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Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods

by

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During the last few years there has been an increased demand for more efficient solution methods for nonlinear rational expectations models. The demand has come from economic researchers with diverse research goals and modelling strategies. Empirical researchers studying real business cycle models are finding serious limitations to the simple representative agent models with convenient, but unrealistic, functional forms for the utility functions and without the possibility of distortions or externalities. Researchers focusing on monetary models are finding it necessary to solve large nonlinear stochastic systems in order to apply rational expectations techniques to practical problems of monetary policy, including international monetary policy. Finance economists interested in dynamic "consumption-beta" models, or generalizations of the Lucas asset pricing model, are finding it necessary to go beyond simple analytical models in order to confront the theory with the data. More generally, it appears that more efficient solution procedures are necessary if macroeconomists and financial economists are to be able to apply recent theoretical advances to practical policy or other applied problems.

The purpose of this paper is to report on a comparison of several alternative solution techniques for nonlinear rational expectations models. All the techniques are currently under development and the comparison is preliminary. Most of the

techniques rely on high speed computer technology or will eventually need this technology when they are moved beyond simple test problems. The availability of such technology is one of the factors that makes the techniques promising for applied problems.

The comparison is based on the results of an ongoing study group[1] in which individual researchers are comparing and evaluating alternative solution techniques. As part of this comparison, researchers were asked to solve a common set of problems using different solution techniques. Five different solution techniques and one of the common problems are considered in this paper. The problem is a simple representative agent, optimal, stochastic growth model designed to describe the behavior of aggregate consumption and the capital stock. Though simple, the problem does not have an analytic solution except in special cases. Hence the solution results are of interest in their own right in addition to enabling a comparison of alternative methods. Current plans of the research group are to extend this comparison to several other solution techniques and several other common problems which reflect alternative modelling applications.

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[1] The group is called the Nonlinear Rational Expectations Modelling Group and is supported by the National Bureau of Economic Research. Participants in the group meetings thus far have included Lawrence Christiano, Darrell Duffie, Ray Fair, Joseph Gagnon, Lars Hansen, Beth Ingram, Kenneth Judd, Pamela Labadie, David Luenberger, Rodolfo Manuelli, Albert Marcet, Ellen McGrattan, David Runkle, John Rust, Thomas Sargent, Christopher Sims, Kenneth Singleton, John Taylor, and George Tauchen.

The first section of the paper describes the stochastic growth model, the second section briefly describes the solution methods, and the third presents the results of the different solution methods, the fourth discusses economic issues raised by the solutions. Suggestions for future research are discussed in the concluding section.

1. The Stochastic Growth Model.[2]

Let C_t be consumption and K_t be the capital stock. Agents are assumed to maximize

$$(1) \quad E \sum_{t=0}^{\infty} (1-\tau)^{-1} C^{(1-\tau)} \beta^t$$

subject to

$$(2) \quad C_t + K_t - K_{t-1} = \theta_t K_{t-1}^{\alpha}$$

and to the side condition that $K_t > 0$, all t. Note that equation (2) implies that there is no depreciation of the capital stock. A slightly more general formulation would have some depreciation in which a coefficient less than one would multiply the lagged value of the capital stock in equation (2). Agents at time t choose K_t and C_t. Agents are assumed to know the history of all variables dated t and earlier when they choose decision variables

[2] Chris Sims proposed this common problem as well as the other common problems that the Group is examining.

dated t.

The stochastic process for θ_t is given by

$$\ln \theta_t = \rho \ln \theta_{t-1} + \epsilon_t \tag{3}$$

where ϵ is a serially uncorrelated, normally distributed random variable with mean 0 and constant variance σ_ϵ.

For this problem, decision rules for consumption C_t and the capital stock K_t in any period t are given by the functions $f(K_{t-1}, \theta_t)$ and $g(K_{t-1}, \theta_t)$ of the capital stock in period t-1 and the random shock in period t. Exact solutions for f and g are not known for this problem. If the utility function is logarithmic (τ=1) and there is full depreciation rather than zero depreciation as in equation (2), then there is a simple closed form solution for (see, for example, Sargent (1987), p.122). Hence, for this problem the functions f and g must be evaluated numerically.

To compare the different solution methods, the functions C and K were evaluated for four cases of parameter values. These cases are defined as follows:

Case 1:	β = .95	τ = .5
Case 2:	β = .95	τ = 1.5
Case 3:	β = .98	τ = .5
Case 4:	β = .98	τ = 1.5

with α = .33, ρ = .95, and σ_ϵ = .1 for all cases. Note that

these values of the coefficient of relative risk aversion (τ) do not allow for too much curvature of the utility function. Note also that the technology shock has a very large variance indicating a high degree of uncertainty.

For each of these cases, the functions

$$K_1 = f(K_0, \theta_1)$$
$$C_1 = g(K_0, \theta_1)$$

were evaluated for several values of K_0 and θ_1. The grid of values for the tabulation of the function f and g were:

θ_1 =	.4	.7	1.0	1.3	1.6	
K_0 =	5	10	15	20	25	for Cases 1 and 2
	20	40	60	100	120	for Cases 3 and 4

Hence, with 4 cases and 2 functions (f and g) there are a total of eight 5x5 tables that describe the decision rules for each solution technique.

For the same 4 cases, stochastic simulations of the growth model were also performed. For these simulations, shocks on ϵ_t were drawn from a normal distribution with mean zero and standard deviation .1. From a single realization of a 100 period simulation, time series plots of C_t and K_t were generated and the summary statistics were tabulated. The reason for tabulating

these statistics in addition to the functional forms is that in many applications it is these statistics that are compared with the data. Errors of approximation in the functional forms f and g could cancel out or be insignificant in the calculations of these statistics. The following summary statistics from the simulations are reported in this paper:

(a) Contemporaneous covariance matrix of $(C_t, K_t)'$

(b) Univariate autoregressions of C_t and K_t; AR(1), AR(2), AR(3), and AR(4).

(c) Bivariate autoregressions of $(C_t, K_t)'$; report VAR(1), VAR(2), VAR(3), and VAR(4).

The results for the 5 different solution methods are reported in Section 3.

2. The Solution Methods.

The different solution methods along with the researcher who carried out the solutions and the simulations are listed below:

Method	Researcher
(1) Simple Grid	Lawrence Christiano
(2) Quadrature Grid	George Tauchen
(3) Linear-Quadratic	Ellen McGrattan Lawrence Christiano
(4) Backsolving	Beth Ingram
(5) Extended Path	Joseph Gagnon

(In the revised draft of this paper the results from other researchers will be added. In particular, calculations made by Pamela Labadie, Albert Marcet, and Christopher Sims which were discussed but not yet tabulated for the study group are not included in this draft. Sims' results are based on the backsolving method, but are implemented slightly differently than Ingram's solutions. See Labadie (1986) and Marcet (1988) for a discussion of the other methods. Several new participants in the group are also planning to provide solutions.)

Details of how these methods work can be found in the references to this paper. A very brief summary is presented here.

(1) **Simple Grid.** The basic idea here is to approximate the continuous distribution of θ (or ln θ) by a discrete distribution over a grid, and by assuming that the values of K also lie on a grid. By making the grid finer, the actual solution for K and C can be approximated arbitrarily closely. These approximations result in a discrete state space dynamic optimization model which can be solved using standard numerical iteration methods. The finer the grid is, the more expensive will be the computation for this method. Higher dimensions for the control variable increase computation time greatly, but for the test problem there is only one dimension and computing time is not a problem. For more details on this method and how it was employed to solve the optimization problem in equation (1), see Christiano (1988).

(2) **Quadrature Grid.** This method also discretizes the state space, but it is more efficient than the simple grid in that a quadrature rule is used to discretize the state space. Tauchen has applied this method successfully in several problems. See Tauchen (1986) for a description of the method and for a discussion of some applications.

(3) **Linear-Quadratic.** This method approximates the control problem in equation (1) with a standard linear quadratic control problem to which linear decision rules for K and C are optimal and can be computed easily by iterating on the matrix Riccati equations. The linear decision rules are then treated as approximations to the exact solutions. The approximation is made by first substituting the constraint (2) into the objective function (1) and then making a quadratic approximation of the utility function at each time period. The approximation is taken about the steady state values of the problem. This method was used by Kydland and Prescott (1972). Its application to the problem considered in this paper is described by Christiano (1988).

In preparing calculations for the linear quadratic method reported this paper, Christiano did two alternative approximations, one in which ln(K) was treated as a control variable and another in which K was treated as a control variable. The two solutions are referred to as logLQ and linLQ respectively. It turns out that the results are very similar for the two methods. McGrattan's linear quadratic results are based

on treating K as the control variable and therefore are referred to as the linLQ method in this paper.

(4) **Backsolving.** This method was proposed by Christopher Sims and is described in Sims (1984). The idea is to assume convenient distributions and stochastic processes for the control variables rather than for the exogenous inputs. In other words, it is frequently easier to work backwards from assumptions about the control variables (say K and C) to properties of the exogenous variables. The common control problem described above is not ideally suited for the backsolving method because it specifies a specific distribution for the exogenous shock θ. The backsolving method would be more appropriate for an empirical application where the distribution for a shock like θ need not be assumed at the start, but rather could come out of the backsolving routine. In applying this method and calculating the time series statistics for consumption and capital, Ingram, therefore, assumed a distribution for consumption which does not necessarily evolve from the log normal distribution for θ described above. For this reason, the tables reported in the next section on the statistical properties for consumption and capital for the backsolving method are not strictly comparable with the other methods.

(5) **Extended Path.** This method is described in Fair and Taylor (1982). When applied to the optimal control problems like the one in equation (1), it works by solving the nonlinear dynamic first order conditions that are implied from the discrete

time calculus of variations formulation of the problem. (See Gagnon and Taylor (1986).) These first order conditions at time t involve conditional expectations of K_{t+1}. These future expectations are solved out iteratively in order to solve the first order conditions, thereby obtaining the solution for K_t. In order to avoid the problem of taking expectations through nonlinear functions, it is, in general, necessary to perform stochastic simulations of these simulations. In the results with this method reported here by Gagnon, only deterministic iterations were performed.

3. A Comparison of the Results.

Eight sets of solution results were reported for the five different algorithms. Three sets of solution results were reported for the linear-quadratic method: the linLQ by Christiano, the linLQ by McGrattan, and the logLQ by Christiano. One set of solutions was provided for each of the other methods by the researchers mentioned above. The information provided for each of these solutions sets is described in the following table:

Method	Stochastic Sim. Plots	Decision Rules	Summary Stats.
(1) Simple Grid	No	No*	No
(2) Quadrature	Yes	Yes	Yes
(3a) linLQ - Christiano	Yes	Yes	Yes
(3b) linLQ - McGrattan	Yes	Yes	Yes
(3c) logLQ - Christiano	Yes	Yes	Yes
(4) Backsolving	No	No	Yes
(5) Extended Path	Yes	Yes	No

*Christiano's results for a simple grid were only used to judge

the accuracy of the linear quadratic method. Actual values of the decision rules for the simple grid are not available at the time of this writing.

In the results reported at the end of the paper, the figures showing the stochastic simulation plots and the tables for the decision rules and the summary statistics are labeled by the numbers 1, 2, 3a, 3b, 3c, 4, and 5 as in the above summary. Note that some figure numbers and table numbers are missing because of the missing results shown in the above table.

Time Series Plots of the Stochastic Simulation

Figures 2, 3a, 3b, 3c and 5 show the realizations of a single stochastic simulation. Note that each chart represents a different set of draws so that the actual realizations will be much different for each method. Only the general patterns should appear similar for the different methods. Note that, except for Figure 2 (Tauchen's quadrature method), the scales for consumption and capital are different. (Different levels, but not different units, of the scales are implicit in Gagnon's extended path plots which take the mean out.) On an absolute basis, the level of consumption is, of course, much less than the level of the capital stock. The fluctuations in consumption are also smaller than the fluctuations in the capital stock.

All the methods show a high degree of contemporaneous correlation between consumption and the level of capital. All the methods show that most of the variance in both consumption

and capital is in the low frequencies. Consumption seems to have a surprisingly large low frequency component which is not evident in the capital stock in these charts. Note the close similarity between the linLQ and the logLQ where the same set of shocks is drawn.

Decision Rules

Decision rules for consumption and capital, for the same methods presented in the figures, are shown in Tables 2-D, 3a-D, 3b-D, 3c-D, and 5-D. Note first that the results for the two independent calculations of the linLQ are identical. This is, of course, not surprising, but provides a useful check on the results. The logLQ results are somewhat different from the linLQ results when examined through the decision rules.

The linear quadratic solutions are surprisingly similar to the quadrature grid solutions given the apparent large nonlinearities in this problem. The values for the quadrature method reported in the table are interpolated from the grid values that automatically emerge from the method, so that there is some question about the accuracy of these numbers as estimates of the exact solution. Nevertheless, given the small computation time for the linear quadratic approximations, these preliminary results are very promising for the linLQ or the logLQ method.

One puzzle about both the linear quadratic method and the quadrature methods, is that the response of consumption to the technology shock is surprisingly non-monotonic: over some

regions, lower values of the technology shock actually increase consumption, over other regions lower values more plausibly decrease consumption. This result may reflect the inaccuracies of the interpolation method used in the quadrature method, and needs to be examined further.

There is also a broad similarity between the results for the extended path method and the quadrature method. Given the relatively low cost of the extended path method, these results are promising, especially for application in higher dimension problems or in problems that are mixtures of optimization equations and other equations. Note that the extended path method does not have the non-monotonicity property mentioned above. The decision rule for consumption shows that consumption is a positive function of the technology shock over the entire region of initial capital stocks and technology shocks.

Summary Statistics from Stochastic Simulation

The covariance matrix and various autoregressions for consumption and the capital stock are given in Tables 2-S, 3a-S, 3b-S, 3c-S, and 4-S for the quadrature method, the linear quadratic methods, and the backsolving method. Because the decision functions are identical for the two linLQ methods (Christiano and McGrattan), the use of a different series of shocks gives some measure of the "Monte Carlo" error in estimating these moments from a single stochastic simulation. Although the stochastic simulation period is fairly long (100

periods), the simulations were not necessarily started from the same initial conditions, and, given the importance of very low frequencies, the simulation period is probably not long enough to have much confidence in the law of large numbers.

Looking first at the contemporaneous covariance matrix between consumption and capital, it is clear that part of the difference between the estimates is due to the Monte Carlo error in the simulation of 100 draws. A comparison of the two linear quadratic (linLQ) methods shows a large difference between the contemporaneous covariance matrices. The estimates of the variances and covariances are different by more than a factor of two in some cases. The autoregression and the vector autoregressions have even larger discrepancies for the two linLQ solutions. Recall that these two solutions have exactly the same functional forms for the values shown in the tables and since they are linear, the decision rules are identical. Nevertheless, the fact that different shocks are drawn and the simulation is started from different initial conditions leads to differences in the summary time series statistics.

Given the probable size of this Monte Carlo error, it is not possible to conclude much from the differences between the summary statistics for the other methods. There are differences between the quadrature method and the LQ methods, but these are not much larger in magnitude than the differences between the two linLQ solutions. On the other hand, there is a difference of an order of magnitude between the elements of the variance-

covariance matrix in the backsolving method and the other methods. This difference is not likely to be due to Monte Carlo error. More likely it is due to the implicit assumption that the technology shock follows a different stochastic process than that used in the other methods.

4. Economic Issues Raised by the Results.

Despite the Monte Carlo error, all the autoregression results reveal a high degree of serial dependence for consumption and capital and a high degree of correlation between consumption and capital. These properties were also evident from the time series charts for the quadrature, linear quadratic, and extended path methods. The summary statistics show that the same properties emerge for the backsolving method.

One puzzle in the results is that consumption appears to be smoother for lower values of the coefficient of relative risk aversion. This effect is most clear for the linear quadratic methods and the quadrature methods, but is difficult to detect in the extended path and backsolving methods. If income were exogenous, then lower values of risk aversion should lead to more volatile consumption so the result is either due to an interaction with the endogenous production function or to errors in the solution methods.

The decision rules for the linear quadratic method have a higher coefficient on the technology shock when the coefficient of relative risk aversion is lower. This is, of course,

consistent with the difference in consumption volatility noted in the time series charts. It is important to determine whether this result is due to an error in the approximation methods. Another, perhaps related, economic puzzle was noted above: over some regions consumption increases when the technology shock goes down (for fixed values of the initial capital stock). This result, which is a property of the linear quadratic and the quadrature methods, may also be due to an error of approximation. In the case of the quadrature method, it may simply be due to the need to interpolate to get the values of the decision variables for the specified value of the arguments of the decision functions.

5. Conclusion.

The conclusions from this preliminary comparison of different solution techniques for nonlinear rational expectations models can be summarized briefly as follows.

(1) The linear quadratic methods and the extended path methods give decision rules that are very close to the "exact" solution as represented here by the quadrature grid method. Given the relatively low computation times for these two methods and their relatively easy generalization to higher dimensions, it is important to establish whether this property holds up in other problems, or whether there are hidden simplifications in this problem that are favorable for the linear quadratic and the extended path methods.

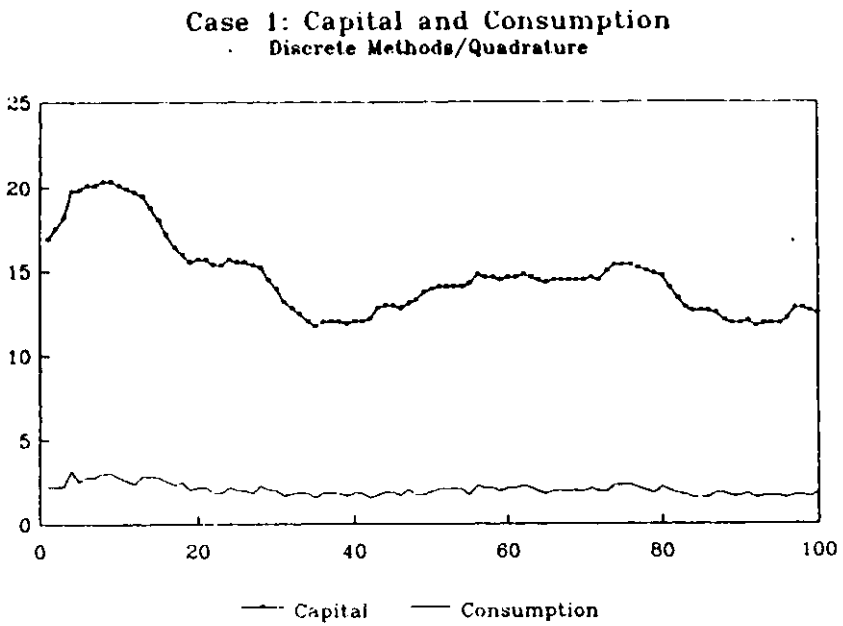
(2) Certain features of the decision rules calculated by some of the methods--for example, the property that consumption is not monotonic in the technology shock or the property that consumption is more volatile for higher values of risk aversion for the linear quadratic method or for the quadrature method--indicate that there may still be some economically significant approximation errors in the solution algorithms. Some of these errors are probably easy to avoid in most applications (such as the need to interpolate the results in order to evaluate the decision rules of different values), but further study is necessary to determine their cause or whether, in fact, they are errors.

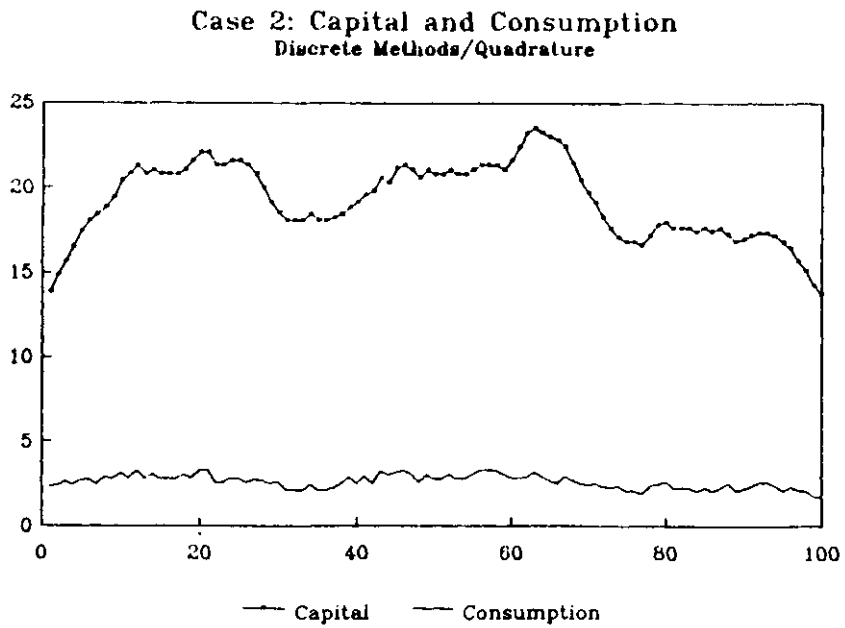
(3) In order to compare the properties of different estimates of parameters that emerge from the different solution techniques, it is necessary to use very large samples or many Monte Carlo simulations. The statistical results of this preliminary study are not accurate enough to draw inferences about the effective difference between the estimators based on the different methods.

The most important objective of future research should be to examine the properties of these and other solution methods on more difficult and more nonlinear problems. Solving the same problems for larger values of the coefficient of relative risk aversion would be a first step. It is also important to consider the problem of estimating the parameter values (for example, the value of ρ in this model) from actual data. It is for such

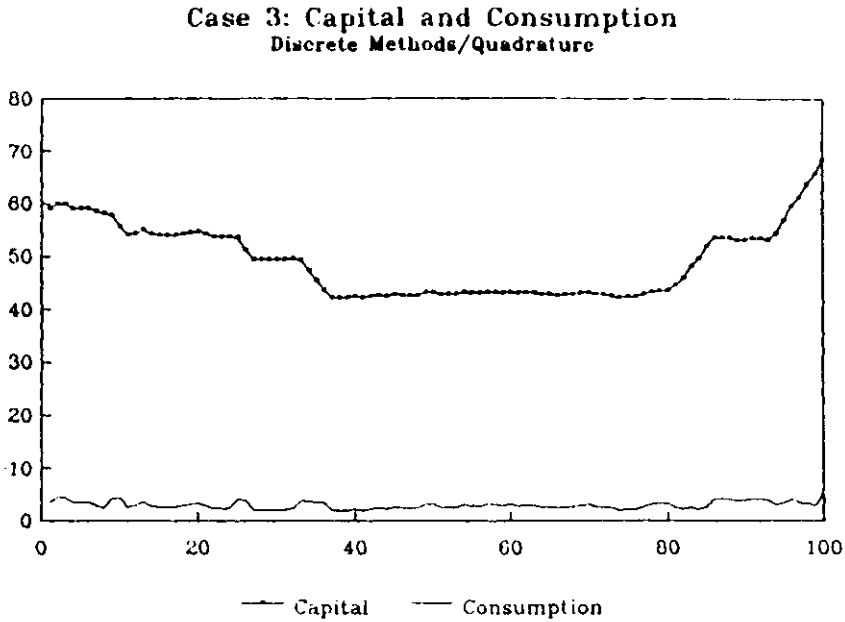
problems that some methods, such as the backsolving method, are designed. The results of this study indicate that to compare the properties of different estimators will probably require large Monte Carlo sample sizes.

Figure 2.


Case 1: Capital and Consumption
Discrete Methods/Quadrature
25
20
15
10
5
0
0
20
40
60
80
100
Capital
Consumption


Case 2: Capital and Consumption
Discrete Methods/Quadrature
25
20
15
10
5
0
0
20
40
60
80
100
Capital
Consumption

Quadrature Grid



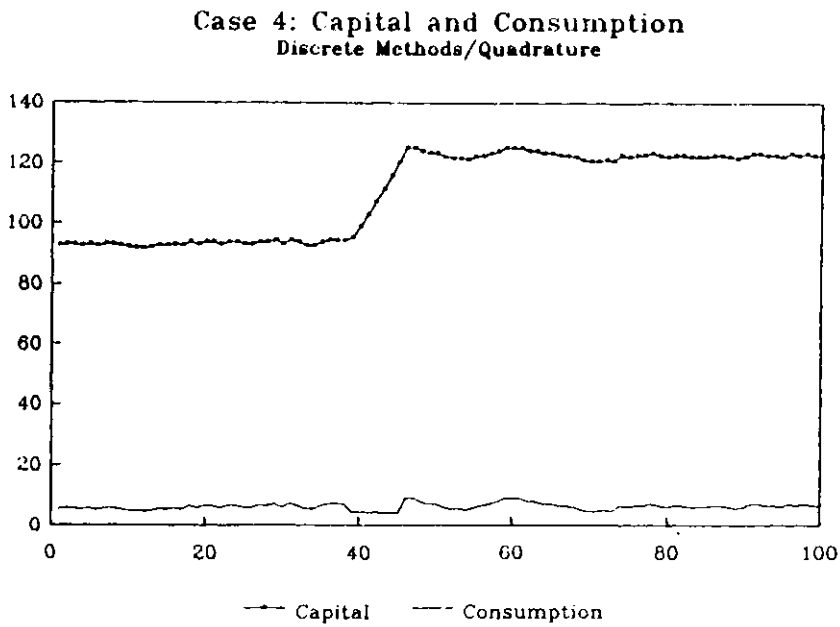
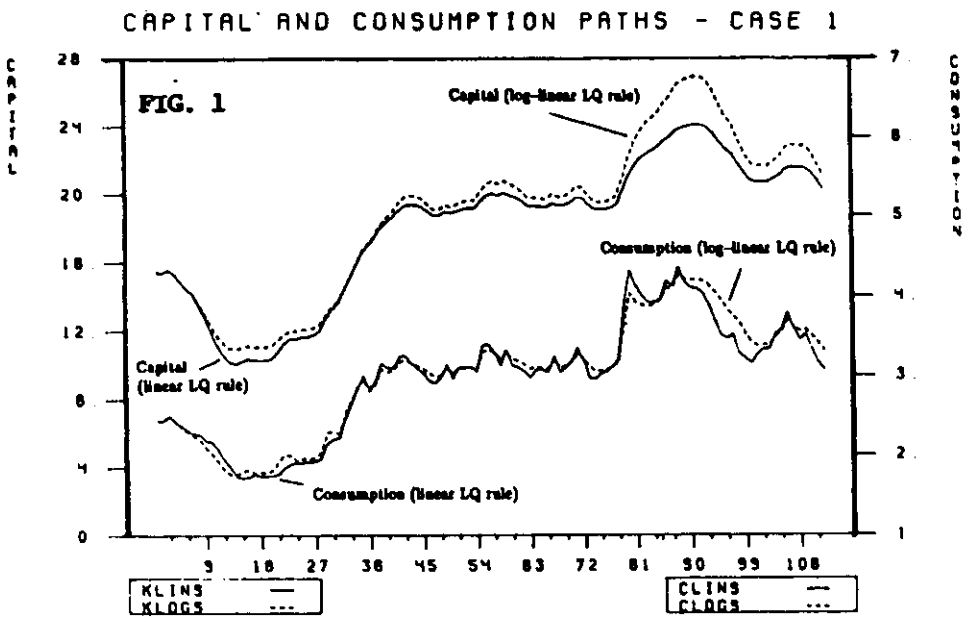
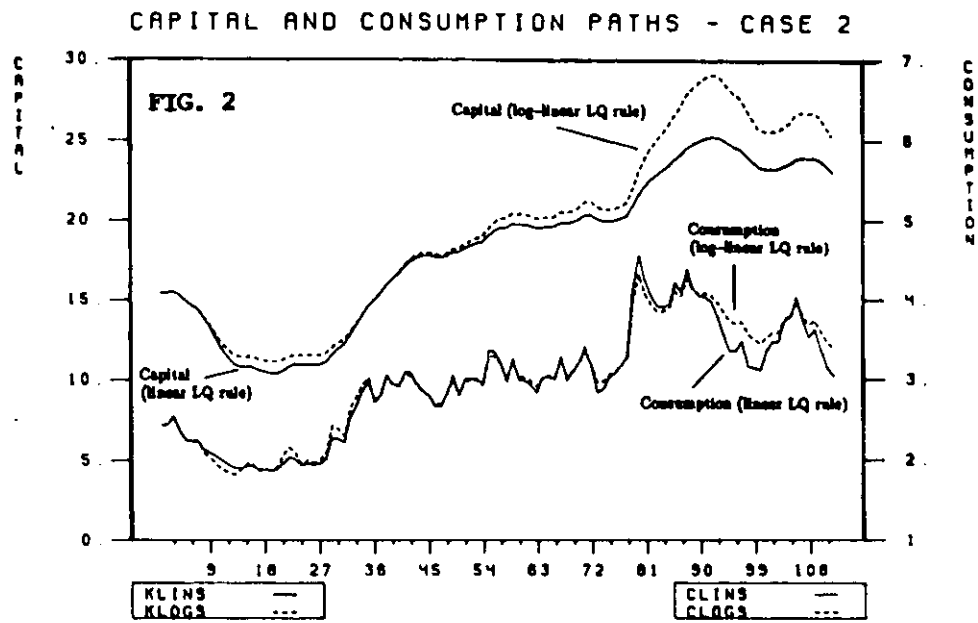
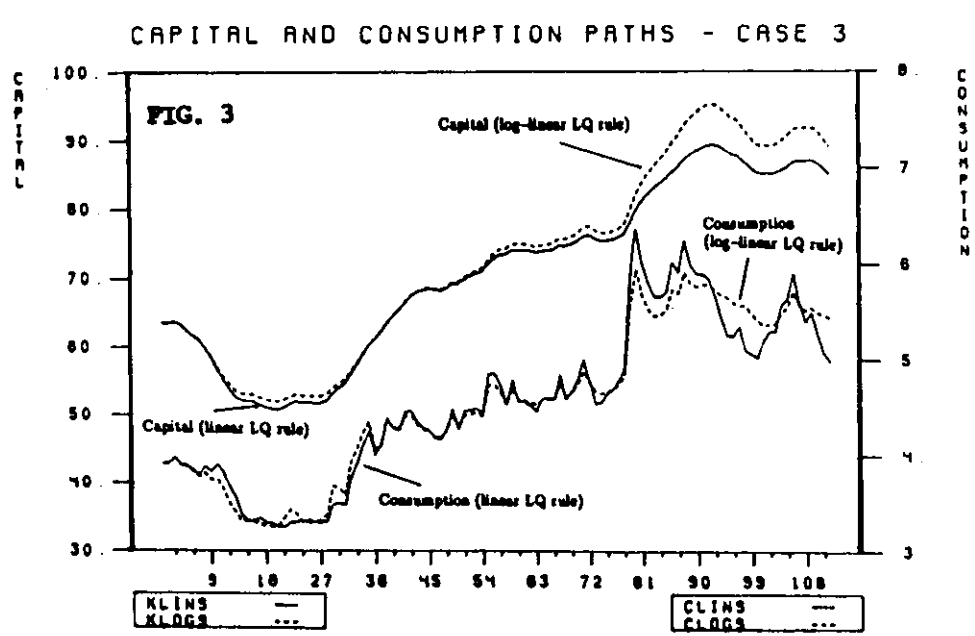


Figure 3a. (linLQ - Christiano) and Figure 3c. (logLQ)







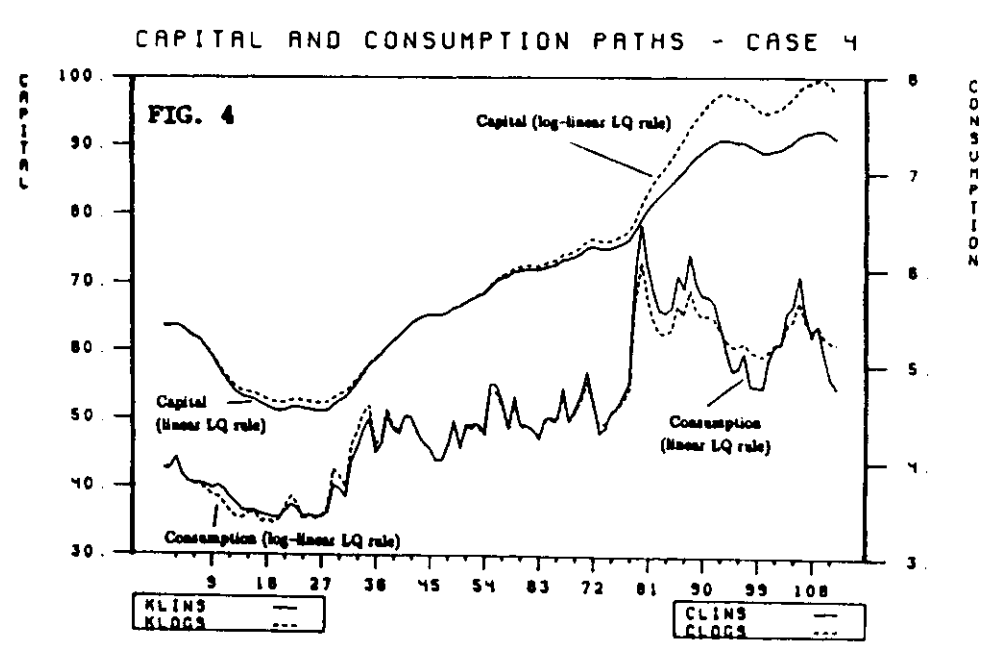
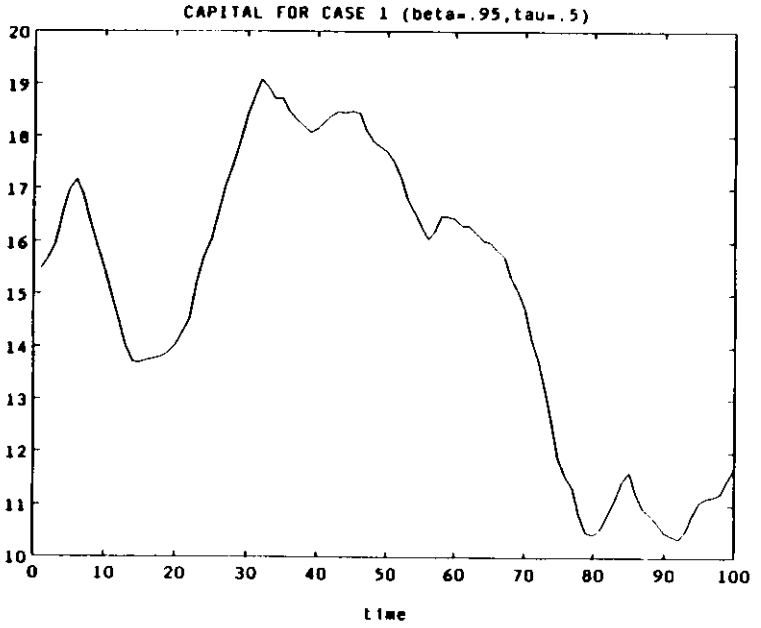
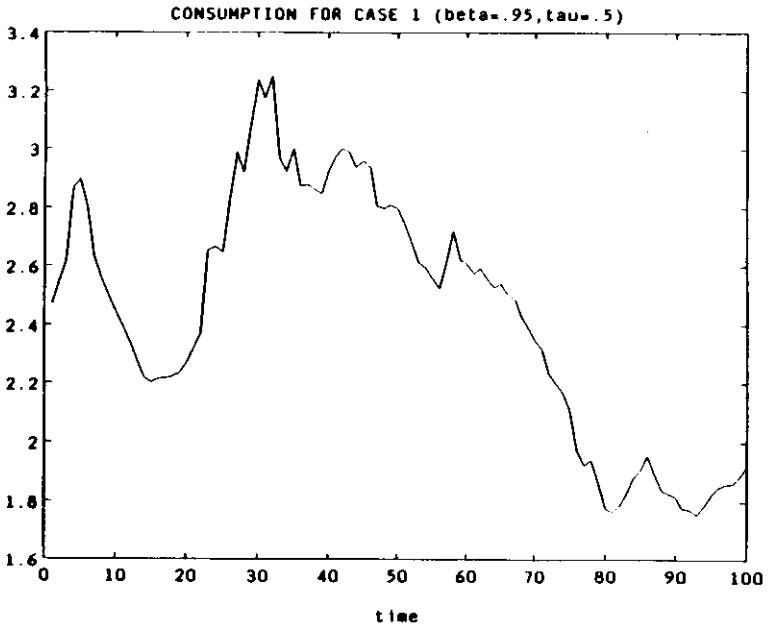
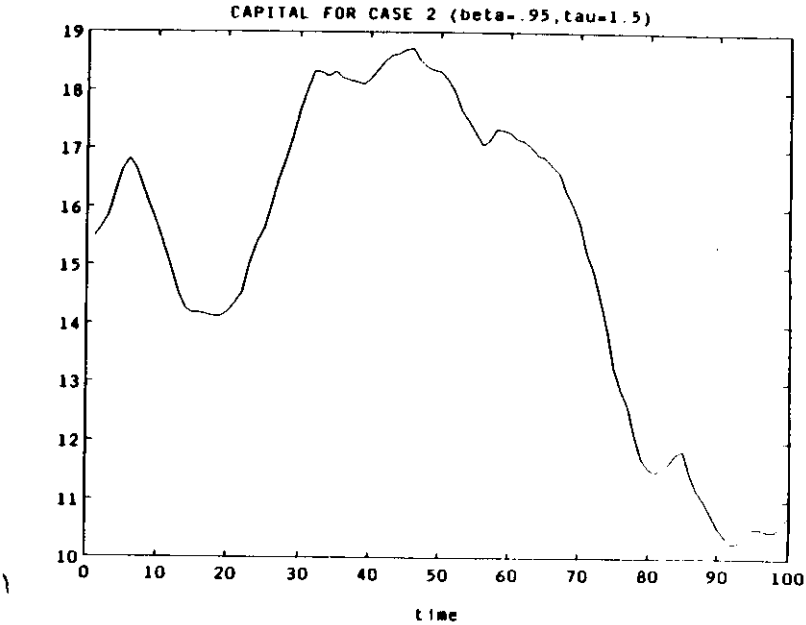


Figure 3b.


CAPITAL FOR CASE 1 (beta=.95,tau=.5)
20
19
18
17
16
15
14
13
12
11
10
0
10
20
30
40
50
60
70
80
90
100
time


CONSUMPTION FOR CASE 1 (beta=.95,tau=.5)
3.4
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
0
10
20
30
40
50
60
70
80
90
100
time

linLQ - McGrattan



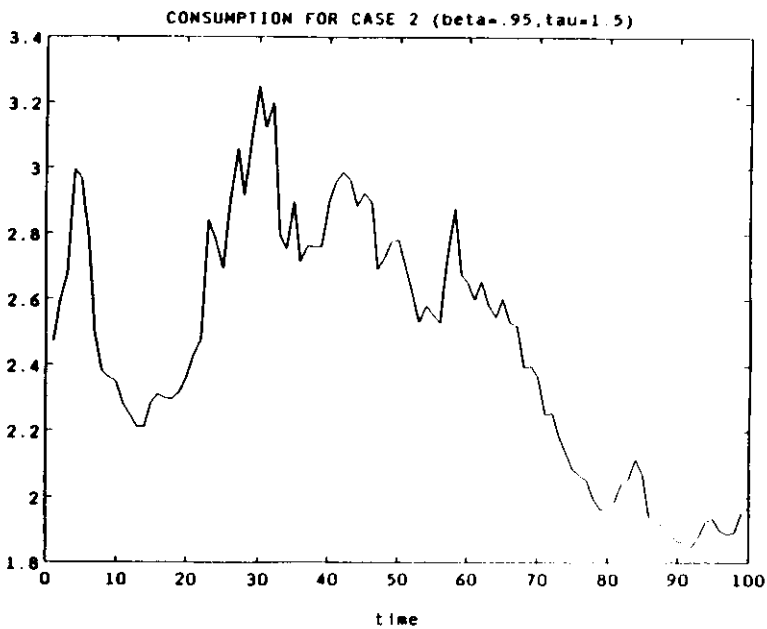
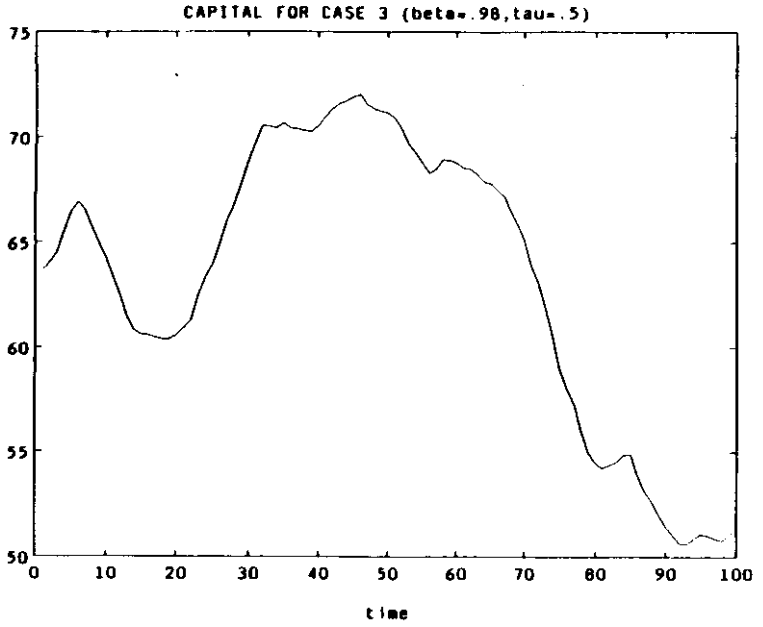
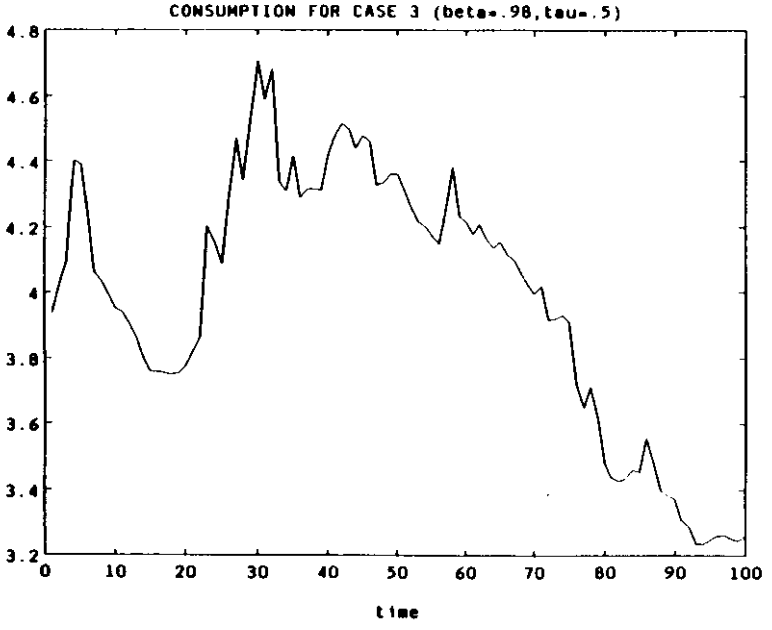
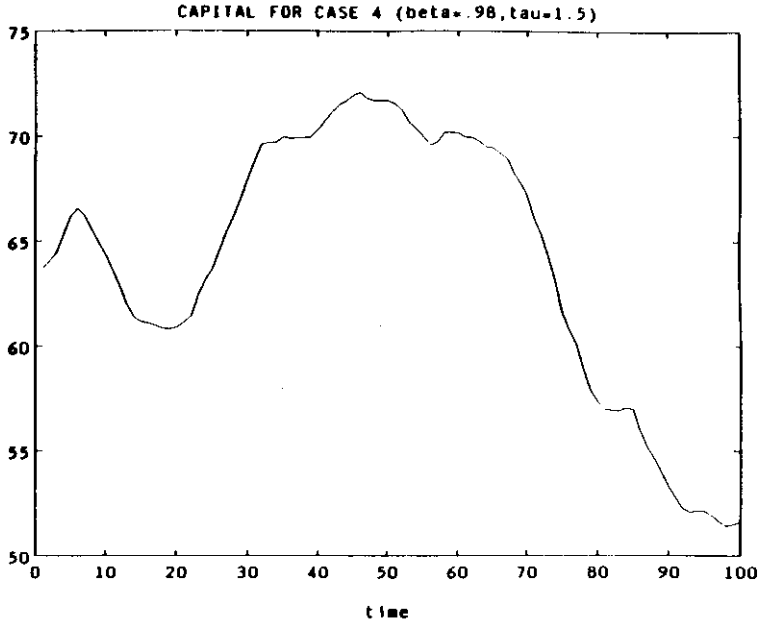


Figure 3b. linLQ -





McGrattan (continued)



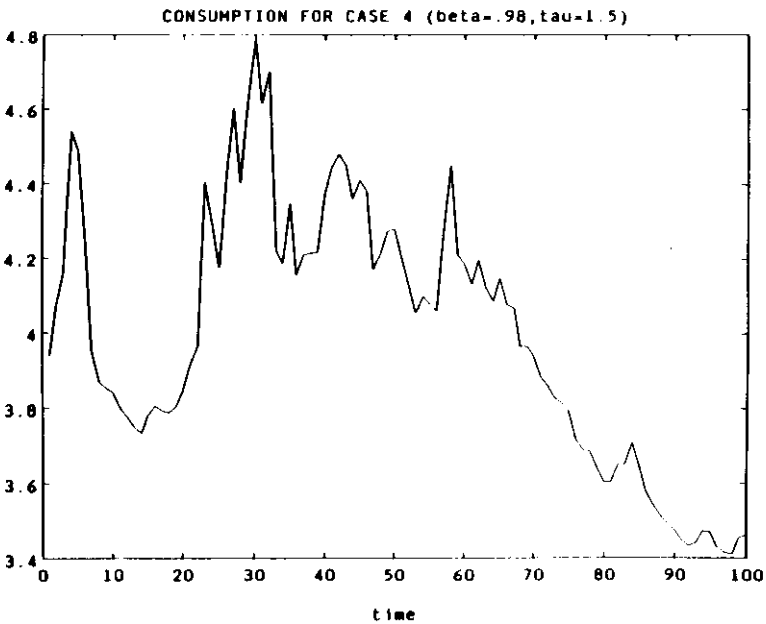
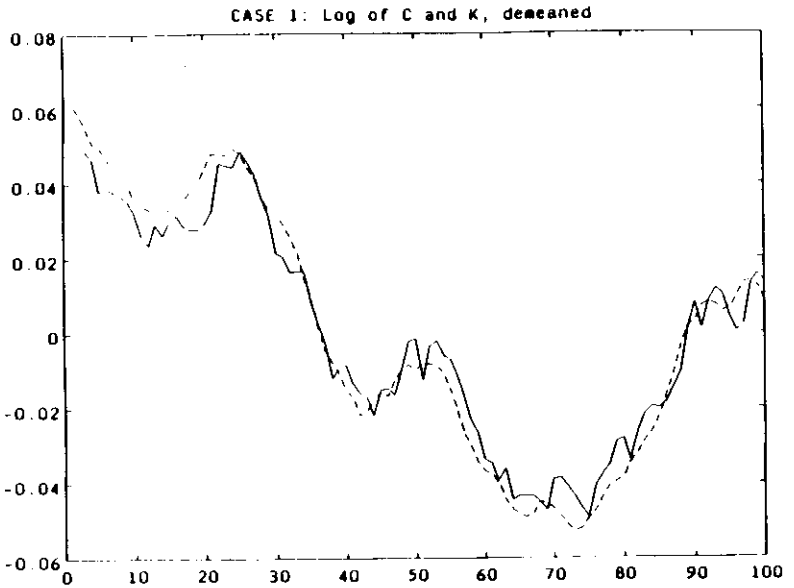
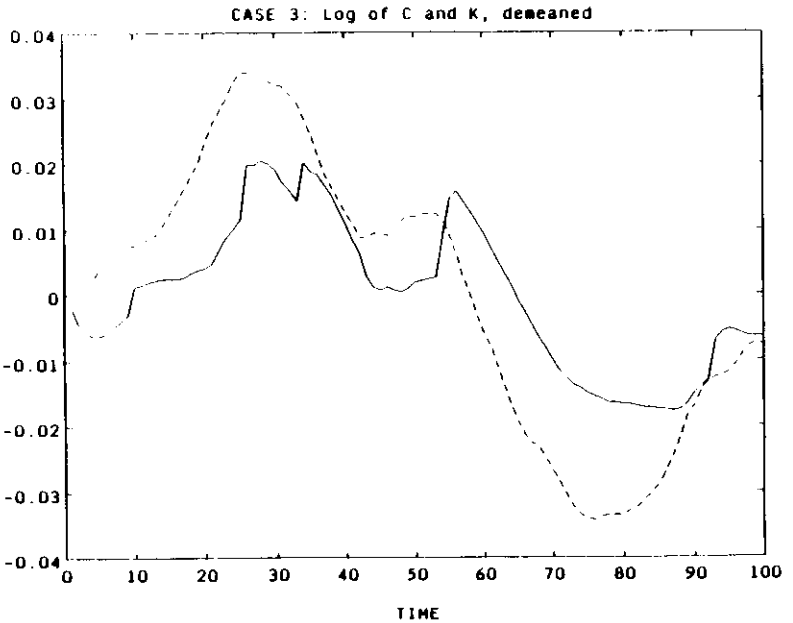
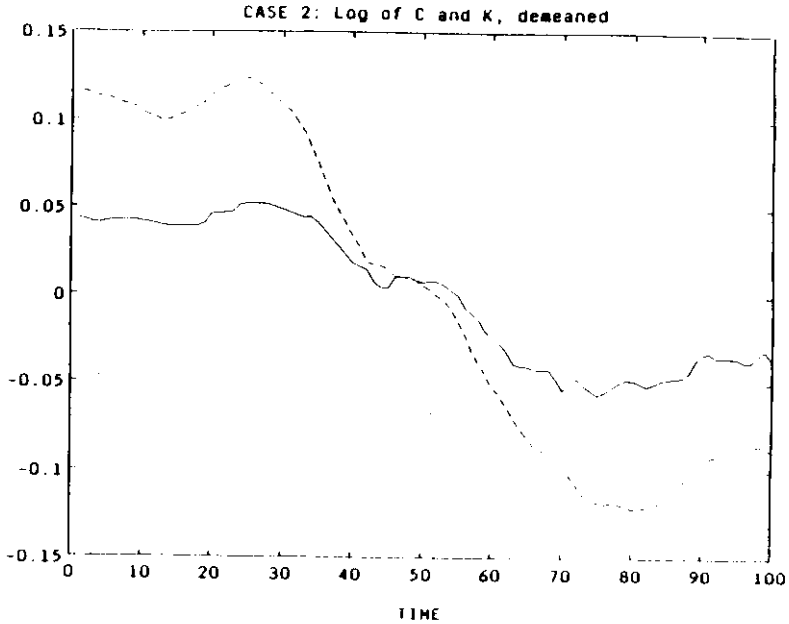


Figure 5.


CASE 1: Log of C and K, demeaned
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0
10
20
30
40
50
60
70
80
90
100


CASE 3: Log of C and K, demeaned
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
0
10
20
30
40
50
60
70
80
90
100
TIME

Extended Path



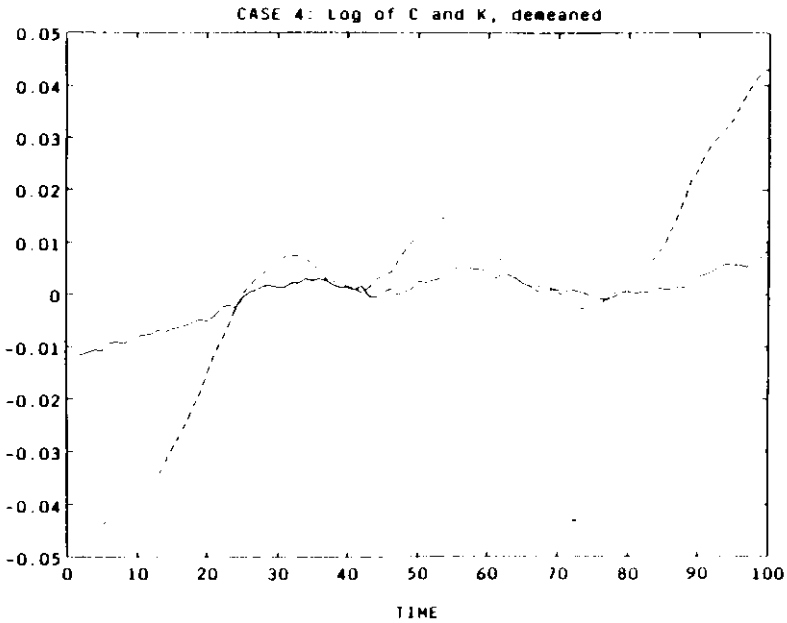


Table 2-D. Decision Rules for Quadrature Grid

Capital Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	3.96	5.18	5.72	6.11	6.52
10	10.38	10.05	10.42	10.93	11.45
15	13.50	14.40	15.00	15.73	16.30
20	20.92	19.20	19.45	20.04	20.88
25	23.04	23.80	24.00	25.00	25.37

Consumption Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	1.72	1.01	0.98	1.10	1.20
10	0.47	1.45	1.72	1.85	1.97
15	2.47	2.31	2.44	2.44	2.61
20	0.15	2.68	3.24	3.45	3.42
25	3.12	3.23	3.89	3.76	4.26

Capital Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	3.80	4.98	5.43	5.68	6.03
10	9.60	9.92	10.13	10.47	10.92
15	14.40	14.36	15.00	15.69	15.86
20	19.20	19.20	20.00	20.21	20.97
25	24.00	24.00	25.00	25.00	26.03

Consumption Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	1.88	1.21	1.27	1.54	1.69
10	1.26	1.58	2.01	2.31	2.50
15	1.58	2.35	2.44	2.48	3.06
20	1.87	2.68	2.69	3.28	3.33
25	2.16	3.02	2.89	3.76	3.60

Table 2-D. (continued)

Capital Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	20.00	20.50	20.94	21.70	22.75
40	30.46	40.00	40.21	41.67	42.31
60	57.60	59.50	60.00	60.56	62.94
100	76.80	76.80	80.00	80.00	81.19
120	96.00	96.00	100.11	100.00	100.00

Consumption Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	1.07	1.38	1.75	1.79	1.55
40	10.89	2.36	3.17	2.73	3.09
60	3.94	3.20	3.86	4.46	3.24
100	4.90	6.17	4.25	5.52	5.60
120	5.83	7.20	4.46	5.94	7.31

Capital Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	20.00	20.00	20.75	21.02	21.85
40	35.19	40.00	40.00	41.84	41.67
60	62.64	60.18	60.00	60.00	62.92
100	76.80	79.33	80.00	80.00	81.26
120	96.00	95.60	100.00	100.00	100.00

Consumption Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	1.07	1.88	1.93	2.48	2.45
40	6.17	2.36	3.38	2.55	3.74
60	1.10	2.52	3.86	5.02	3.26
100	4.90	3.64	4.25	5.52	5.54
120	5.83	7.60	4.57	5.94	7.31

Table 3a-D. Decision Rules for Linear Quadratic–Christiano

Capital Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.07	5.13	5.80	6.30	6.69
10	8.69	9.75	10.42	10.91	11.31
15	13.31	14.36	15.04	15.53	15.92
20	17.93	18.98	19.65	20.15	20.54
25	22.54	23.60	24.27	24.77	25.16

Consumption Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	1.61	1.06	0.90	0.91	1.03
10	2.16	1.75	1.72	1.86	2.11
15	2.67	2.35	2.41	2.64	2.99
20	3.15	2.90	3.03	3.34	3.76
25	3.61	3.43	3.62	3.99	4.47

Capital Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.11	4.90	5.40	5.76	6.06
10	8.92	9.71	10.21	10.58	10.87
15	13.74	14.52	15.02	15.39	15.68
20	18.55	19.33	19.83	20.20	20.49
25	23.36	24.14	24.64	25.01	25.30

Consumption Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	1.57	1.29	1.30	1.45	1.67
10	1.93	1.79	1.93	2.20	2.55
15	2.24	2.19	2.43	2.79	3.23
20	2.53	2.55	2.86	3.30	3.81
25	2.80	2.88	3.25	3.75	4.33

Table 3a-D. (continued)

Capital Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	18.28	20.16	21.36	22.24	22.94
40	37.66	39.54	40.74	41.62	42.32
60	57.04	58.92	60.11	61.00	61.69
100	95.79	97.67	98.87	99.75	100.45
120	115.17	117.05	118.25	119.13	119.80

Consumption Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	2.79	1.72	1.33	1.25	1.36
40	3.69	2.82	2.64	2.77	3.09
60	4.51	3.79	3.75	4.02	4.49
100	6.04	5.53	5.70	6.19	6.87
120	6.77	6.35	6.61	7.18	7.94

Capital Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	18.04	19.65	20.67	21.42	22.01
40	37.74	39.34	40.36	41.11	41.71
60	57.43	59.04	60.06	60.81	61.40
100	96.82	98.42	99.45	100.20	100.79
120	116.52	118.12	119.14	119.89	120.49

Consumption Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	3.03	2.24	2.02	2.08	2.29
40	3.61	3.02	3.02	3.28	3.70
60	4.11	3.67	3.81	4.21	4.78
100	5.01	4.77	5.12	5.75	6.52
120	5.42	5.28	5.71	6.42	7.28

Table 3b-D. Decision Rules for Linear Quadratic–McGrattan

Capital Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.07	5.13	5.80	6.30	6.69
10	8.69	9.75	10.42	10.91	11.31
15	13.31	14.36	15.04	15.53	15.92
20	17.93	18.98	19.65	20.15	20.54
25	22.54	23.60	24.27	24.77	25.15

Consumption Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	1.61	1.06	0.90	0.91	1.03
10	2.16	1.75	1.72	1.86	2.11
15	2.67	2.35	2.41	2.64	2.99
20	3.15	2.90	3.03	3.34	3.76
25	3.61	3.43	3.62	3.99	4.47

Capital Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.11	4.90	5.40	5.76	6.06
10	8.92	9.71	10.21	10.58	10.87
15	13.74	14.52	15.02	15.39	15.68
20	18.54	19.33	19.83	20.20	20.49
25	23.36	24.14	24.64	25.01	25.30

Consumption Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	1.57	1.29	1.30	1.45	1.67
10	1.93	1.70	1.93	2.20	2.55
15	2.24	2.19	2.43	2.79	3.22
20	2.53	2.55	2.86	3.30	3.81
25	2.80	2.88	3.25	3.75	4.33

Table 3b-D. (continued)

Capital Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	18.28	20.16	21.36	22.24	22.94
40	37.66	39.54	40.74	41.62	42.32
60	57.04	58.92	60.11	61.00	61.69
100	95.79	97.67	98.87	99.75	100.45
120	115.17	117.05	118.25	119.13	119.82

Consumption Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	2.79	1.72	1.33	1.25	1.36
40	3.69	2.82	2.64	2.77	3.09
60	4.51	3.79	3.75	4.02	4.49
100	6.04	5.53	5.70	6.19	6.87
120	6.77	6.35	6.61	7.18	7.94

Capital Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	18.04	19.65	20.67	21.42	22.01
40	37.74	39.34	40.36	41.11	41.71
60	57.43	59.04	60.06	60.81	61.40
100	96.82	98.43	99.45	100.20	100.79
120	116.52	118.12	119.14	119.89	120.49

Consumption Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	3.03	2.24	2.02	2.08	2.29
40	3.61	3.02	3.02	3.28	3.70
60	4.11	3.67	3.81	4.21	4.78
100	5.01	4.77	5.12	5.75	6.52
120	5.42	5.28	5.71	6.42	7.28

Table 3c-D. Decision Rules for Log-Linear Quadratic

Capital Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.88	5.22	5.45	5.63	5.77
10	9.25	9.90	10.34	10.68	10.95
15	13.45	14.40	15.04	15.53	15.92
20	17.54	18.78	19.61	20.25	20.77
25	21.55	23.08	24.10	24.88	25.52

Consumption Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	0.80	0.97	1.25	1.58	1.95
10	1.61	1.60	1.80	2.10	2.47
15	2.53	2.31	2.41	2.65	2.99
20	3.53	3.10	3.07	3.24	3.53
25	4.60	3.95	3.79	3.88	4.11

Capital Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.80	5.05	5.22	5.34	5.45
10	9.36	9.84	10.17	10.41	10.61
15	13.82	14.54	15.02	15.38	15.67
20	18.23	19.18	19.81	20.28	20.67
25	22.60	23.77	24.55	25.14	25.62

Consumption Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	0.88	1.14	1.48	1.87	2.28
10	1.50	1.65	1.97	2.37	2.81
15	2.15	2.17	2.43	2.80	3.24
20	2.84	2.70	2.88	3.21	3.63
25	3.56	3.25	3.34	3.62	4.01

Table 3c-D. (continued)

Capital Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	19.76	20.35	20.73	21.02	21.25
40	38.67	39.83	40.58	41.15	41.60
60	57.28	58.99	60.11	60.95	61.62
100	93.95	96.77	98.60	99.98	101.08
120	112.11	115.46	117.66	119.29	120.61

Consumption Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	1.32	1.53	1.95	2.47	3.04
40	2.68	2.54	2.79	3.24	3.80
60	4.27	3.71	3.75	4.07	4.56
100	7.87	6.43	5.97	5.96	6.23
120	9.84	7.93	7.20	7.02	7.16

Capital Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	19.54	20.03	20.36	20.60	20.79
40	38.66	39.64	40.28	40.76	41.14
60	57.63	59.10	60.05	60.77	61.34
100	95.31	97.73	99.31	100.49	101.43
120	114.05	116.96	118.85	120.26	121.38

Consumption Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	1.54	1.85	2.33	2.90	3.51
40	2.69	2.72	3.09	3.63	4.26
60	3.91	3.60	3.81	4.25	4.84
100	6.52	5.47	5.26	5.45	5.88
120	7.89	6.44	6.01	6.05	6.38

Table 5-D. Decision Rules for Extended Path

Capital Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.98	5.31	5.68	6.08	6.48
10	9.55	9.95	10.40	10.88	11.37
15	14.09	14.53	15.04	15.57	16.12
20	18.61	19.07	19.62	20.20	20.81
25	23.11	23.61	24.19	24.81	25.45

Consumption Values for Case 1 with $\beta = 0.95$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	0.71	0.88	1.02	1.14	1.25
10	1.30	1.55	1.74	1.90	2.05
15	1.89	2.18	2.41	2.61	2.79
20	2.46	2.81	3.07	3.29	3.49
25	3.05	3.42	3.70	3.97	4.18

Capital Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	4.98	5.16	5.40	5.66	5.92
10	9.72	9.94	10.23	10.56	10.89
15	14.43	14.69	15.02	15.38	15.76
20	19.12	19.40	19.77	20.17	20.60
25	23.81	24.12	24.51	24.95	25.40

Consumption Values for Case 2 with $\beta = 0.95$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
5	0.70	1.03	1.30	1.56	1.80
10	1.14	1.55	1.90	2.22	2.53
15	1.55	2.02	2.43	2.80	3.15
20	1.96	2.48	2.92	3.32	3.71
25	2.35	2.90	3.38	3.82	4.23

Table 5-D. (continued)

Capital Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	20.96	21.16	21.57	22.08	22.65
40	40.01	40.38	40.93	41.59	42.30
60	58.91	59.40	60.07	60.81	61.62
100	96.55	97.21	98.03	98.94	99.91
120	115.33	116.06	116.95	117.92	118.95

Consumption Values for Case 3 with $\beta = 0.98$ and $\tau = 0.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	0.11	0.72	1.20	1.42	1.65
40	1.34	1.99	2.45	2.81	3.11
60	2.63	3.30	3.80	4.21	4.56
100	5.28	5.99	6.54	7.00	7.40
120	6.61	7.34	7.91	8.39	8.81

Capital Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	20.97	21.10	21.37	21.72	22.11
40	40.11	40.38	40.78	41.25	41.77
60	59.18	59.54	60.03	60.57	61.18
100	97.22	97.70	98.32	99.01	99.76
120	116.23	116.76	117.43	118.17	118.97

Consumption Values for Case 4 with $\beta = 0.98$ and $\tau = 1.5$

	Values of θ_1				
k_0	0.4	0.7	1.0	1.3	1.6
20	0.11	0.78	1.32	1.78	2.20
40	1.24	1.98	2.60	3.14	3.64
60	2.36	3.16	3.83	4.45	5.00
100	4.61	5.50	6.25	6.93	7.56
120	5.72	6.64	7.43	8.14	8.80

Table 2-S. Summary Statistics for Quadrature Grid

Covariance Matrices of C_t, K_t, Cases 1-4

0.78	5.52
5.52	41.79

0.79	6.83
6.83	66.95

1.84	20.50
20.50	361.16

1.68	20.75
20.75	520.90

Case 1: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.93			
0.63	0.33		
0.57	0.23	0.17	
0.57	0.21	0.13	0.05

Case 1: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
0.99			
1.71	-0.72		
1.59	-0.43	-0.16	
1.57	-0.46	-0.06	-0.06

Case 2: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.93			
0.67	0.29		
0.63	0.19	0.15	
0.61	0.17	0.10	0.09

Case 2: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.00			
1.68	-0.68		
1.54	-0.34	-0.20	
1.54	-0.37	-0.09	-0.07

Table 2-S. (continued)

Case 3: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.79			
0.58	0.27		
0.54	0.18	0.15	
0.53	0.16	0.10	0.10

Case 3: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.00			
1.65	-0.65		
1.54	-0.38	-0.16	
1.54	-0.40	-0.07	-0.06

Case 4: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.82			
0.66	0.20		
0.63	0.10	0.14	
0.62	0.10	0.10	0.08

Case 4: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.00			
1.61	-0.61		
1.52	-0.39	-0.14	
1.51	-0.42	-0.02	-0.07

Table 2-S. (continued)

Case 1: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t								
K_t								
C_t								
K_t								
C_t								
K_t								
C_t	0.03	0.19	0.00	0.01	0.30	-0.19	0.02	-0.01
K_t	-1.05	0.99	0.02	0.00	1.98	-0.98	0.01	-0.02

Case 2: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t								
K_t								
C_t								
K_t								
C_t								
K_t								
C_t	0.21	0.35	-0.03	0.00	0.36	-0.35	0.04	0.00
K_t	-1.00	1.00	-0.03	0.00	1.99	-1.01	0.02	0.00

Case 3: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t								
K_t								
C_t								
K_t								
C_t								
K_t								
C_t	0.33	0.26	0.01	0.00	0.21	-0.20	0.02	0.00
K_t	-1.01	0.94	0.01	0.00	1.95	-0.94	-0.02	0.01

Case 4: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t								
K_t								
C_t								
K_t								
C_t								
K_t								
C_t	0.46	0.27	0.05	-0.01	0.24	-0.23	-0.02	0.02
K_t	-1.02	0.96	-0.02	0.02	1.95	-0.97	0.05	-0.02

Table 3a-S. Summary Statistics for Linear Quadratic–Christiano

Covariance Matrices of C_t, K_t, Cases 1-4

0.48	2.72	0.47	2.95	0.69	9.81	0.59	9.27
2.72	16.50	2.95	22.30	9.81	156.02	9.27	187.60

Case 1: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.98			
1.18	-0.20		
1.18	-0.20	-0.01	

Case 2: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.96			
1.05	-0.09		
1.06	-0.15	0.06	

Case 3: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.98			
1.10	-0.13		
1.11	-0.19	0.06	

Case 4: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.96			
1.04	-0.09		
1.05	-0.17	0.08	

Table 3a-S. (continued)

Case 1: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.96				0.00			
K_t	0.91				0.85			
C_t	0.93	-0.17			0.15	-0.12		
K_t	0.05	-0.06			1.88	-0.87		
C_t	0.98	-0.24	0.02		0.11	-0.03	-0.05	
K_t	-0.11	-0.06	0.09		1.95	-1.00	0.06	
C_t	0.96	-0.18	0.00	0.03	0.13	-0.10	0.05	-0.03
K_t	-0.16	0.01	0.09	-0.06	1.20	-1.11	0.15	-0.02

Case 2: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.94				0.00			
K_t	0.72				0.90			
C_t	0.88	-0.13			0.20	-0.17		
K_t	-0.03	-0.05			1.94	-0.94		
C_t	0.98	-0.26	0.03		0.07	0.10	-0.14	
K_t	-0.06	-0.04	0.04		2.00	-1.05	0.06	
C_t	0.94	-0.19	0.00	-0.04	0.12	-0.06	0.05	-0.07
K_t	-0.09	0.00	0.04	-0.03	2.05	-1.17	0.16	-0.03

Case 3: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.90				0.01			
K_t	1.27				0.92			
C_t	0.95	-0.18			0.07	-0.05		
K_t	-0.04	-0.10			1.92	-0.91		
C_t	1.00	-0.24	0.01		0.03	0.02	-0.04	
K_t	-0.10	-0.11	0.11		1.97	-1.02	-0.05	
C_t	0.98	-0.20	0.00	-0.03	0.04	-0.02	0.02	-0.02
K_t	-0.15	-0.05	0.13	-0.07	2.01	-1.11	0.13	-0.02

Case 4: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.90				0.00			
K_t	1.18				0.95			
C_t	0.91	-0.15			0.09	-0.08		
K_t	-0.03	-0.08			1.95	-0.95		
C_t	1.00	-0.27	0.03		0.01	0.08	-0.08	
K_t	-0.06	-0.09	0.08		1.99	-1.04	0.05	
C_t	0.97	-0.22	0.02	-0.03	0.03	0.01	0.00	-0.03
K_t	-0.11	-0.06	0.10	-0.05	2.04	-1.13	0.11	-0.02

Table 3b-S. Summary Statistics for Linear Quadratic–McGrattan

Covariance Matrices of C_t, K_t, Cases 1-4

0.80	1.15
1.15	7.72

0.14	0.94
0.94	7.48

0.16	2.64
2.64	47.15

0.12	1.86
1.86	40.45

Case 1: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.99			
1.23	-0.25		
1.22	-0.20	-0.04	
1.17	-0.14	0.34	-0.39

Case 1: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.00			
1.85	-0.86		
1.94	-1.07	0.12	
1.89	-1.13	0.41	-0.19

Case 2: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.97			
1.03	-0.06		
1.03	-0.16	0.10	
1.06	-0.14	0.35	-0.30

Case 2: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.01			
1.88	-0.89		
1.97	-1.09	0.11	
1.93	-1.26	0.42	-0.19

Table 3b-S. (continued)

Case 3: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.99			
1.01	-0.02		
1.02	-0.15	0.12	
1.04	-0.11	0.39	-0.33

Case 3: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.01			
1.89	-0.90		
1.98	-1.09	0.10	
1.93	-1.17	0.42	-0.19

Case 4: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.95			
0.95	0.00		
0.96	-0.14	0.15	
0.99	-0.11	0.36	-0.28

Case 4: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
1.01			
1.90	-0.91		
1.99	-1.09	0.10	
1.94	-1.17	0.42	-0.20

Table 3b-S. (continued)

Case 1: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.96				0.01			
K_t	1.47				0.78			
C_t	0.77	-0.09			1.18	-0.13		
K_t	0.04	-0.13			1.83	-0.83		
C_t	0.77	-0.16	0.09		0.17	-0.20	-0.01	
K_t	-0.07	0.10	-0.06		1.94	-1.08	0.14	
C_t	0.78	-0.18	0.27	-0.21	0.16	-0.12	0.00	-0.01
K_t	-0.12	0.24	-0.06	-0.18	1.95	-1.25	0.43	-0.13

Case 2: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.97				0.00			
K_t	1.15				0.86			
C_t	0.75	-0.09			0.25	-0.21		
K_t	0.05	-0.13			1.89	-0.89		
C_t	0.74	-0.13	0.08		0.26	-0.24	-0.01	
K_t	-0.06	-0.01	0.08		2.01	-1.15	0.14	
C_t	0.73	-0.07	0.17	-0.17	0.26	-0.33	0.18	-0.07
K_t	-0.10	0.21	-0.06	-0.16	2.04	-1.44	0.65	-0.24

Case 3: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.86				0.00			
K_t	2.21				0.89			
C_t	0.73	-0.07			0.08	-0.06		
K_t	0.05	-0.19			1.88	-0.88		
C_t	0.74	-0.16	0.12		0.08	-0.05	-0.01	
K_t	-0.07	0.09	-0.10		1.97	-1.09	0.12	
C_t	0.76	-0.19	0.28	-0.18	0.06	-0.04	-0.01	0.00
K_t	-0.11	0.25	-0.12	-0.21	1.98	-1.26	0.44	-0.16

Case 4: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.91				0.00			
K_t	2.07				0.92			
C_t	0.71	-0.08			0.12	-0.11		
K_t	0.07	-0.21			1.92	-0.92		
C_t	0.72	-0.14	0.09		0.12	-0.10		
K_t	-0.06	-0.07	0.09		2.00	-1.12	0.11	
C_t	0.73	-0.14	0.22	-0.16	0.11	-0.12	0.04	-0.18
K_t	-0.10	0.22	-0.08	-0.24	2.01	-1.34	0.56	-0.23

Table 3c-S. Summary Statistics for Log-Linear Quadratic

Covariance Matrices of C_t, K_t, Cases 1-4

0.49	3.14	0.48	3.67	0.66	11.10	0.52	10.40
3.14	20.60	3.67	31.70	11.10	195.40	10.40	250.20

Case 1: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.99			
1.31	-0.33		
1.31	-0.34	0.01	

Case 2: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.98			
1.06	-0.09		
1.07	-0.19	0.10	

Case 3: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.99			
1.10	-0.11		
1.12	-0.26	0.13	

Case 4: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
0.97			
1.02	-0.05		
1.03	-0.20	0.15	

Table 3c-S. (continued)

Case 1: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	1.17				-0.03			
K_t	2.75				0.58			
C_t	0.83	-0.23			0.18	-0.12		
K_t	0.18	-0.24			1.81	-0.81		
C_t	0.92	-0.43	0.16		0.14	-0.05	-0.05	
K_t	0.26	-0.47	0.35		1.80	-0.80	-0.03	
C_t	0.93	-0.52	0.30	-0.07	0.15	-0.01	-0.12	0.04
K_t	0.26	-0.62	0.52	0.01	1.81	-0.74	-0.18	0.07

Case 2: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.98				0.00			
K_t	1.20				0.86			
C_t	0.87	-0.11			0.16	-0.14		
K_t	0.11	-0.08			1.84	-0.85		
C_t	0.93	-0.25	0.11		0.11	-0.05	-0.05	
K_t	0.03	-0.01	0.10		1.93	-1.06	0.12	
C_t	0.98	-0.51	0.41	-0.03	0.08	0.20	-0.50	0.22
K_t	0.78	-0.32	0.43	-0.01	1.89	-0.77	-0.40	0.26

Case 3: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.91				0.00			
K_t	1.80				0.93			
C_t	0.80	-0.12			0.11	-0.10		
K_t	-0.01	-0.11			1.95	-0.94		
C_t	0.87	-0.28	0.10		0.07	-0.01	-0.05	
K_t	-0.11	-0.04	0.16		2.03	-1.16	0.12	
C_t	0.89	-0.37	0.22	-0.04	0.07	0.04	-0.15	-0.06
K_t	-0.08	-0.28	0.43	-0.02	2.02	-1.01	-0.18	0.16

Case 4: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	1.17				-0.03			
K_t	2.75				0.58			
C_t	0.83	-0.23			0.18	-0.12		
K_t	0.18	-0.24			1.81	-0.81		
C_t	0.92	-0.43	0.16		0.13	-0.05	-0.05	
K_t	0.26	-0.47	0.35		1.80	-0.80	-0.03	
C_t	0.93	-0.52	0.30	-0.07	0.15	-0.01	-0.12	0.04
K_t	0.26	-0.62	0.52	0.01	1.81	-0.74	-0.18	0.07

Table 4-S. Summary Statistics for Back-Solving

Covariance Matrices of C_t, K_t, Cases 1-4

17.90	23.00
23.00	55.61

0.03	0.25
0.25	55.61

19.90	179.30
179.30	2955.57

0.22	19.81
19.81	2955.57

Case 1: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
1.00			
0.99	0.00		
0.99	-0.07	0.08	
0.10	-0.07	-0.07	0.16

Case 1: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
0.99			
1.00	-0.01		
1.00	-0.17	0.16	
1.00	-0.17	0.14	0.02

Case 2: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
1.00			
1.02	-0.16		
1.01	-0.09	0.08	
1.00	-0.08	-0.07	0.15

Case 2: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
0.99			
1.00	-0.01		
1.00	-0.17	0.16	
1.00	-0.17	0.14	0.02

Table 4-S. (continued)

Case 3: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
1.00			
0.99	0.00		
0.99	-0.07	0.08	
0.97	-0.07	-0.07	0.16

Case 3: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
0.90			
0.69	0.22		
0.64	0.07	0.20	
0.62	0.07	0.08	0.15

Case 4: Univariate Autoregressions for Consumption, C_t

Lags			
C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}
1.00			
1.02	-0.02		
1.01	-0.09	0.08	
1.00	-0.08	-0.07	0.15

Case 4: Univariate Autoregressions for Capital, K_t

Lags			
K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
0.90			
0.69	0.22		
0.64	0.07	0.20	
0.62	0.07	0.08	0.15

Table 4-S. (continued)

Case 1: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	0.98				0.02			
K_t	0.06				0.91			
C_t	0.89	0.10			0.11	-0.10		
K_t	-0.15	0.22			1.03	-0.14		
C_t	0.89	-0.02	0.12		0.10	-0.04	-0.05	
K_t	-0.15	-0.16	0.37		1.01	-0.06	-0.03	
C_t	0.89	-0.04	-0.07	0.20	0.08	-0.03	0.03	-0.07
K_t	-0.15	-0.22	-0.11	0.54	0.95	-0.03	0.24	-0.27

Case 2: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	1.00				0.00			
K_t	0.68				0.90			
C_t	0.94	0.06			0.00	0.00		
K_t	-4.17	4.87			1.01	-0.11		
C_t	0.94	-0.04	0.11		0.00	0.00	0.00	
K_t	-3.65	-5.40	9.66		0.99	-0.06	-0.02	
C_t	0.94	-0.06	-0.07	0.19	0.00	0.00	0.00	0.00
K_t	-3.75	-6.62	-3.17	14.16	0.94	-0.04	0.23	-0.23

Case 3: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	1.05				-0.01			
K_t	1.39				0.58			
C_t	1.34	-0.30			-0.03	0.02		
K_t	8.72	-7.42			0.15	0.46		
C_t	1.35	-0.31	0.00		-0.03	0.14		
K_t	9.14	-4.78	-3.15		0.09	0.25	0.28	
C_t	1.33	-0.29	-0.24	0.23	-0.03	0.01	0.01	0.00
K_t	9.06	-4.55	-6.33	2.95	0.08	0.25	0.32	0.00

Case 4: Bivariate Autoregressions for C_t, K_t

	Independent Variables							
Dep.	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	K_{t-1}	K_{t-2}	K_{t-3}	K_{t-4}
C_t	1.01				0.00			
K_t	10.51				0.66			
C_t	1.27	-0.26			0.00	0.00		
K_t	204.50	-194.52			0.26	0.42		
C_t	1.28	-0.27	0.01		0.00	0.00	0.00	
K_t	216.51	-124.58	-82.56		0.20	0.22	0.27	
C_t	1.27	-0.26	-0.18	0.18	0.00	0.00	0.00	0.00
K_t	218.09	-119.37	-151.35	61.31	0.19	0.22	0.26	0.04

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