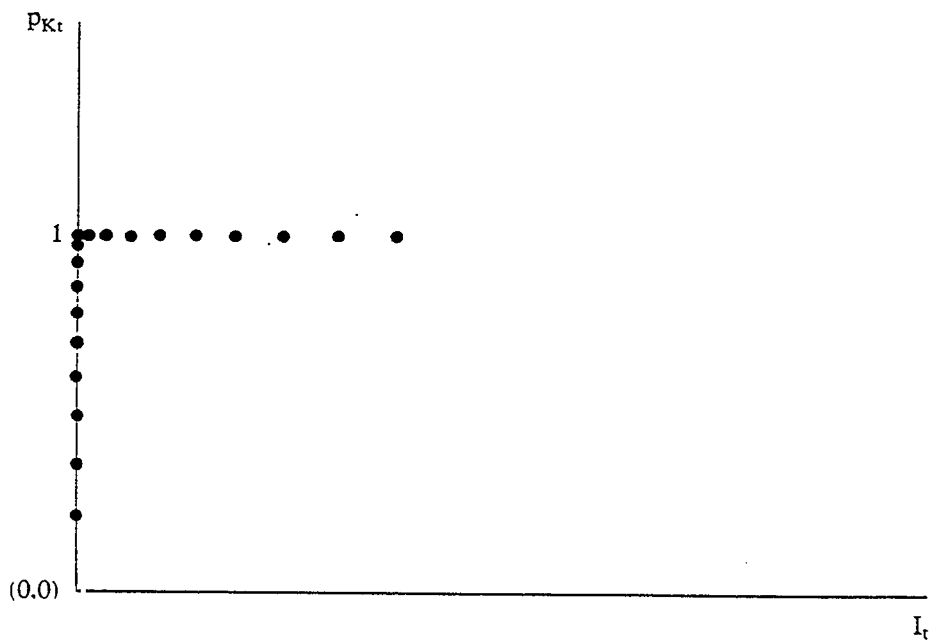


Figure 2

The Population Probability Distribution of (p_{Kt}, I_t)

binding, the relative price of old and new capital is unity. Consequently, the model implies (stationary) distributions of the relative price p_{Kt} and gross investment per man I_t of the kind indicated by the scatter of points in Figure 2. The bivariate distribution of p_{Kt} and I_t is a function of the probability distribution of $(\theta_t, \varepsilon_t)$, and of the parameters of preferences and technology. It is in the sense of Figure 2 that the model of this paper delivers a relationship between p_{Kt} and I_t. A main purpose of the paper is to study the sense of this relationship.

2. A Market Interpretation of the Model

Production is governed by

$$y_t = f(k_t)\theta_t$$

where y is output per man, and k_t is capital per man at t; θ_t is a positive random variable distributed independently and identically at all dates t. We assume that f( ) is twice continuously differentiable and satisfies

$$f'(k) > 0, \ f''(k) < 0$$

$$f'(0) = \infty, \ f'(\infty) = 0.$$

The economy is inhabited by a large number of competitive, price-taking, infinite-lived consumers. The number of consumers in the economy is assumed constant over time. Each consumer inelastically supplies one unit of labor each period. All consumers are alike and have bounded one-period utility function $u(c_t, \varepsilon_t)$, which we assume is twice continuously differentiable. Here c_t is consumption per man and ε_t is a random shock to preferences. We assume that ε_t is independently and identically distributed across time. We assume

$u(c,\varepsilon) < M$ for all c, ε for some $M > 0$

$u_c(c,\varepsilon) > 0$, $u_{cc}(c,\varepsilon) < 0$, $u_\varepsilon(c,\varepsilon) > 0$

$u_{c\varepsilon}(c,\varepsilon) > 0$

$u_c(0,\varepsilon) = \infty$, $u_c(\infty,\varepsilon) = 0$.

We assume that the random processes θ_t and ε_s are distributed independently of each other at all dates. We assume further that the cumulative distribution function

$$F(\theta,\varepsilon) = \text{Prob}\{\theta_t \leq \theta,\ \varepsilon_t \leq \varepsilon\} \quad \forall t$$

assigns probability one to the rectangle $\{\theta_a \leq \theta \leq \theta_b,\ \varepsilon_b \leq \varepsilon \leq \varepsilon_a\}$ where $0 < \theta_a$ is the lowest possible value of θ and $\theta_b > \theta_a$ is the highest value of θ. Here ε_b is the _lowest_ possible value of ε while ε_a is the _highest_. (The reason for our asymmetric notation will become clear later.) We assume that $F(\theta,\varepsilon)$ is absolutely continuous with respect to (θ,ε) at all points _except_ at the point (θ_a,ε_a). We assume that the point $\{\theta_a,\varepsilon_a\}$ has positive probability:

$$\text{Prob}\{\theta_a,\varepsilon_a\} > 0.$$

This is a condition borrowed from Mirman [16] and Mirman and Zilcha [18] that is sufficient to bound the support of the stationary distribution of capital per man strictly away from zero. Except at the point (θ_a,ε_a), $F(\theta,\varepsilon)$ is assumed to possess a strictly positive and continuous probability density function.

Finally, we impose the following condition on $u(\cdot,\cdot)$ and $f(\cdot)$: for any function of K, $\xi(K)$ with range $[0,1]$,

$$(0) \qquad \lim_{K \to 0} \frac{\delta K u_{cc}[f(K)\theta_a - \delta\xi(K)\cdot K, \varepsilon_a]}{u_c[f(K)\varepsilon_a]} = 0$$

where $1 > \delta > 0$ is the fixed rate of physical depreciation. The reader can verify that this condition is satisfied for, e.g., $f(k) = k^{\alpha}$ with $\alpha\varepsilon(0,1)$ and $u(c,\varepsilon) = \varepsilon \ln c$.

We assume that in a given period, all agents draw the same $(\theta_t, \varepsilon_t)$. Since all agents are assumed alike in the sense that they have the same utility functions and have access to the same technology and market opportunities, we shall assume that there is a single representative consumer who supplies one unit of labor each period. The consumer views himself as a perfect competitor and views economy-wide outcomes as independent of his own actions. This means that we must distinguish between the economy-wide state, which the consumer takes as given, and the consumer's own state variables, the evolution of some of which are a matter of choice to the consumer. In equilibrium, the economy-wide state variables equal the representative consumer's state variables, but the consumer is assumed to ignore this.[3/]

The state of the economy at time t can be characterized by the values of $(K_t, \theta_t, \varepsilon_t)$ where K_t is the economy-wide capital-labor ratio at the beginning of period t, θ_t is the random shock to productivity realized in period t, and ε_t is the random shock to preferences realized in period t. The state of the individual consumer at time t is characterized by his stock of capital at the beginning of t, k_t, and also the same shocks ε_t and θ_t that affect all agents' preferences and opportunities. The consumer's supply of labor is identically one, so that k_t also equals his capital-labor ratio. At the beginning of time t, the consumer rents his capital k_t to firms and receives during the period a competitively determined rental r_t, measured in output per unit of capital per unit time. Capital depreciates at the fixed rate δ, so that at the end of period t, the firm returns only $(1-\delta)k_t$ units of capital to the consumer. During period t the consumer can buy or sell claims to existing capital to be carried into period (t+1) at a competitively determined relative price p_{Kt} measured in units of new output per unit of capital. According to one possible interpretation, the relative price p_{Kt} is precisely Tobin's q. During period t households also buy newly produced output, consuming an amount c_t and carrying an amount i_t into next period as capital. The relative price of newly produced capital goods in terms of consumption goods is unity. Finally, the consumer inelastically supplies one unit of labor and is paid a competitively determined real wage w_t measured in output per unit labor.

The consumer's problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, \varepsilon_t), \quad 0 < \beta < 1 \tag{1}$$

where E_0 is the mathematical expectation operator conditional on information available at time 0, subject to the sequence of budget constraints for t=0, 1, 2, . . .

$$c_t + p_{Kt}k_t^d + i_t \leq w_t + r_t k_t + (1-\delta)p_{Kt}k_t$$

$$k_{t+1} = k_t^d + i_t, \; i_t \geq 0, \; c_t \geq 0, \; k_t^d \geq 0$$

where

δ = rate of depreciation of capital, $0 < \delta < 1$.

c_t = consumption per unit labor.

k_t^d = amount of old capital held at end of period t.

i_t = amount of newly produced goods to be used as capital.

k_t = amount of capital per unit of labor at beginning of period t.

The consumer seeks to maximize (1) with respect to the choice of stochastic processes for c_t, i_t, and k_t^d given the information he has at each period and given the constraints that he faces. To make the consumer's problem well posed, we suppose that the equilibrium relative prices in the system can be expressed as continuous functions of the economy-wide state variables, so that

$$
\begin{aligned}
r_t &= r(K_t,\theta_t,\varepsilon_t) \\
(2)\qquad p_{Kt} &= p_K(K_t,\theta_t,\varepsilon_t) \\
w_t &= w(K_t,\theta_t,\varepsilon_t).
\end{aligned}
$$

We suppose that the functions are such that they yield positive values of w_t, r_t, and p_{Kt} for all values of (K,θ,ε). We assume that the representative agent in the economy knows the three functions listed in (2) and that at time t he knows the values of θ_t, ε_t, and the economy-wide capital stock K_t. We also suppose that K_t follows the law of motion

$$(3)\qquad K_{t+1} = h(K_t,\theta_t,\varepsilon_t)$$

where h is a continuous function. We assume that this aggregate law of motion is known to the representative agent and is perceived by the agent to be independent of his own decisions. Let us denote the four functions in (2) and (3) as ϕ.

For a given selection of the four functions in (2) and (3), the household's problem is equivalent with finding an optimal value function $J(k,\theta,\varepsilon;K,\phi)$ which solves the functional equation

$$(4)\qquad J(k,\theta,\varepsilon;K,\phi) = \max_{i\geq 0,\ k^d\geq 0} \{u(w(\cdot)+r(\cdot)k+(1-\delta)p_K(\cdot)k-p_K(\cdot)k^d-i,\varepsilon)$$

$$+ \beta\int J(k^d+i,\theta',\varepsilon';h(K,\theta,\varepsilon),\phi)dF(\theta',\varepsilon')\}.$$

Here the functions $w(\cdot)$, $r(\cdot)$, and $p_K(\cdot)$ have as arguments (K,θ,ε). For a given selection of the functions in ϕ, it is possible to prove that the functional equation has a unique, continuous bounded solution $J(k,\theta,\varepsilon;K,\phi)$.[4/] The right side of (4) can be shown to be uniquely attained by continuous functions[5/]

$$c = c(k,\theta,\varepsilon;K,\phi)$$

$$i + k^d = i(k,\theta,\varepsilon;K,\phi) + k^d(k,\theta,\varepsilon;K,\phi).$$

It can also be proved that $J(\cdot)$ is strictly concave in k for fixed (θ,ε), and that J has a continuous and bounded partial derivative with respect to k.[6/]

The first-order necessary conditions for the maximization problem on the right side of (4) are[7/]

$$(5)\qquad k^d:\ -u_c(c,\varepsilon)p_K(K,\theta,\varepsilon) + \beta\int J_k(k^d+i,\theta',\varepsilon';h(K,\theta,\varepsilon),\phi)dF(\theta',\varepsilon') \le 0,$$

$$= 0 \text{ if } k^d > 0$$

$$(6)\qquad i:\ -u_c(c,\varepsilon) + \beta\int J_k(k^d+i,\theta',\varepsilon';h(K,\theta,\varepsilon),\phi)dF(\theta',\varepsilon') \le 0,$$

$$= 0 \text{ if } i > 0.$$

The partial derivative of $J(\cdot)$ with respect to k can be calculated from (4) to be[8/]

$$(7)\qquad J_k(k,\theta,\varepsilon;K,\phi) = u_c(c(k,\theta,\varepsilon;K,\phi),\varepsilon)[r(K,\theta,\varepsilon)+(1-\delta)p_K(K,\theta,\varepsilon)].$$

Conditions (5) and (6) tell something about the sense in which there is a "q theory" of investment in the present model. Use (7) to write (5) and (6) as

$$(8)\qquad -u_c(c_t,\varepsilon_t)p_K(K_t,\theta_t,\varepsilon_t) + \beta\int u_c(c_{t+1},\varepsilon_{t+1})$$

$$[r(K_{t+1},\theta_{t+1},\varepsilon_{t+1})+(1-\delta)p_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})]\,dF(\theta_{t+1},\varepsilon_{t+1}) \le 0,$$

$$= 0 \text{ if } k_t^d > 0$$

(9) $$-u_c(c_t,\varepsilon_t) + \beta\int u_c(c_{t+1},\varepsilon_{t+1})$$

$$[r(K_{t+1},\theta_{t+1},\varepsilon_{t+1})+(1-\delta)p_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})]\,dF(\theta_{t+1},\varepsilon_{t+1}) \leq 0,$$

$$= 0 \text{ if } i_t > 0.$$

Now in equilibrium, k_t^d must exceed zero, so that (8) will be satisfied with equality. It then follows that $p_K(K,\theta,\varepsilon) \leq 1$. The marginal condition (9) shows that i_t will be >0 only if $p_{Kt} = p_K(K_t,\theta_t,\varepsilon_t) = 1$. However, notice that the marginal conditions (8) and (9) necessarily involve the agent's perceptions of the distribution of one-period-ahead values of the rental $r_{t+1} = r(K_{t+1},\theta_{t+1},\varepsilon_{t+1})$ and the relative price $p_{Kt+1} = p_K(K_{t+1},\theta_{t+1},\varepsilon_{t+1})$. In general, the agent's choice of i_t depends on all of the current state variables that help determine the conditional distribution of future values of p_K and r. In a limited sense, the first-order conditions (8) and (9) do provide some foundation for the "q theory" of investment demand. But it is really the function $p_K(K,\theta,\varepsilon)$, or put differently, p_K as a stochastic process, and not only the currently realized value of p_{Kt} that influences investment at time t.

The marginal conditions (8) and (9) make it clear that _some_ carefully spelled out view about the stochastic processes (laws of motion) of K_{t+1}, r_{t+1}, and p_{Kt+1} must be attributed to agents in order for the decision problem to be well specified. The restriction that we have imposed, that agents' perceptions of those laws of motion are accurate, is the hypothesis of _rational expectations_.

We think of production as being determined by competitive firms which rent capital and hire labor to maximize profits

$$\pi = n^d f(k)\theta - w(\cdot)n^d - r(\cdot)k\cdot n^d$$

where $f(k)$ is output per man, k is the capital-labor ratio of the representative firm, and n^d is the employment level of the representative firm. The first-order necessary conditions for a maximum of profits are

$$f'(k)\theta = r(K,\theta,\varepsilon)$$

$$f(k)\theta - kf'(k)\theta = w(K,\theta,\varepsilon).$$

We can now give a definition of equilibrium.

<u>Definition</u>: An equilibrium is a five-tuple of functions $r(K,\theta,\varepsilon)$, $p_K(K,\theta,\varepsilon)$, $w(K,\theta,\varepsilon)$, $h(K,\theta,\varepsilon)$, and $J(k,\theta,\varepsilon;K,\phi)$ such that--

i. The functional equation (4) is satisfied with the right-hand side being attained by the continuous function $i(k,\theta,\varepsilon;K,\phi)$ + $k^d(k,\theta,\varepsilon;K,\phi)$.

ii. $i(K,\theta,\varepsilon;K,\phi) + k^d(K,\theta,\varepsilon;K,\phi) \equiv h(K,\theta,\varepsilon)$.

iii. The marginal conditions for firms are satisfied with

$$f'(K)\theta = r(K,\theta,\varepsilon)$$

$$f(K)\theta - Kf'(K)\theta = w(K,\theta,\varepsilon).$$

Condition (i) says that consumers are maximizing expected utility, given the random processes they are facing, which includes the Markov process (law of motion) for the economy-wide capital-labor ratio K. Condition (ii) says that the consumer's perceptions of the law of motion for the aggregate K turn out to be correct; that is, those perceptions are implied by the representative agent's solution of the maximum problem on the right side of (4). Condition (iii) states that firms are on their demand schedules for factors and that the factor markets always clear.

We shall follow Lucas and Prescott [14] by studying the equilibrium of the model only indirectly by studying the planning problem that reproduces the competitive equilibrium. In the next section we study the version of the Cass-Koopmans planning model that is isomorphic with the market model of this section and which generates as a shadow price for capital the correct function $p_K(K,\theta,\varepsilon)$.

3. The Planning Model

The planning problem is to choose a contingency plan for I_t which maximizes

(10) $$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t,\varepsilon_t)$$

subject to

$$C_t + I_t \leq f(K_t)\theta_t$$

$$C_t \geq 0, \; I_t \geq 0$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

where

C_t = consumption per man.

I_t = gross investment per man.

K_t = capital per man.

Solving the planning problem is equivalent with solving the following functional equation in the optimum value function $v(K,\theta,\varepsilon)$

(11) $$v(K,\theta,\varepsilon) = \max_{I\geq 0}\{u(f(K)\theta-I,\varepsilon)+\beta\int v((1-\delta)K+I,\theta',\varepsilon')dF(\theta',\varepsilon')\}.$$

The solution $v(K,\theta,\varepsilon)$ gives the maximum value of (10) starting from state (K,θ,ε) at time 0. Associated with the functional equation (10) is the operator T defined by

$$(12)\qquad Ta(K,\theta,\varepsilon) = \max_{I\geq 0}\{u(f(K)\theta-I,\varepsilon)+\beta\int a((1-\delta)K+I,\theta',\varepsilon')dF(\theta',\varepsilon')\}.$$

Let L^{3+} be the space of bounded continuous functions mapping R^{3+} into the real line. Then it is readily verified that T maps bounded functions into bounded functions. Application of the "maximum theorem" of Berge [3, p. 215, 216] shows that T maps continuous functions a into continuous functions Ta. Therefore, T is an operator on the space of bounded continuous functions L^{3+}, mapping bounded continuous functions into bounded continuous functions.

As a norm on L^{3+}, take

$$||a_1-a_2|| = \sup_{K,\theta,\varepsilon}|a_1(K,\theta,\varepsilon)-a_2(K,\theta,\varepsilon)|$$

where $a_1\varepsilon L^{3+}$, $a_2\varepsilon L^{3+}$. With this norm, the space $(L^{3+},||\cdot||)$ is complete, so that the contraction mapping theorem is potentially applicable.9/

It can be verified that the operator T satisfies Blackwell's [5] pair of sufficient conditions for T to be a contraction operator:

i. T is monotone, i.e., if $a_1(K,\theta,\varepsilon) \geq a_2(K,\theta,\varepsilon)$ for all $(K,\theta,\varepsilon)\varepsilon R^{3+}$, then $Ta_1(K,\theta,\varepsilon) \geq Ta_2(K,\theta,\varepsilon)$ for all $(K,\theta,\varepsilon)\varepsilon R^{3+}$.

ii. For all constants γ and all $a\varepsilon L^{3+}$, $T(a+\gamma) = Ta + \beta\gamma$.

By virtue of Blackwell's [5] theorem 5, satisfaction of (i) and (ii) implies that T is a contraction mapping. Therefore, application of the contraction mapping theorem proves:10/

Proposition 1: The functional equation $v(K,\theta,\varepsilon) = Tv(K,\theta,\varepsilon)$ has a unique continuous bounded solution $v(K,\theta,\varepsilon)$. Furthermore, given any $v_0\varepsilon L^{3+}$, $\lim_{n\to\infty} T^n v_0 = v$ where the convergence is in the sup norm. This implies that the convergence is uniform.

It is also possible to prove:

Proposition 2: The value function $v(K,\theta,\varepsilon)$ is strictly concave in K for each fixed pair (θ,ε).

This follows because T maps concave functions into strictly concave functions.

We also have:

Proposition 3: The value function $v(K,\theta,\varepsilon)$ is uniquely attained by the single-valued policy function $I = I(K,\theta,\varepsilon)$. The function $I(K,\theta,\varepsilon)$ is continuous.

Uniqueness of the maximizing value of I is implied by the strict concavity of $u(\cdot)$ in C and of $v(\cdot)$ in K. Continuity of the policy function $I(\cdot)$ is implied by the "maximum theorem" of Berge [5, p. 215-216].

Now choose $v^0(K,\theta,\varepsilon)$ to be nondecreasing in K, strictly concave in K, and continuously differentiable in K. Define $v^{j+1}(K,\theta,\varepsilon) = Tv^j(K,\theta,\varepsilon)$. We shall show that $v^{j+1}(K,\theta,\varepsilon)$ is continuously differentiable in K for each fixed (θ,ε), provided that $v^j(K,\theta,\varepsilon)$ is continuously differentiable in K for each fixed (θ,ε). Consider

$$(13) \qquad v^{j+1}(K,\theta,\varepsilon) = \max_{I^j \geq 0} \{u(f(K)\theta - I^j,\varepsilon) + \beta\int v^j((1-\delta)K+I^j,\theta',\varepsilon')dF(\theta',\varepsilon')\}$$

and assume that $v^j(K,\theta,\varepsilon)$ is nondecreasing in K, concave and continuously differentiable in K for each fixed (θ,ε). The first-order necessary condition for the maximum problem on the right-hand side is[11/]

$$(14) \qquad -u_c(f(K)\theta - I^j,\varepsilon) + \beta\int v_K^j((1-\delta)K+I^j,\theta',\varepsilon')dF(\theta',\varepsilon') \leq 0,$$

$$= 0 \text{ if } I^j > 0.$$

Let $\tilde{I}^j = \tilde{g}^j(K,\theta,\varepsilon)$ be the solution of (14) with equality replacing the inequality, so that $\tilde{g}^j(K,\theta,\varepsilon)$ would be the optimal rate of investment given terminal reward function $v^j(\cdot)$ if the inequality constraint $I^j \geq 0$ were not present. Then the optimum rate of investment I^j implied by (14) is

$$I^j = I^j(K,\theta,\varepsilon) = \max(0,\tilde{g}^j(K,\theta,\varepsilon)).$$

That $I^j(K,\theta,\varepsilon)$ is continuous is implied by the maximum theorem of Berge. We consider three sets for (K,θ,ε):

i. The set of (K,θ,ε) such that $I^j > 0$.

ii. The set of (K,θ,ε) such that $I^j = 0$ and $\tilde{g}^j(K,\theta,\varepsilon) < 0$.

iii. Points (K,θ,ε) such that $\tilde{g}^j(K,\theta,\varepsilon) = 0$.

On the first set of points (K,θ,ε) such that $I^j > 0$, Benveniste and Scheinkman's [2] theorem implies that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable in K with derivative given by

$$(15)\qquad v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta-I^j(K,\theta,\varepsilon),\varepsilon)[f'(K)\theta+(1-\delta)].$$

On the second set of points (K,θ,ε) such that $I^j(K,\theta,\varepsilon) = 0$ and $\tilde{g}^j(K,\theta,\varepsilon) < 0$, $I^j(K,\theta,\varepsilon)$ is differentiable in K with derivative zero. Then direct calculations on (13) show that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable with respect to K and that

$$(16)\qquad v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta,\varepsilon)f'(K)\theta$$
$$+ \beta(1-\delta)\int v_K^j((1-\delta)K+I^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Now consider the third set of points such that $I^j(K,\theta,\varepsilon) = 0 = \tilde{g}^j(K,\theta,\varepsilon)$. We shall initially assume that $\tilde{g}^j(K,\theta,\varepsilon)$ is decreasing in K in the neighborhood of the point K in set (iii), so that the situation is as depicted in Figure (3). We shall then indicate how to modify the argument to handle the

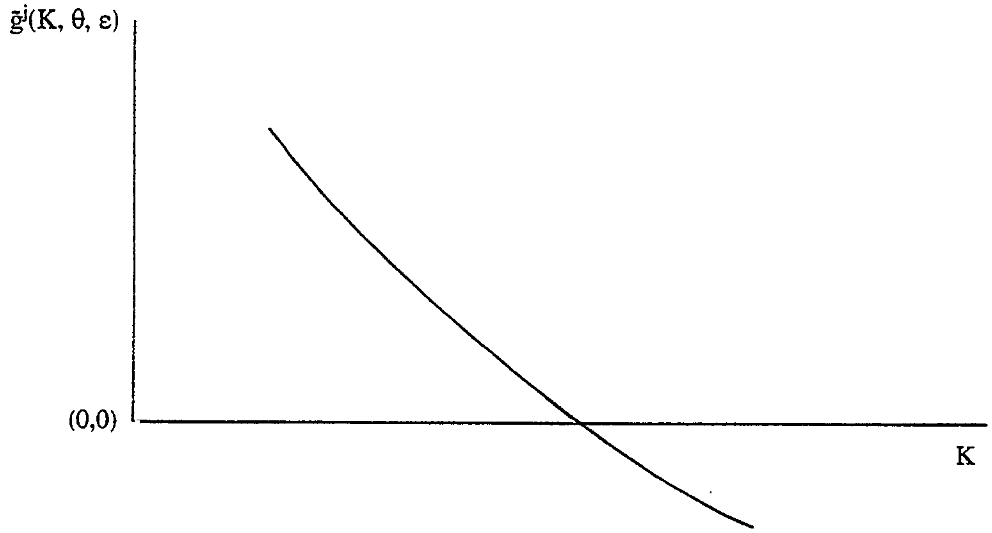


Figure 3

Hypothetical Investment Function

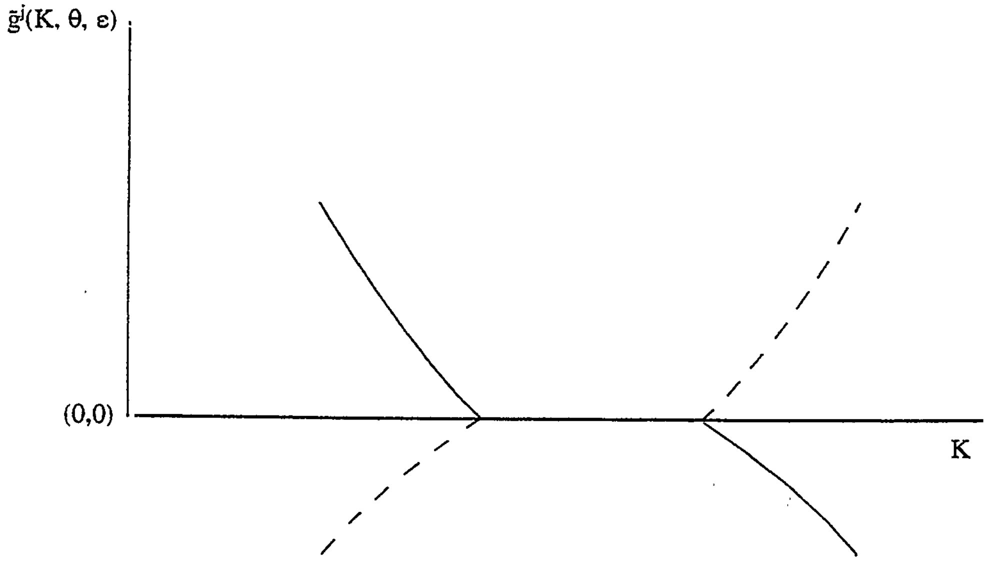


Figure 4

Hypothetical Investment Functions

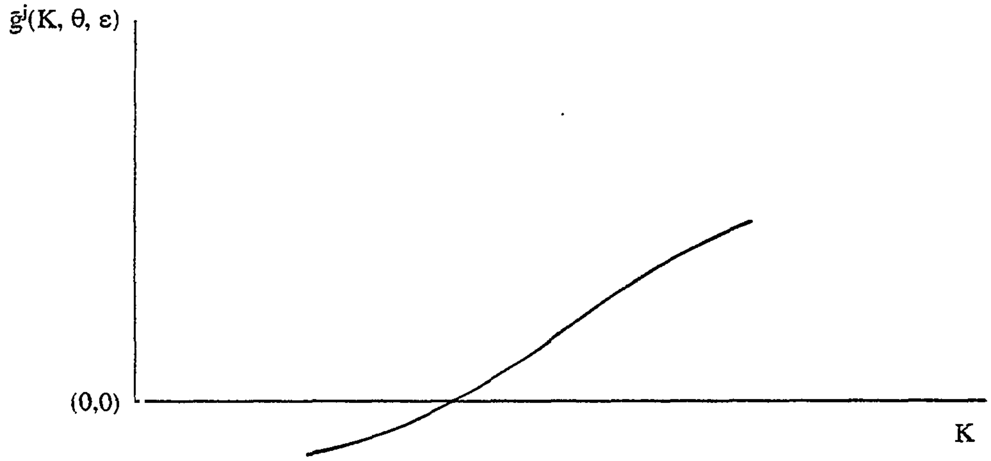


Figure 5

Hypothetical Investment Function

possibility that $\tilde{g}^j(K,\theta,\varepsilon)$ is constant, as in Figure (4), or increasing in K, as in Figure (5) in the neighborhood of the point in set (iii). Now assuming that $\tilde{g}^j(K,\theta,\varepsilon)$ is decreasing in K in a neighborhood of K, notice that $I^j(K,\theta,\varepsilon)$ has a right-hand derivative with respect to K equal to zero. The argument used for set (ii) implies that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable from the right at points in set (iii), with a right-hand derivative given by formula (16). We now undertake to show that $v^{j+1}(K,\theta,\varepsilon)$ is also differentiable from the left in set (iii) and that the left-hand derivative is also given by (16).

First, note that in region (i) since $I^j > 0$, the first-order necessary condition (14) holds with equality. Substituting (14) with equality into (15) yields

$$(17)\qquad v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta - I^j(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$

$$+\ \beta(1-\delta)\int v_K^j((1-\delta)K+I^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Thus, (17) holds for sets (i) and (ii) and also gives the right-hand derivative on set (iii). We wish to show that the left-hand derivative of $v^{j+1}(K,\theta,\varepsilon)$ exists at points in set (iii) and also equals (17). Let (K,θ,ε) be in set (iii), and let $\Delta > 0$. We know that $v^{j+1}(K,\theta,\varepsilon)$ is continuous on the closed interval $[K-\Delta,K]$ and is differentiable on the open interval $(K-\Delta,K)$, each point of which has been assumed to be in set (i). By the mean value theorem for derivatives, there exists a point ξ belonging to the open interval $(K-\Delta,K)$ for which

$$\frac{v^{j+1}(K,\theta,\varepsilon) - v^{j+1}(K-\Delta,\theta,\varepsilon)}{\Delta} = v_K^{j+1}(\xi,\theta,\varepsilon).$$

Taking the limit as Δ goes to zero proves that the left-hand derivative of $v^{j+1}(K,\theta,\varepsilon)$ at (K,θ,ε) exists and equals the limit of the derivatives $v_K^{j+1}(\xi,\theta,\varepsilon)$ as ξ approaches K from the left. From (15) or (17), we know that this latter limit exists since the right-hand side of (15) or (17) is continuous in K. Therefore, we have that the left-hand derivative of $v^{j+1}(K,\theta,\varepsilon)$ at K exists and equals the right side of (17), as does the right-hand derivative. In summary, it follows that for (K,θ,ε) in all three regions, the partial derivative of $v^{j+1}(K,\theta,\varepsilon)$ with respect to K exists and is given by (17).

Now if $\tilde{g}^j(K,\theta,\varepsilon)$ had been assumed to be increasing in K in a neighborhood of K in set (iii), a symmetrical argument would establish that $v^{j+1}(K,\theta,\varepsilon)$ is differentiable with derivative obeying (17). The arguments used above with respect to the left- and right-hand derivatives with respect to K would simply have to be exchanged.

Next, if $\tilde{g}^j(K,\theta,\varepsilon)$ is constant in K in a neighborhood of (K,θ,ε) in set (iii), the same argument as used in region (ii) would apply.

Since the function $I^j(K,\theta,\varepsilon)$ is continuous and has a slope with respect to K that is bounded from above and below (see Appendix B), it suffices to consider the cases depicted in Figure 3.

This establishes:

Proposition 4: Choose $v^0(K,\theta,\varepsilon)$ to be nondecreasing and concave in K with bounded and continuous partial derivative in K. Generate the sequence $v^j(K,\theta,\varepsilon) = T^j v^0(K,\theta,\varepsilon)$. For all $j \geq 0$, $v^{j+1}(K,\theta,\varepsilon)$ is continuously differentiable with respect to K with a partial derivative $v_K^{j+1}(K,\theta,\varepsilon)$ satisfying equation (17).

Equation (17) and the first-order necessary condition (14) imply the inequality

$$(18) \qquad v_K^{j+1}(K,\theta,\varepsilon) \leq u_c(f(K)\theta - I^j(K,\theta,\varepsilon),\varepsilon)[f'(K)\theta + (1-\delta)]$$

$$= \text{ if } I^j(K,\theta,\varepsilon) > 0.$$

In Appendix A, it is proved that $I^{j+1}(K,\theta,\varepsilon) \geq I^{j}(K,\theta,\varepsilon)$ for all $j \geq 0$. Then replacing $I^{j}(K,\theta,\varepsilon)$ with the (pointwise) limit function $I(K,\theta,\varepsilon)$ gives the inequality

$$v_K^{j+1}(K,\theta,\varepsilon) \leq u_c(f(K)\theta - I(K,\theta,\varepsilon),\varepsilon)[f'(K)\theta+(1-\delta)].$$

This establishes that for each fixed (K,θ,ε), $v_K^{j+1}(K,\theta,\varepsilon)$ is a bounded sequence. In Appendix A, it is proved that $v_K^{j+1}(K,\theta,\varepsilon) \geq v_K^{j}(K,\theta,\varepsilon)$ for all $j \geq 0$. Together with the boundedness of $v_K^{j}(K,\theta,\varepsilon)$, this proves that the sequence $v_K^{j}(K,\theta,\varepsilon)$ converges pointwise to a limit function, call it $\tilde{v}_K(K,\theta,\varepsilon)$.

It is now our aim to establish that the limit function $\tilde{v}_K(K,\theta,\varepsilon)$ is the partial derivative with respect to K of the value function $v(K,\theta,\varepsilon)$. We shall proceed by first restricting the domain along the K axis in a natural way, and then by arguing that in this domain $v_K^{j}(K,\theta,\varepsilon)$ converges uniformly to $\tilde{v}_K(K,\theta,\varepsilon)$.

In Appendix A, it is proved that there is an $\eta > 0$ such that for all $K \varepsilon (0,\eta]$ and for all (θ,ε), $I^{j}(K,\theta,\varepsilon) + (1-\delta)K > K$ for all $j \geq 1$. In particular, for all $K\varepsilon(0,\eta]$ and for all (θ,ε), $I(K,\theta,\varepsilon) + (1-\delta)K > K$. It follows that if the system starts out with any $K > 0$, eventually capital will have to remain forever within the interval $[\eta,\overline{\overline{K}}]$. Here $\overline{\overline{K}}$ is the "maximum sustainable capital stock" that solves

$$f(\overline{\overline{K}})\theta_b = \delta\overline{\overline{K}}.$$

The value $\overline{\overline{K}}$ is the steady state capital stock associated with the policy of consuming nothing and with always drawing the best technology shock θ_b. So $\overline{\overline{K}}$ is the stationary point of the difference equation

$$K_{t+1} = (1-\delta)K_t + f(K_t)\theta_b,\ K_0 > 0.$$

Our assumptions about $f(\cdot)$ guarantee that this difference equation has a unique stationary point.

From now on we shall assume that $K\varepsilon[\eta,\bar{\bar{K}}]$.

Now choose $v^0(K,\theta,\varepsilon) = v_K^0(K,\theta,\varepsilon) \equiv 0$, and generate the sequence $v_K^j(K,\theta,\varepsilon)$ according to (17). We have

$$\text{(19)} \qquad v_K^1(K,\theta,\varepsilon) = u_c(f(K)\theta,\varepsilon)f'(K)\theta$$

$$v_K^2(K,\theta,\varepsilon) = u_c(f(K)\theta-I^1(K,\theta,\varepsilon),\varepsilon)f'(K)$$

$$+\ \beta(1-\delta)\int u_c[f[(1-\delta)K+I^1(K,\theta,\varepsilon)]\theta',\varepsilon']f'(K)\theta'dF(\theta',\varepsilon').$$

$$\vdots$$

Notice that since $I^j(K,\theta,\varepsilon) \geq I^{j-1}(K,\theta,\varepsilon)$ (see Appendix A), it follows that on the domain $[\eta,\bar{\bar{K}}]$, we have

$$\text{(20)} \qquad u_c(f(K)\theta-I^j(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$

$$\leq u_c(f(K)\theta-I(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$

$$\leq \max_{\substack{K\varepsilon[\eta,\bar{\bar{K}}] \\ \theta\varepsilon[\theta_a,\theta_b] \\ \varepsilon\varepsilon[\varepsilon_b,\varepsilon_a]}} u_c(f(K)\theta-I(K,\theta,\varepsilon),\varepsilon)f'(K)\theta \equiv A < \infty.$$

We can continue the recursions (19) to get a series that can be written in the form

$$\text{(21)} \qquad v_K^n(K,\theta,\varepsilon) = h_1(K,\theta,\varepsilon) + h_2(K,\theta,\varepsilon)+\ldots+h_n(K,\theta,\varepsilon)$$

where

$$h_j(K,\theta,\varepsilon) \geq 0 \text{ for all } j \geq 1$$

and where

$$h_j(K,\theta,\varepsilon) \leq [\beta(1-\delta)]^{j-1}A.$$

That is, applying inequality (20) to (19), we have

$$v_K^1(K,\theta,\varepsilon) \leq A$$

$$v_K^2(K,\theta,\varepsilon) \leq A\{1+\beta(1-\delta)\}$$

$$v_K^3(K,\theta,\varepsilon) \leq A\{1+\beta(1-\delta)+\beta^2(1-\delta)^2\}$$

$$\vdots$$

$$v_K^j(K,\theta,\varepsilon) \leq A \sum_{j=0}^{j} [\beta(1-\delta)]^{j-1}.$$

It follows by the Weierstrass M-test (see Apostol [1, p. 223]) that the series (21) converges uniformly. Therefore, $v_K^j(K,\theta,\varepsilon)$ converges uniformly to $\tilde{v}_K(K,\theta,\varepsilon)$. So we have proved:

Proposition 5: Choose $v^0(K,\theta,\varepsilon) = v_K^0(K,\theta,\varepsilon) \equiv 0$. Then $v_K^j(K,\theta,\varepsilon)$ exists for all $j \geq 1$. On the domain $[\eta,\bar{\bar{K}}]$ the sequence of functions $v_K^j(K,\theta,\varepsilon)$ converges uniformly to a bounded and continuous function $\tilde{v}_K(K,\theta,\varepsilon)$.

Now choose $v^0(K,\theta,\varepsilon) = v_K^0(K,\theta,\varepsilon) \equiv 0$, and generate $v^j(K,\theta,\varepsilon) = T^j v^0(K,\theta,\varepsilon)$ and $v_K^j(K,\theta,\varepsilon)$ from (17). From the uniform convergence of $v_K^j(K,\theta,\varepsilon)$ to $\tilde{v}_K(K,\theta,\varepsilon)$ on $[\eta,\bar{\bar{K}}]$ we have (see Apostol [1, p. 238-239]),

Proposition 6: The value function $v(K,\theta,\varepsilon)$ is continuously differentiable in K with $v_K(K,\theta,\varepsilon) = \tilde{v}_K(K,\theta,\varepsilon)$. The partial derivative obeys the equation

$$(22) \qquad v_K(K,\theta,\varepsilon) = u_c(f(K)\theta - I(K,\theta,\varepsilon),\varepsilon)f'(K)\theta + \beta(1-\delta)\int v_K((1-\delta)K+I(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Proposition 6 implies that the first-order necessary condition for the maximum problem on the right side of (11) is

$$(23)\qquad -u_c(f(K)\theta - I, \varepsilon) + \beta\int v_K((1-\delta)K+I, \theta', \varepsilon')dF(\theta', \varepsilon') \leq 0,$$

$$= 0 \text{ if } I > 0.$$

Proposition 6 implies that obvious candidates for the equilibrium price functions $r(K,\theta,\varepsilon)$, $p_K(K,\theta,\varepsilon)$, and $w(K,\theta,\varepsilon)$ are

$$r(K,\theta,\varepsilon) = f'(K)\theta$$

$$(24)\qquad w(K,\theta,\varepsilon) = f(K)\theta - Kf'(K)\theta$$

$$p_K(K,\theta,\varepsilon) = \{u_c(f(K)\theta - I(K,\theta,\varepsilon),\varepsilon)\}^{-1}$$

$$\cdot\ \beta\int v_K((1-\delta)K + I(K,\theta,\varepsilon), \theta', \varepsilon')dF(\theta', \varepsilon').$$

It can be verified that with these price functions and with $h(K,\theta,\varepsilon)$ taken to be given by

$$(25)\qquad h(K,\theta,\varepsilon) \equiv (1-\delta)K + I(K,\theta,\varepsilon)$$

the market model of Section 2 is in equilibrium with the representative consumer's choice of $i(k,\theta,\varepsilon;K,\phi) + k^d(k,\theta,\varepsilon;K,\phi)$ equaling $I(K,\theta,\varepsilon) + (1-\delta)K$, the planner's capital accumulation plan, and with the representative consumer's choice of consumption equaling $f(K)\theta - I(K,\theta,\varepsilon)$. This can be verified by noting first that with (24), firms' marginal conditions are satisfied. Second, note that with (24), (25), and the proposed choices of $i(\ ) + k^d(\ )$, the marginal conditions for the representative agent in the market problem exactly match the planner's marginal condition (23). For example, with the suggested substitutions condition (9) becomes

$$-u_c(f(K_t)\theta_t - I(K_t, \theta_t, \varepsilon_t), \varepsilon_t)$$

$$+\beta \int\{u_c(f(K_{t+1})\theta_{t+1} - I(I_{t+1}, \theta_{t+1}, \varepsilon_{t+1}), \varepsilon_{t+1})$$

$$[f'(K_{t+1})\theta_{t+1} + (1-\delta)u_c(f(K_{t+1})\theta_{t+1} - I(K_{t+1}, \theta_{t+1}, \varepsilon_{t+1}), \varepsilon_{t+1})^{-1}$$

$$\beta\int v_K((1-\delta)K_{t+1} + I(K_{t+1}, \theta_{t+1}, \varepsilon_{t+1}), \theta_{t+2}, \varepsilon_{t+2})dF(\theta_{t+2}, \varepsilon_{t+2})]\}$$

$$dF(\theta_{t+1}, \varepsilon_{t+1}) \leq 0$$

with equality if $I(K_t, \theta_t, \varepsilon_t) > 0$. But notice that from (22), the term in braces simply equals $v_K(K_{t+1}, \theta_{t+1}, \varepsilon_{t+1})$. Therefore the above inequality becomes

$$-u_c(f(K_t)\theta_t - I(K_t, \theta_t, \varepsilon_t), \varepsilon_t)$$

$$+\beta\int v_K(K_{t+1}, \theta_{t+1}, \varepsilon_{t+1})dF(\theta_{t+1}, \varepsilon_{t+1}) \leq 0; \ = 0 \text{ if } I_t > 0.$$

This is equivalent with (23), as claimed.

Martingale Properties

From (17) and the first-order necessary condition (23) we have

$$v_K(K_t, \theta_t, \varepsilon_t) = v_K((1-\delta)K_{t-1} + I(K_{t-1}, \theta_{t-1}, \varepsilon_{t-1}), \theta_t, \varepsilon_t)$$

$$\geq [f'(K_t)\theta_t + (1-\delta)]\,\beta\int v_K((1-\delta)K_t + I(K_t, \theta_t, \varepsilon_t), \varepsilon_{t+1}, \theta_{t+1})dF(\theta_{t+1}, \varepsilon_{t+1})$$

with equality for $I(K_t, \theta_t, \varepsilon_t) > 0$. Integrating both sides with respect to $dF(\theta_t, \varepsilon_t)$ gives

$$\beta \int v((1-\delta)K_{t-1} + I(K_{t-1}, \theta_{t-1}, \varepsilon_{t-1}), \theta_t, \varepsilon_t)dF(\theta_t, \varepsilon_t)$$

$$\geq \beta\int [f'(k_t)\theta_t + (1-\delta)]\,\beta\int v_K((1-\delta)K_t + I(K_t, \theta_t, \varepsilon_t), \varepsilon_{t+1}, \theta_{t+1})$$

$$dF(\theta_{t+1}, \varepsilon_{t+1})dF(\theta_t, \varepsilon_t)$$

or

$$(26)\qquad p_{Kt-1} \geq E_{t-1}\{[(1-\delta)+f'(K_t)\theta_t]\cdot\beta p_{Kt}\,\frac{u_c(c(K_t,\theta_t,\varepsilon_t),\varepsilon_t)}{u_c(c(K_{t-1},\theta_{t-1},\varepsilon_{t-1}),\varepsilon_{t-1})}\}$$

where $c(K, \theta, \varepsilon) \equiv f(K)\theta - I(K, \theta, \varepsilon)$. Expression (26) shows that even adjusted for "dividends" and time preference, the relative price of existing capital is not a martingale, for essentially the same reason that the martingale property fails to hold in the models of Lucas [11] and Danthine [7]: the presence of corners, making (26) an inequality, and the presence of risk aversion, which is reflected in the failure of $u_c(\ )$ to be constant as a function of consumption. The same message emphasized by Lucas and Danthine is carried by the present model: failure of the relative price p_K to be a martingale does not reflect on whether or not markets are in equilibrium.

4. Sample Economies

The preceding section shows that aggregate investment I_t and the relative price of existing capital p_{Kt} can each be expressed as continuous functions of the aggregate state $(K_t,\theta_t,\varepsilon_t)$

$$p_{Kt} = P_K(K_t,\theta_t,\varepsilon_t)$$

$$I_t = I(K_t,\theta_t,\varepsilon_t).$$

It was shown that each of these functions reflects <u>all</u> of the parameters of the economy. In particular, the forms of <u>both</u> $p_K(\cdot)$ and $I(\cdot)$ depend on (i) the form of the utility function $u(c,\varepsilon)$, (ii) the form of the production function $f(K)\theta$, and (iii) the nature of the distribution of random shocks $F(\theta_t,\varepsilon_t)$. Thus, while the model can be seen to imply a pattern of covariation between I_t and p_{Kt}, the nature of that covariation reflects consumers' preferences, technology, and the probability distribution of the shocks θ and ε.

To make this point more formally, let

$$P(K'|K) = \text{Prob}\{K_{t+1} \leq K' | K_t = K\}$$

$$= \int_{A(K',K)} dF(\theta,\varepsilon)$$

where

$$A(K',K) = \{(\theta,\varepsilon):(1-\delta)K + I(K,\theta,\varepsilon) \leq K'\}.$$

Here the stochastic kernel $P(K'|K)$ defines a first-order Markov process for capital per man. Let

$$\psi_0(K) = \text{Prob}\{K_0 \leq K\}$$

be given. In Appendix C, it is proved that the Markov process for K possesses a unique stationary distribution $\psi(K)$ which is approached by iterations on

$$\psi_{t+1}(K') = \int P(K'|K)d\psi_t(K)$$

where $\psi_{t+1}(K') = \text{Prob}\{K_{t+1} \leq K'\}$. The stationary distribution $\psi(K)$ uniquely solves

$$\psi(K') = \int P(K'|K)d\psi(K).$$

The stationary distribution is approached for <u>any</u> initial distribution $\psi_0(K)$ assigning positive probability to positive capital.

Since (ε,θ) is a serially independent process, it follows that (K,θ,ε) are mutually independent contemporaneously. Therefore, the stationary moments of p_K and I can be calculated, for example, by

$$E(I \cdot p_K) = \iint p_K(K,\theta,\varepsilon) \cdot I(K,\theta,\varepsilon)dF(\theta,\varepsilon)d\psi(K)$$

$$E(p_K^2) = \iint p_K^2(K,\theta,\varepsilon)dF(\theta,\varepsilon)d\psi(K).$$

It is then clear that, for example, the regression coefficient of I on p_K, is in general a function of all of the parameters in the model. Further, the strong law of large numbers for Markov processes stated by Doob [8] tells us that sample moments such as

$$\frac{1}{T}\sum_{t=1}^{T} I_t p_{Kt},\ \frac{1}{T}\sum_{t=1}^{T} p_{Kt}^2,\ \frac{1}{T}\sum_{t=1}^{T} p_{Kt},\ \text{etc.},$$

converge with probability one to the corresponding moments of the stationary distribution EIp_K, Ep_K^2, Ep_K, etc., respectively.

We carried out some calculations designed to illustrate how the regression of I on p_K depends on various parameters. We assumed that the distributions of ε_i and θ_i were concentrated on two points with

$$\text{Prob}\{\theta=\theta_1\} = p_1$$

$$\text{Prob}\{\theta=\theta_2\} = 1-p_1 \equiv p_2$$

$$\text{Prob}\{\varepsilon=\varepsilon_1\} = q_1$$

$$\text{Prob}\{\varepsilon=\varepsilon_2\} = 1-q_1 \equiv q_2$$

$$\text{Prob}\{\theta=\theta_i, \varepsilon=\varepsilon_j\} = p_i q_j,\ i=1,\ 2;\ j=1,\ 2.$$

We specified a grid of admissible points along the capital-labor axis, restricting the planner to choose among this finite set of feasible points, call it $\overline{K}$. The functional equation for the optimal value function is

$$v(K_a, \theta_i, \varepsilon_j) = \max_{\substack{I\geq 0 \\ I+(1-\delta)K_a \in\overline{K}}} \{u(f(K_a)\theta_i - I, \varepsilon_j) + \beta\sum_s\sum_m v((1-\delta)K_a+I, \theta_s, \varepsilon_m)p_s q_m\} \tag{27}$$

where $K_a \varepsilon \overline{K}$. Notice that next period's capital stock $I+(1-\delta)K_a$ is required to belong to the set $\overline{K}$. The grid of feasible points $\overline{K}$ was chosen as follows. Where the grid contains n points and $\tilde{K}$ was chosen as the highest capital-labor ratio in the grid, we chose

$$K_{n-j+1} = (1-\delta)^{j/z} \cdot \tilde{\tilde{K}} \qquad j=1, \ldots, n$$

where z is a positive integer. Notice that the grid is chosen so that the "corner points" $(1-\delta)K$ are included. In practice, $\tilde{\tilde{K}}$ and z were chosen so that the grid at least covered the set of ergodic states for the capital-labor ratio.

We solved the functional equation (27) by in effect iterating on the "T mapping" described in the discussion of Proposition 1. In practice we used an algorithm described by Bertsekas [4, p. 237-241] to speed up the convergence. We are constrained to consider variations in the investment rate of Δ where Δ is the distance between adjacent points in $\overline{K}$. The necessary condition for the maximum problem on the right side of (27) is that for $\hat{I}$ optimal

$$u(f(K_a)\theta_i-\hat{I}, \varepsilon_j) + \beta \sum_s \sum_m v((1-\delta)K_a+\hat{I}, \theta_s, \varepsilon_m)p_s q_m$$

$$\geq u(f(K_a)\theta_i-(\hat{I}+\Delta), \varepsilon_j) + \beta \sum_s \sum_m v((1-\delta)K_a+(\hat{I}+\Delta), \theta_s, \varepsilon_m)p_s q_m$$

for all $\Delta > 0$ and for all $\hat{I} + \Delta \geq 0$ or $\Delta \geq - \hat{I}$, where $\hat{I} + (1-\delta)K_a \varepsilon \overline{K}$ and $\hat{I} + \Delta + (1-\delta)K_a \varepsilon \overline{K}$. The optimizing I thus satisfies the condition that it is the largest value of I for which

$$(28) \qquad \frac{u(f(K_a)\theta_i-I, \varepsilon_j)-u(f(K_a)\theta_i-(I+\Delta), \varepsilon_j)}{} \\ - \frac{\beta \sum_s \sum_m (v((1-\delta)K_a+I+\Delta, \theta_s, \varepsilon_m)-v((1-\delta)K_a+I, \theta_s, \delta_m))p_s q_m}{\Delta}$$

for all admissible $\Delta > 0$. For the smallest admissible Δ, we take the left side of (28) as our estimate of $u_c(c, \varepsilon_j)$, while we take the right side as our estimate of $p_K \cdot u_c(c, \varepsilon)$. We form our estimate of $p_K(K_a, \theta_i, \varepsilon_j)$ by dividing the latter by the former. The optimum policy function $I(K_a, \theta_i, \varepsilon_j)$ is obtained as a by-product of solving for the optimal value function.

We generated the stochastic matrix associated with the Markov process for K from

$$P_{ij} = \text{Prob}\{K_{t+1}=K_i \mid K_t=K_j\}$$

$$= \text{Prob}\{I(K_j, \theta, \varepsilon)+(1-\delta)K_j=K_i\}$$

$$= \sum_{s,m\varepsilon S} p_s q_m$$

where $S = \{(s,m): I(K_j,\theta_s,\varepsilon_m)+(1-\delta)K_j=K_i\}$. An (nxn) stochastic matrix P with elements P_{ij} was formed, with n being the number of points in the set of admissible capital stocks $\overline{K}$. Then the stationary distribution of K was determined by taking any column of $\lim_{t\to\infty} P^t$ (in the limit the columns of P^t are all the same, if P possesses a unique stationary distribution). For the stationary distribution of K_t we denote

$$\text{Prob}\{K_t=K_i\} = \pi_i, \quad K_i\varepsilon\overline{K}, \quad i=1, \ldots, n.$$

We calculated the population moments of I and p_K from, e.g.,

$$EI(K, \theta, \varepsilon) = \sum_{h=1}^{n} \sum_{i=1}^{2} \sum_{j=1}^{2} I(K_h, \theta_i, \varepsilon_j)\pi_h p_i q_j$$

$$EI(K, \theta, \varepsilon)\cdot p_K(K, \theta, \varepsilon) = \sum_{h=1}^{n} \sum_{i=1}^{2} \sum_{j=1}^{2} I(K_h, \theta_i, \varepsilon_j)p_K(K_h, \theta_i, \varepsilon_j)\pi_h p_i q_j.$$

Table 1

$u(c) = \varepsilon\cdot\ln c$	$\delta = .05$
$f(K) = K^{(.25)}$	$\delta = .95$

<u>Economy 1</u> (64 points in $\overline{K}$)

$P\{\theta=.9\} = .5,\ P\{\theta=1.1\} = .5$

$P\{\theta=.9\} = .75,\ P\{\theta=1.1\} = .25$

$$\frac{\text{cov}(I,p_K)}{\text{var } p_K} = 1.5798$$

$$\frac{\text{cov}(I,p_K)}{\sqrt{\text{var } I\cdot\text{var } p_K}} = .4213$$

$$\frac{\text{cov}(I,p_K)}{\text{var } I} = .1123$$

<u>Economy 2</u> (64 points in $\overline{K}$)

$P\{\theta=.9\} = .5,\ P\{\theta=1.1\} = .5$

$P\{\theta=.9\} = .5,\ P\{\theta=1.1\} = .75$

$$\frac{\text{cov}(I,p_K)}{\text{var } p_K} = .8653$$

$$\frac{\text{cov}(I,p_K)}{\sqrt{\text{var } I\cdot\text{var } p_K}} = .0934$$

$$\frac{\text{cov}(I,p_K)}{\text{var } I} = .0101$$

<u>Economy 3</u>

$P\{\theta=.9\} = .5,\ P\{\theta=1.1\} = .5$

$P\{\varepsilon=.9\} = .5,\ P\{\varepsilon=1.1\} = .5$

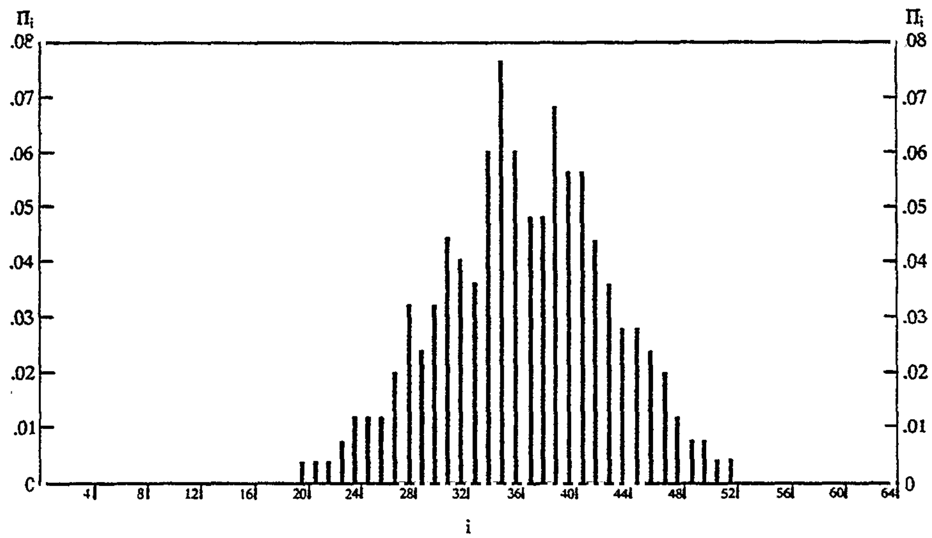
	(32 states in $\bar{K}$)	(48 states in $\bar{K}$)	(64 states in $\bar{K}$)	(80 states in $\bar{K}$)
$\frac{\text{cov}(I,p_K)}{\text{var } p_K}$	= 1.31913	2.1889	2.5057	2.7491
$\frac{\text{cov}(I,p_K)}{\sqrt{\text{var } I \cdot \text{var } p_K}}$	= .3115	.4572	.4933	.5261
$\frac{\text{cov}(I,p_K)}{\text{var } I}$	= .0735	.0955	.0971	.1007

Table 1 gives examples for an economy in which $u(c) = \varepsilon \ln c$ and $f(K) = K^{.25}$, $\beta = .95$, and $\delta = .05$. The set $\bar{K}$ included sixty-four states, except where otherwise noted. For the parameters of economy 3, we have calculated the sample moments for alternative $\bar{K}$'s including 48, 64, and 80 states. These calculations for increasingly fine grids on K are interesting if one views these finite economies as approximations to the continuous-state economy analyzed in previous sections. From the behavior of these moments with increasingly fine grids, our grids are evidently not yet fine enough to approximate the corresponding continuous-state economies very well. An alternative way to view these calculations is not as giving approximations but exact evaluations of the population moments of the indicated finite-state economies. The three economies are identical except that they are characterized by different distributions of the shock to preferences $\{\varepsilon\}$. Notice the effects of alterations in the distribution on the population values of the regression coefficient of I on p_K, given by $\text{cov}(I,p_K)/\text{var } p_K$, and on the correlation coefficient between I and p_K, given by $\text{cov}(I,P_K)/\sqrt{\text{var } I \cdot \text{var } p_K}$. The table illustrates how, in the jargon of macroeconomists, shifts in the distribution of the consumption function cannot be expected to leave the regression of I on p_K unaltered. Figure 6 depicts the population discrete density function giving the unique stationary distribution associated with economy 3 with 64 states.

Figure 6

Stationary Distribution on K for Economy 3

K (1) (= capital/labor ratio in state 1) = 2.118238
K (64) = 4.7514702



State	Π_i	State	Π_i	State	Π_i
1	.000000	26	.014426	51	.004128
2	.000000	27	.019485	52	.004685
3	.000001	28	.032343	53	.001323
4	.000002	29	.024945	54	.001244
5	.000003	30	.030884	55	.000386
6	.000009	31	.043868	56	.000338
7	.000012	32	.039169	57	.000103
8	.000034	33	.035644	58	.000086
9	.000047	34	.058756	59	.000048
10	.000134	35	.076250	60	.000012
11	.000184	36	.059213	61	.000003
12	.000185	37	.048279	62	.000001
13	.000327	38	.049524	63	.000000
14	.000693	39	.068342	64	.000000
15	.000680	40	.054608		
16	.001093	41	.056144		
17	.002452	42	.044587		
18	.002349	43	.037053		
19	.003157	44	.029894		
20	.004323	45	.027909		
21	.004464	46	.024824		
22	.006848	47	.022069		
23	.008950	48	.012234		
24	.012308	49	.008601		
25	.012164	50	.008171		

These examples illustrate how in such an economy, the regression of investment on p_K does not recover the law governing the demand to accumulate capital. The problem is <u>not</u> a failure to correct for simultaneous equation bias, say by using instrumental variables, nor is it a failure to include enough lagged values of q. In these economies it would be impossible to recover a structural investment schedule by pursuing such modifications.

It is straightforward to describe econometric procedures that would permit recovery of the economy's structural parameters from time series data on y_t, K_t, and p_{Kt}. It would be necessary to specify functional forms for $u(c, \varepsilon)$ and $f(K)\,\theta$, as well as a form for the distribution $F(\theta, \varepsilon)$. Then for each point in the space of parameters determining β, δ, $u(\cdot,\cdot)$, $f(\ )$, and $F(\ ,\ )$, there is a unique pair of functions $I(K,\theta,\varepsilon)$ and $p_K(K,\theta,\varepsilon)$. The likelihood function of a vector of time series on (y_t,K_t,p_{Kt}) can then be characterized as a function of the free parameters of $\{\beta,\ \delta,\ u(\cdot,\cdot),\ f(\cdot),\ \text{and}\ F(\cdot,\cdot)\}$. The method of maximum likelihood could then be used to estimate the structural parameters of the economy. As of now, such procedures would be very expensive even for the very simple economy that we have described. They would be prohibitively expensive for any "realistic" model.

Of course, in our sample economies the least squares regression of I on p_K is predicted to remain the same so long as the distributions of all shocks remain unaltered. It is possible to construct examples, as we have in Table 1, in which p_K explains a large part of the variation in investment. But one wants a structural model of investment in order to be able to analyze interventions in the forms of alterations in certain random processes, in particular, in processes describing various aspects of fiscal policy. It is for analyzing such policy changes that our analysis suggests that it will be inadequate to rely on the maintenance of historical patterns between I and p_K.

5. Concluding Remarks

The following two features of our model deserve brief discussion: first, whenever p_K is less than unity, the aggregate rate of investment is zero; and second, it is impossible for p_K ever to be above unity. It is easy to conceive of variations on the present model in which aggregate investment is positive even when an aggregate index corresponding to p_K is less than unity. For example, consider a model with two goods, x and y, both of which are consumed while good y can also be used to augment the capital stock of industries x and y. Assume that new output of y can be costlessly allocated across consumption, investment in industry x, or investment in industry y. But once in place, capital in industries x and y cannot be consumed. This setup will give rise to two distinct prices of existing capital in industries x and y, say, p_{Kx} and p_{Ky}, respectively, relative to newly produced capital. Investment in industry x will be positive only if p_{Kx} is unity, and investment in industry y will be positive only if p_{Ky} is unity. But aggregate investment can be positive when an aggregate index of the price of existing capital relative to newly produced capital is less than equity. Conceptually, analysis of such a model is no more complicated than the one-sector model studied in this paper; it is only much more cumbersome notationally.

The second peculiarity of our model, the inability of p_K to rise above unity, stems from the asymmetry in the "friction" that we have posited. That is, the technological rigidity that we have posited impedes rapid decreases in the capital stock, but not increases. It remains to be seen what would be the implications of general equilibrium versions of the cost-of-change model of Lucas [12], Gould [10], and Treadway [23] which posit more or less symmetrical costs of adjustment.

There is little reason to believe that modifications along either of these lines would alter the basic message of this paper: that the same "frictions" or "adjustment costs" that make it possible for p_K or q to diverge from unity also establish a presumption that agents' investment decisions at time t are not expressible in any simple way as a function of p_{Kt}.

Appendix A

Properties of $v^j(K,\theta,\varepsilon)$ and $I^j(K,\theta,\varepsilon)$ Sequences

We consider a sequence of n+1 period problems, n=0, 1, 2, ..., with value functions satisfying

$$v^{n+1}(K,\theta,\varepsilon) = \max E_0 \sum_{t=0}^{n} \beta^t u(C_t,\varepsilon_t)$$

where the maximization is subject to

$$C_t + I_t \leq f(K_t)\theta_t$$

$$C_t \geq 0, \; I_t \geq 0$$

$$K_{t+1} = (1-\delta)K_t + I_t.$$

The sequence $v^{n+1}(K,\theta,\varepsilon)$ is generated by iterating on $v^0(K,\theta,\varepsilon) \equiv 0$ with T defined by equation (12), i.e.,

$$\text{(A1)} \qquad v^{n+1}(K,\theta,\varepsilon) = \max_{I \geq 0}\{u(f(K)\theta-I,\varepsilon)+\beta\int v^n((1-\delta)K+I,\theta',\varepsilon')dF(\theta',\varepsilon')\}$$

or

$$v^{n+1}(K,\theta,\varepsilon) = Tv^n(K,\theta,\varepsilon) = T^{n+1}v^0(K,\theta,\varepsilon).$$

The right-hand side of (A1) is uniquely attained by the policy function $I = I^n(K,\theta,\varepsilon)$, so that $I^n(K, \theta, \varepsilon)$ is the optimal first-period choice of investment for the (n+1) period problem. It is known that as $n \rightarrow \infty$, $I^n(K,\theta,\varepsilon)$ converges pointwise to $I(K,\theta,\varepsilon)$, where recall that $I(K,\theta,\varepsilon)$ is the optimal investment policy function for the infinite horizon problem.

We can now prove:

Proposition A1: For all $n \geq 1$ and all (K,θ,ε),

$$I^{n}(K,\theta,\varepsilon) \geq I^{n-1}(K,\theta,\varepsilon)$$

and

$$v_K^{n}(K,\theta,\varepsilon) \geq v_K^{n-1}(K,\theta,\varepsilon).$$

Proof: Starting with $n = 0$ and $v^0(K,\theta,\varepsilon) = 0$, we have

$$v^1(K,\theta,\varepsilon) = \max_{I^0 \geq 0} \{u(f(K)\theta - I^0, \varepsilon)\} = u(f(K)\theta, \varepsilon)$$

where $I^0(K,\theta,\varepsilon) = 0$ attains $v^1(K,\theta,\varepsilon)$. Further, we have

$$v_K^1(K,\theta,\varepsilon) = u_c(f(K)\theta, \varepsilon) f'(K)\theta. \tag{A2}$$

For $n = 1$, we have

$$v^2(K,\theta,\varepsilon) = \max_{I^1 \geq 0} \{u(f(K)\theta - I^1, \varepsilon) + \beta\int u(f[(1-\delta)+I^1]\theta', \varepsilon')dF(\theta',\varepsilon')\}.$$

Further, we have from (17)

$$v_K^2(K,\theta,\varepsilon) = u_c(f(K)\theta - I^1(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$
$$+ \beta(1-\delta)\int v_K^1((1-\delta)K + I^1(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon'). \tag{A3}$$

Since $I^1(K,\theta,\varepsilon) \geq 0 = I^0(K,\theta,\varepsilon)$, and since $u_c(f(K)\theta - I(K,\theta,\varepsilon),\varepsilon)$ is increasing in I, it follows that, comparing expressions (A2) and (A3) for v_K^1 and v_K^2, we have

$$v_K^2(K,\theta,\varepsilon) \geq v_K^1(K,\theta,\varepsilon).$$

Now consider the first-order necessary condition for the j+1 period problem:

(A4) $$u_c(f(K)\theta - I^j, \varepsilon) \geq \beta\int v_K^j((1-\delta)K+I^j, \theta', \varepsilon')dF(\theta', \varepsilon')$$

$$= \text{ if } I^j > 0.$$

Since the right-hand side is decreasing in I^j and the left-hand side is increasing in I^j, it follows that if $v_K^j(K, \theta, \varepsilon) \geq v_K^{j-1}(K, \theta, \varepsilon)$, then $I^j(K, \theta, \varepsilon) \geq I^{j-1}(K, \theta, \varepsilon)$. (Refer to Figure (A1) where we have graphed the situation off corners where $v_K^{n-1} \geq v_K^{n-2}$.)

Now assume that $v_K^{n-1}(K, \theta, \varepsilon) \geq v_K^{n-2}(K, \theta, \varepsilon)$ and that as a consequence $I^{n-1}(K, \theta, \varepsilon) \geq I^{n-2}(K, \theta, \varepsilon)$. Then we have that

$$\begin{aligned}
v_K^n(K, \theta, \varepsilon) &= u_c(f(K)\theta - I^{n-1}(K, \theta, \varepsilon), \varepsilon)f'(K)\theta \\
&\quad + \beta(1-\delta)\int v_K^{n-1}((1-\delta)K+I^{n-1}(K, \theta, \varepsilon), \theta', \varepsilon')dF(\theta', \varepsilon') \\
&\geq u_c(f(K)\theta - I^{n-2}(K, \theta, \varepsilon), \varepsilon)f'(K)\theta \\
&\quad + \beta(1-\delta)\int v_K^{n-2}((1-\delta)K+I^{n-2}(K, \theta, \varepsilon), \theta', \varepsilon')dF(\theta', \varepsilon') \\
&= v_K^{n-1}(K, \theta, \varepsilon)
\end{aligned}$$

where the middle inequality follows because u_c is increasing in I and $I^{n-2}(K, \theta, \varepsilon) \leq I^{n-1}(K, \theta, \varepsilon)$ and from Figure (A1). So we have $v_K^n(K, \theta, \varepsilon) \geq v_K^{n-1}(K, \theta, \varepsilon)$. But from the first-order necessary condition (A4) or Figure A1, we have that $v_K^n(K, \theta, \varepsilon) \geq v_K^{n-1}(K, \theta, \varepsilon)$ implies $I_K^n(K, \theta, \varepsilon) \geq I_K^{n-1}(K, \theta, \varepsilon)$. So we have proved that if $v_K^{n-1}(K, \theta, \varepsilon) \geq v_K^{n-2}(K, \theta, \varepsilon)$, then $v_K^n(K, \theta, \varepsilon) \geq v_K^{n-1}(K, \theta, \varepsilon)$ and $I^n(K, \theta, \varepsilon) \geq I^{n-1}(K, \theta, \varepsilon)$. Since we have proved that $v_K^2(K, \theta, \varepsilon) \geq v_K^1(K, \theta, \varepsilon)$, by induction it follows that $v_K^n(K, \theta, \varepsilon) \geq v_K^{n-1}(K, \theta, \varepsilon)$ and that $I^n(K, \theta, \varepsilon) \geq I^{n-1}(K, \theta, \varepsilon)$ for all $n \geq 1$.

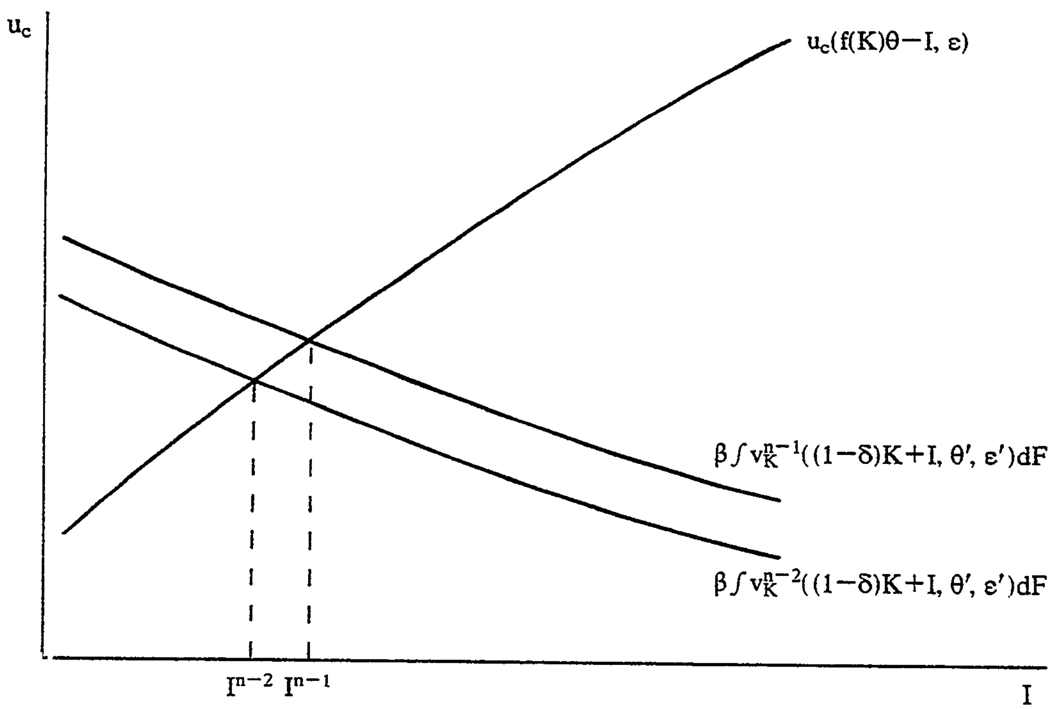


Figure A1

Determination of Investment in Finite-Horizon Problems

We now consider the two-period problem, which we have seen is associated with the functional equation

$$v^2(K,\theta,\varepsilon) = \max_{I^1 \geq 0} \{u(f(K)\theta - I^1, \varepsilon) + \beta \int u[f[(1-\delta)K+I^1]\theta', \varepsilon']dF(\theta',\varepsilon')\}.$$

The first-order necessary condition for the problem on the right-hand side is

(A5) $$u_c(f(K)\theta - I^1, \varepsilon) \geq \beta \int u_c(f[(1-\delta)K+I^1]\theta', \varepsilon')f'[(1-\delta)K+I^1]\theta' dF(\theta',\varepsilon')$$

$$= \text{ if } I^1 > 0.$$

Recall that θ_a is the minimum possible value of θ, and that ε_a is the maximum possible value of ε, and that we have assumed that the point $(\theta_a, \varepsilon_a)$ is assigned positive probability by $F(\cdot,\cdot)$. Then (A5) implies

(A6) $$u_c(f(K)\theta_a - I^1(K,\theta_a,\varepsilon_a), \varepsilon_a)$$

$$\geq \beta u_c\{f[(1-\delta)K+I(K,\theta_a,\varepsilon_a)]\cdot\theta_a, \varepsilon_a\}\cdot f'[(1-\delta)K+I^1(K,\theta_a,\varepsilon_a)]\cdot$$

$$\theta_a \cdot \text{Prob}(\theta_a,\varepsilon_a).$$

We are now in a position to prove the following:

Proposition A2: There is a $\eta > 0$ such that for all K such that $0 < K \leq \eta$, $b^1(K,\theta_a,\varepsilon_a) = (1-\delta)K + I^1(K,\theta_a,\varepsilon_a) > K$.

Proof: If there were no such $\eta > 0$, then we could find a sequence $\{K_n\}$ decreasing to zero with $K_{n+1} = b^1(K_n,\theta_a,\varepsilon_a) \leq K_n$. We shall assume that such a sequence can be found, and show that the assumption leads to a contradiction. Letting $\{K_n\}$ be the sequence in question, we note that $I^1(K_n,\theta_a,\varepsilon_a) \leq \delta K_n$ by assumption. Inequality (A6) implies

$$u_c(f(K_n)\theta_a - I^1(K_n,\theta_a,\varepsilon_a), \varepsilon_a)$$

$$\geq \beta u_c(f(K_{n+1})\theta_a, \varepsilon_a)\cdot f'(K_{n+1})\theta_a \cdot \text{Prob}(\theta_a,\varepsilon_a)$$

or

$$\frac{u_c(f(K_n)\theta_a - I^1(K_n,\theta_a,\varepsilon_a),\varepsilon_a)}{u_c(f(K_{n+1})\theta_a,\varepsilon_a)} \geq \beta f'(K_{n+1})\theta_a \cdot \text{Prob}(\theta_a,\varepsilon_a).$$

Since $u_{cc} < 0$ and $I^1(K_n,\theta_a,\varepsilon_a) \leq \delta K_n$, and since by assumption $f(K_{n+1}) \leq f(K_n)$, we have $u_c[f(K_n)\theta_a - \delta K_n, \varepsilon_a] \geq u_c[f(K_n)\theta_a - I(K_n,\theta_a,\varepsilon_a),\varepsilon_a]$ and $u_c[f(K_n)\theta_a,\varepsilon_a] \leq u_c[f(K_n)\theta_a,\varepsilon_a]$. Combining these with the above inequality gives

$$\text{(A7)} \qquad \frac{u_c[f(K_n)\theta_a - \delta K_n,\varepsilon_a]}{u_c[f(K_n)\theta_a,\varepsilon_a]} \geq \beta f'(K_n)\theta_a \cdot \text{Prob}(\theta_a,\varepsilon_a).$$

By Taylor's theorem, there exists a $\xi_n \in [0,1]$ such that the left-hand side of (A7) can be expressed as

$$\frac{u_c[f(K_n)\theta_a,\varepsilon_a] - \delta K_n u_{cc}[f(K_n)\theta - \xi_n \delta K_n,\varepsilon_a]}{u_c[f(K_n)\theta_a,\varepsilon_a]}.$$

Then by property (0) assumed for $u(\cdot,\cdot)$ and $f(\cdot)$, it follows that the limit as $K_n \downarrow 0$ of the left-hand side of (A7) is unity. But the limit of the right-hand side as $K_n \downarrow 0$ is plus infinity. Therefore, we have been led into a contradiction. This proves the proposition.

Taken together, propositions A1 and A2 imply:

<u>Proposition A3</u>: There is a $\eta > 0$ such that for all K such that $0 < K \leq \eta$, $b(K,\theta_a,\varepsilon_a) = (1-\delta)K + I(K,\theta,\varepsilon) > K$. Further, since $I(K,\theta,\varepsilon)$ is nondecreasing in θ and nonincreasing in ε, it follows that $b(K,\theta,\varepsilon) = (1-\delta)K + I(K,\theta,\varepsilon) > K$ for K such that $0 < K \leq \eta$ and all (θ,ε).

Propositions A2 and A3 and their proofs are relatively straightforward modifications of theorems and proofs in Mirman [16].

Appendix B

Restrictions on 'Slopes'

The evolution of the aggregate capital stock is governed by the stochastic difference equation

$$K_{t+1} = (1-\delta)K_t + I(K_t, \theta_t, \varepsilon_t) \equiv b(K_t, \theta_t, \varepsilon_t).$$

In studying the "stability" of this difference equation, we will need some information about the slopes of b with respect to K, θ, and ε. The following argument is taken from Lucas.[12/] Rewrite the functional equation (11) as

$$\text{(B1)} \qquad v(K,\theta,\varepsilon) = \max_{y \geq (1-\delta)K} \{u[(1-\delta)K+f(K)\theta-y, \varepsilon]+\beta\int v(y, \theta', \varepsilon')dF(\theta', \varepsilon')\}$$

where the right-hand side is uniquely attained by

$$y = b(K,\theta,\varepsilon) \equiv I(K,\theta,\varepsilon) + (1-\delta)K.$$

Let us choose $v^0(K,\theta,\varepsilon)$ to be continuous, bounded, strictly concave, and twice differentiable in K. Then it follows that for all $j \geq 1$, $v^j(K,\theta,\varepsilon) = T^j v^0(K,\theta,\varepsilon)$ is twice differentiable in K (almost everywhere). This property is useful in establishing restrictions on the "slopes" of $b(K,\theta,\varepsilon)$. To establish this property, assume that $v^j(K,\theta,\varepsilon)$ is almost everywhere twice differentiable in K. Let $b^j(K,\theta,\varepsilon)$ attain $v^{j+1}(K,\theta,\varepsilon)$. Then off corners, the first-order necessary conditions for the maximization of $\{u[(1-\delta)K+f(K)\theta-y,\varepsilon]+\beta\int v^j(y, \theta', \varepsilon')dF(\theta', \varepsilon')\}$ are satisfied with equality. Differentiating the first-order condition shows that off corners, $b^j(K,\theta,\varepsilon)$ is differentiable with

$$\text{(B2)} \qquad \frac{\partial b^j}{\partial K} = \frac{u_{cc}\cdot[(1-\delta)+f'(K)\theta]}{u_{cc}+\beta\int v^j_{KK}(y, \theta', \varepsilon')dF(\theta', \varepsilon')} > 0$$

(B3) $$\frac{\partial I^j}{\partial \varepsilon} = \frac{\partial b^j}{\partial \varepsilon} = \frac{u_{c\varepsilon}}{u_{cc} + \beta\int v_{KK}^j(y, \theta', \varepsilon')dF(\theta', \varepsilon')} < 0$$

(B4) $$\frac{\partial I^j}{\partial \theta} = \frac{\partial b^j}{\partial \theta} = \frac{u_{cc}f(K)}{u_{cc} + \beta\int v_{KK}^j(y, \theta', \varepsilon')dF(\theta', \varepsilon')} > 0$$

Also, notice that since $c^j(K, \theta, \varepsilon) = f(K)\theta - I^j(K, \theta, \varepsilon)$ and since $\partial b^j/\partial\theta = \partial I^j/\partial\theta$, we have

$$\frac{\partial c^j(K,\theta,\varepsilon)}{\partial \theta} = f(K) - \frac{\partial b^j(K,\theta,\varepsilon)}{\partial \theta}$$

$$= \frac{f(K)\beta\int v_{KK}^j(y,\theta',\varepsilon')dF(\theta',\varepsilon')}{u_{cc}+\beta\int v_{KK}^j(y,\theta',\varepsilon')dF(\theta',\varepsilon')} > 0.$$

The terms $\int v_{KK}^j(y,\theta',\varepsilon')dF(\theta',\varepsilon')$ are well defined by the assumed (almost everywhere) twice differentiability of v^j and the assumption that $F(\theta,\varepsilon)$ has a continuous density function and so assigns zero probability to points where $v^j(K,\theta,\varepsilon)$ is not twice differentiable. Where $b^j(K, \theta, \varepsilon) = (1-\delta)K$ and $\tilde{g}^j(K,\theta,\varepsilon) < 0$ (i.e., in our region ii), $b^j(K, \theta, \varepsilon)$ is differentiable with $\partial b^j/\partial K = (1-\delta)$, $\partial b^j/\partial\varepsilon = \partial b^j/\partial\theta = 0$. In region (iii), which is a set of Lebesque measure zero, $b^j(K,\theta,\varepsilon)$ is not differentiable. Now write (17) as

$$v_K^{j+1}(K,\theta,\varepsilon) = u_c(f(K)\theta+(1-\delta)K-b^j(K,\theta,\varepsilon),\varepsilon)f'(K)\theta$$

$$+ \beta(1-\delta)\int v_K^j(b^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon').$$

Differentiating with respect to K gives

$$v_{KK}^{j+1}(K,\theta,\varepsilon) = u_{cc}f'(K)\theta[f'(K)\theta+(1-\delta)-b_K^j(K,\theta,\varepsilon)]$$

$$+ \beta(1-\delta)\int v_{KK}^j(b^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon')\cdot b_K^j(K,\theta,\varepsilon) + u_c f''(K)\theta.$$

Since the right-hand side exists almost everywhere, so does the left. So we have established that if $v^j(K, \theta, \varepsilon)$ is twice differentiable (a.e.) in K, then so is $v^{j+1}(K, \theta, \varepsilon)$. It follows that iterating with T on a $v^0(K, \theta, \varepsilon)$ that is continuous, bounded, strictly concave, and twice differentiable in K gives rise to a sequence $b^j(K, \theta, \varepsilon)$ of approximate policy functions, each member of which satisfies (B2, B3, B4) off corners.

Notice that where $v_{KK}^{j+1}(K, \theta, \varepsilon)$ is attained with $b^j(K,\theta,\varepsilon) > (1-\delta)K$ so that B2-B4 apply, we have

$$v_{KK}^{j+1}(K, \theta, \varepsilon) = u_c f''(K)\theta$$

$$+ u_{cc} f'(K)\theta\left[f'(K)\theta + (1-\delta) - \frac{u_{cc}}{u_{cc} + \beta\int v_{KK}^j dF(\theta', \varepsilon')}((1-\delta) + f'(K)\theta)\right]$$

$$+ \beta(1-\delta)\int v_{KK}^j dF(\theta', \varepsilon') \cdot \frac{u_{cc}}{u_{cc} + \beta\int v_{KK}^j dF(\theta', \varepsilon')}((1-\delta) + f'(K)\theta)$$

or

$$v_{KK}^{j+1}(K, \theta, \varepsilon) = \frac{\beta\int v_{KK}^j dF(\theta', \varepsilon')}{u_{cc} + \beta\int v_{KK}^j dF(\theta', \varepsilon')} u_{cc} \cdot [(1-\delta) + f'(K)\theta]^2 + u_c f''(K)\theta.$$

It follows that off corners

(B5) $$v_{KK}^{j+1}(K, \theta, \varepsilon) \geq u_{cc} \cdot [(1-\delta) + f'(K)\theta]^2 + u_c f''(K)\theta.$$

"On corners," i.e., when $v_{KK}^{j+1}(K, \theta, \varepsilon)$ is attained where $b^j(K, \theta, \varepsilon) = (1-\delta)K$, we have

(B6) $$v_{KK}^{j+1}(K, \theta, \varepsilon) = u_{cc} \cdot [f'(K)\theta]^2 + u_c f''(K)\theta$$

$$+ \beta(1-\delta)^2 \int v_{KK}^j((1-\delta)K, \theta', \varepsilon') dF(\theta', \varepsilon').$$

Evidently, (B5) and (B6) imply that $\int v^j_{KK}(b^j(K,\theta,\varepsilon),\theta',\varepsilon')dF(\theta',\varepsilon)$ is uniformly (in j and K) bounded in absolute value on the compact interval $[K_e,K_u]$, where $K_u > K_e > 0$.

The boundedness of $\int v^j_{KK}dF(\theta',\varepsilon')$ together with (B2), (B3), and (B4) imply that off corners for K in the compact interval $[K_e,K_u]$, $K_u > K_e > 0$, the derivatives $\partial b^j/\partial K$, $\partial b^j/\partial\theta$, $\partial b^j/\partial\varepsilon$ remain uniformly strictly bounded away from zero in the directions given by (B2)-(B4).

The differentiability of $b^j(K,\theta,\varepsilon)$ does not necessarily carry-over to the pointwise limit function $b(K,\theta,\varepsilon)$. However, the restrictions that the derivatives in (B2)-(B4) impose on the finite differences of $b^j(K,\theta,\varepsilon)$ do carry over to $b(\cdot,\cdot,\cdot)$. In particular, we have that off corners

$$b(K_2,\theta,\varepsilon) - b(K_1,\theta,\varepsilon) \geq \alpha_1(K_2-K_1),\ \alpha_1 > 0$$

$$K_1,\ K_2\varepsilon[K_e,K_u]$$

$$b(K,\theta_2,\varepsilon) - b(K,\theta_1,\varepsilon) \leq \alpha_2(\theta_2-\theta_1)$$

$$\alpha_2 < 0$$

$$b(K,\theta,\varepsilon_2) - b(K,\theta,\varepsilon_1) \leq \alpha_3(\varepsilon_2-\varepsilon_1),\ \alpha_3 > 0.$$

We also need to evaluate

$$\frac{d}{d\varepsilon}u_c(f(K)\theta-I(K,\theta,\varepsilon),\varepsilon).$$

On corners, this total derivative equals $u_{c\varepsilon} > 0$. Proceeding formally, we have that off corners

$$\frac{d}{d\varepsilon}u_c(f(K)\theta-I(K,\theta,\varepsilon),\varepsilon) = -u_{cc}\frac{\partial I}{\partial\varepsilon} + u_{c\varepsilon}$$

$$= \frac{u_{c\varepsilon}\beta\int v_{KK}(y,\theta',\varepsilon')dF(\theta',\varepsilon')}{u_{cc}+\beta\int v_{KK}(y,\theta',\varepsilon')dF(\theta',\varepsilon')} > 0.$$

It is to be understood that $\partial I/\partial \varepsilon$ exists only almost everywhere. (To be rigorous, we should derive the inequality in terms of the j^{th} iterates on the policy and value function, and then proceed to the limit as above.)

Appendix C

The Configuration of Fixed Points

This appendix is meant to be read alongside a copy of Mirman [16] or Brock and Mirman [6]. Consider the nonstochastic difference equations

$$K_{t+1} = b(K_t, \theta_a, \varepsilon_a) \equiv I(K_t, \theta_a, \varepsilon_a) + (1-\delta)K_t$$

$$K_{t+1} = b(K_t, \theta_b, \varepsilon_b) \equiv I(K_t, \theta_b, \varepsilon_b) + (1-\delta)K_t.$$

Let K_a be a stationary point of the first difference equation. Let K_b be a stationary point of the second difference equation. Brock and Mirman say that the model has a "stable configuration" of fixed points if it is true that $K_b > K_a$. This means that the $b(K, \theta, \varepsilon)$ functions at worst "look like" those in Figure C1.

We first prove two lemmas that imply that the model has a "stable" configuration of fixed points. The lemmas are the counterparts of important results of Brock and Mirman [6], modified as necessitated by the presence of the "corner" in our problem. The proof of lemma 1 essentially is identical with Brock and Mirman's proof, but the presence of "corners" means that their proof of lemma 2 cannot be used here.

Lemma 1: At a fixed point $K_a = b(K_a, \theta_a, \varepsilon_a)$ the following inequality is satisfied:

$$\text{(C1)} \qquad 1 - \beta(1-\delta) < \beta f'(K_a)E(\theta').$$

Our proof parallels Brock and Mirman, since the "corner" in our problem turns out never to be a consideration at $(K_a, \theta, \varepsilon)$.

Proof: Let K_a satisfy $K_a = b(K_a, \theta_a, \varepsilon_a) \equiv (1-\delta)K_a + I(K_a, \theta_a, \varepsilon_a)$. The first-order necessary condition at K_a is

$$u_c(f(K_a)\theta_a - \delta K_a, \varepsilon_a) = \beta\int v_K(K_a, \theta', \varepsilon')dF(\theta', \varepsilon').$$

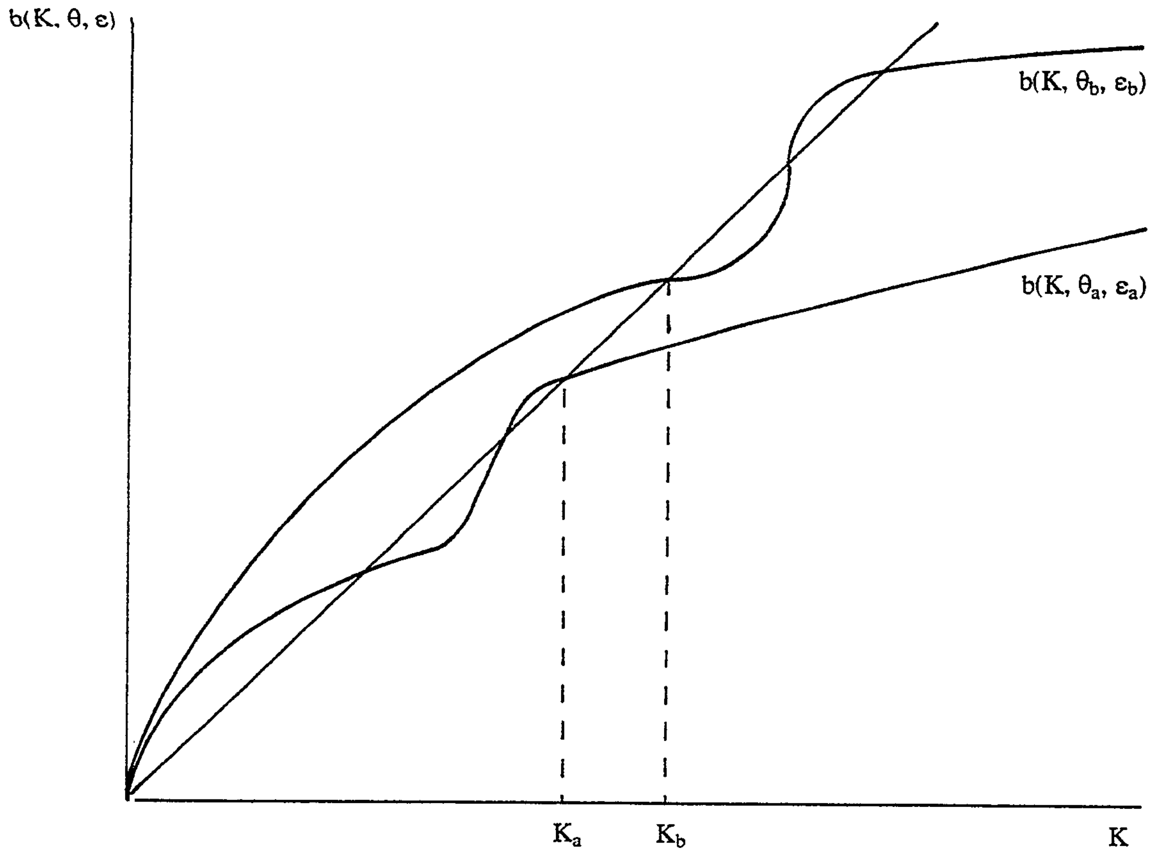


Figure C1

Stable Configuration of Fixed Points

But since $b(K, \theta, \varepsilon)$ is nondecreasing in θ and nonincreasing in ε, it follows that $I(K_a, \theta', \varepsilon') \geq \delta K_a > 0$ for all (θ', ε'). That is, starting from K_a, we will always have an interior solution for I for all values of (θ', ε'). Therefore, at K_a (18) must hold with equality for $j = \infty$, so that we have

$$u_c(f(K_a)\theta_a - \delta K_a, \varepsilon_a)$$

$$= \beta \int u_c(f(K_a)\theta' - I(K_a, \theta', \varepsilon'), \varepsilon')[f'(K_a)\theta' + (1-\delta)]\, dF(\theta', \varepsilon').$$

Since $u_c(f(K_a)\theta' - I(K_a, \theta', \varepsilon'), \varepsilon')$ is increasing in ε' and decreasing in θ' (see Appendix B for the argument with respect to ε'), it follows that

$$u_c(f(K_a)\theta_a - \delta K_a, \varepsilon_a)$$

$$< \beta \int u_c(f(K_a)\theta_a - \delta K_a, \varepsilon_a)[f'(K_a)\theta' + (1-\delta)]\, dF(\theta', \varepsilon').$$

This implies the inequality

$$1 < \beta \int [f'(K_a)\theta' + (1-\delta)]\, dF(\theta', \varepsilon')$$

or

$$\text{(C1)} \qquad 1 - \beta(1-\delta) < \beta f'(K_a)E(\theta').$$

<u>Lemma 2</u>: At a fixed point $K_b = b(K_b, \theta_b, \varepsilon_b)$ the following inequality is satisfied:

$$\text{(C2)} \qquad 1 - \beta(1-\delta) > \beta f'(K_b)E(\theta').$$

<u>Proof</u>: Let K_b be a fixed point of $b(K_b, \theta_b, \varepsilon_b)$, i.e., K_b satisfies $K_b = b(K_b, \theta_b, \varepsilon_b)$. First, notice that (22) implies that

$$\int v_K(K_b, \theta', \varepsilon')dF(\theta', \varepsilon') = \int u_c[f(K_b)\theta' - I(K_b, \theta', \varepsilon'), \varepsilon'] f'(K_b)\theta' dF(\theta', \varepsilon')$$

$$+ \beta(1-\delta) \int\int v_K[(1-\delta)K_b + I(K_b, \theta', \varepsilon'), \theta'', \varepsilon'']\, dF(\theta'', \varepsilon'')dF(\theta', \varepsilon').$$

In the second term on the right, replacing (θ', ε') with $(\theta_b, \varepsilon_b)$ causes $I(K_b, \theta', \varepsilon')$ to increase to δK_b and causes v_K to fall, since v is concave in K, implying

$$\int v_K(K_b, \theta', \varepsilon')dF(\theta', \varepsilon') > \int u_c[f(K_b)\theta' - I(K_b, \theta', \varepsilon'), \varepsilon']f'(K_b)\theta' dF(\theta', \varepsilon')$$

$$+ \beta(1-\delta)\int\int v_K[K_b, \theta'', \varepsilon'']dF(\theta'', \varepsilon'')dF(\theta', \varepsilon')$$

or

$$\int v_K(K_b, \theta', \varepsilon')dF(\theta', \varepsilon') > \int u_c[f(K_b)\theta' - I(K_b, \theta', \varepsilon'), \varepsilon']f'(K_b)\theta' dF(\theta', \varepsilon')$$

$$+ \beta(1-\delta)\int v_K[K_b, \theta'', \varepsilon'']dF(\theta'', \varepsilon'')$$

or

$$\{1-\beta(1-\delta)\}\int v_K(K_b, \theta', \varepsilon')dF(\theta', \varepsilon')$$

$$> \int u_c[f(K_b)\theta' - I(K_b, \theta', \varepsilon'), \varepsilon']f'(K_b)\theta' dF(\theta', \varepsilon')$$

or

(C3) $$\int v_K(K_b, \theta', \varepsilon')dF(\theta', \varepsilon')$$

$$> \frac{1}{1-\beta(1-\delta)}\int u_c[f(K_b)\theta' - I(K_b, \theta', \varepsilon'), \varepsilon']f'(K_b)\theta' dF(\theta', \varepsilon').$$

Now at $(K_b, \theta_b, \varepsilon_b)$, the first-order necessary condition is

$$u_c[f(K_b)\theta_b - \delta K_b, \varepsilon_b] = \beta\int v_K(K_b, \varepsilon', \theta')dF(\theta', \varepsilon').$$

By virtue of inequality (C3), it follows that

$$u_c[f(K_b)\theta_b - \delta K_b, \varepsilon_b]$$

$$> \frac{\beta}{1-\beta(1-\delta)}\int u_c[f(K_b)\theta' - I(K_b, \theta', \varepsilon'), \varepsilon']f'(K_b)\theta' dF(\theta', \varepsilon').$$

Since u_c is increasing in ε and decreasing in θ (see Appendix B for the proof for ε), we have

$$u_c[f(K_b)\theta_b-\delta K_b,\varepsilon_b] > \frac{\beta}{1-\beta(1-\delta)}\int u_c[f(K_b)\theta_b-\delta K_b,\varepsilon_b]f'(K_b)\theta' dF(\theta',\varepsilon')$$

or

(C2) $\quad 1 - \beta(1-\delta) > \beta f'(K_b)E(\theta')$. q.e.d.

We can now prove the following:

Proposition C1: If K_b is a fixed point of $b(K_b,\theta_b,\varepsilon_b)$ and K_a is a fixed point of $b(K_a,\theta_a,\varepsilon_a)$, then $K_a < K_b$.

Proof: Suppose to the contrary that $K_a \geq K_b$. Then, since $f''(\cdot) < 0$, a contradition is implied by inequalities (C1) and (C2). This proves the proposition.

Proposition C1 implies that the maximal fixed point $\overline{K}_a$ of $b(K_a,\theta_a,\varepsilon_a)$ is strictly less than the minimal fixed point $\underline{K}_b$ of $b(K_b,\theta_b,\varepsilon_b)$. This fact in conjunction with the results in Mirman [16, Section 3] are sufficient to imply:

Proposition C2: The stochastic difference equation $K_{t+1} = b(K_t, \theta_t, \varepsilon_t)$ generates a Markov process for $\{K_t\}$ with stochastic kernel $P(K'|K) = \text{Prob}\{K_{t+1} \leq K' | K_t = K\}$ and with a unique stationary distribution $\psi(K) = \text{Prob}\{K_t \leq K\}$. The stationary distribution is approached by iterations on

$$\psi_{t+1}(K') = \int P(K'|K)d\psi_t(K), \qquad t=1,2,\ldots$$

where $\psi_{t+1}(K') = \text{Prob}(K_{t+1} \leq K')$, starting from any initial distribution $\psi_0(K)$ that assigns positive probability to positive capital stocks. Further, $\psi(K)$ assigns probability one to the interval $\overline{K}_a$, $\underline{K}_b$.

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Footnotes

1/ The curves $b(K,\theta_a,\varepsilon_a)$ and $b(K,\theta_b,\varepsilon_b)$ do not necessarily intersect the 45-degree line in the fashion depicted in Figure 1 (see Brock and Mirman [6]). Each curve can intersect the 45-degee line a number of times. But it can be proved (see Appendix C) that the configuration of fixed points is stable in the sense of Brock and Mirman, which implies that in the stochastic stationary state the model does settle down to an interval (K_a,K_b) like that depicted in Figure 1. So our Figure 1 in general is an accurate description of the behavior of the system on $[K_a,K_b]$, but is not totally general outside that range.

2/ For example, if $K_t = K'$ in Figure 1, then for any (θ, ε) pair such that $\theta < \theta'$ and $\varepsilon > \varepsilon'$, $b(K', \theta, \varepsilon) = (1-\delta)K$.

3/ Lucas uses a single representative consumer in exactly this way [11].

4/ The proof of this proposition exactly parallels Lucas's [11] analogous proposition and will be omitted.

5/ The proof parallels Lucas' [11]. Only the sum $i + k^d$ is determined as a continuous function of the state variables $k, \theta, \varepsilon, K$. This is because when $p_K(K,\theta,\varepsilon) = 1$, the agent is indifferent as to the breakdown of $i + k^d$ between i and k^d.

6/ The concavity of $J(\cdot)$ in k can be proved as in Lucas [11]. The differentiability of $J(\cdot)$ can be proved by following an argument analogous to the one used below in Section 3 to prove differentiability of $v(\ )$ with respect to K.

7/ The condition that $u_c(0,\varepsilon) = \infty$ rules out the possibility of corner solutions with $c = 0$.

8/ Calculated using the methods in Section 3 below.

9/ See Naylor and Sell [17].

10/ Propositions 1, 2, and 3 and their proofs mimic analogous propositions in Lucas [11] and Lucas and Prescott [13, 14]. For this reason, we only sketch the proofs.

11/ Again, corner solutions with $c = 0$ are ruled out by the assumed form of the utility function.

12/ From lectures in his Economics 337 class.