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NOTES ON MODELS OF ECONOMIC GROWTH

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Working paper 318

October 1986

Abstract

These notes are intended as a do-it-yourself course in economic growth along lines suggested by Lucas ("On the Mechanics of Economic Development"). We examine in turn the neoclassical growth model; theories of endogenous growth, including learning-by-doing, increasing returns to scale, and externalities; and dynamic comparative advantage in trade. Salient features of growing economies and microeconomic evidence on production processes are used to evaluate alternatives. Excercises supplement the text.

Keywords: neoclassical growth; technical change; learning-by-doing; returns to scale; dynamic comparative advantage.

JEL Classification Numbers: 111, 621, 411.

Backus is on leave from Queen's University and thanks the SSHRC of Canada for financial support.

Contents

1. THE NEOCLASSICAL GROWTH MODEL

1.1 Introduction to Neoclassical Growth

One of the most striking facts of economic life is the diversity of levels and growth rates of per capita income across countries. Our goal is to develop models capable of explaining this diversity. We choose to begin less ambitiously, however, with the neoclassical growth model, whose deficiencies we document below.

The one-sector neoclassical growth model is characterized by preferences

$$(1.1) \qquad \int_0^\infty e^{-\rho t} u(c_t) d_t, \; u' > 0, \; u'' < 0,$$

and technology

$$(1.2) \qquad y_t = c_t + \dot{k}_t = A_t f(k_t), \; f' > 0, \; f'' < 0.$$

On occasion we will also need the Inada conditions,

$$\lim_{c \to 0} u'(c) = \infty, \qquad \lim_{c \to \infty} u'(c) = 0,$$

$$\lim_{k \to 0} f'(k) = \infty, \qquad \lim_{k \to \infty} f'(k) = 0.$$

We have for simplicity ignored inputs other than capital, although one can think of f as depending also on a constant quantity of labor. The important elements for our purposes are the diminishing marginal returns to the reproducible input, capital, and the dynamic behavior of the productivity factor, A_t, which we have yet to specify. We will characterize a competitive equilibrium in this model by deriving an optimal plan for consumption. A method for doing so is outlined in the next section.

1.2 A Digression on Optimal Control

Consider the dynamic control problem

$$(2.1) \qquad \max_{\{u_t\}} \int_0^\infty e^{-\rho t} f(x_t, u_t)dt$$

subject to

$$(2.2) \qquad \dot{x}_t = g(x_t, u_t)$$

and given x_0. We refer to x as the **state** variable and u as the **control**; x_0 will be referred to as the **initial condition**.

We characterize the solution using variational methods the maximum principle). Define the **current-value Hamiltonian**

$$H(x_t, u_t, y_t) = f(x_t, u_t) + y_t\, g(x_t, u_t),$$

where y is the **costate** variable associated with x. Then necessary conditions for an interior optimum are

$$\partial H/\partial u_t = 0$$

$$(2.3) \qquad \dot{y}_t = \rho y_t - \partial H/\partial x_t$$

$$\dot{x}_t = g(x_t, u_t)$$

and the **transversality condition**

$$(2.4) \qquad \lim_{t\to\infty} e^{-\rho t} y_t x_t = 0.$$

If in addition $H^*(x,y) = \max_u H(x,u,y)$ is concave in x, given y, then any plan satisfying (2.3) and (2.4) is optimal.

Consider finally a useful special case. For some problems equations (2.3) have a unique stationary point (x^*,y^*) defined by $\dot{x} = \dot{y} = 0$. Given concavity of H^*, any solution to (2.3) that converges to this point must be optimal since the transversality condition is automatically satisfied.

1.3 Optimal Growth without Technical Change

We are now in a position to derive the empirical implications of the neoclassical growth model. The first step is to characterize the optimal plan using the maximum principle. Using the connection between Pareto optima and competitive equilibria in these models, we can interpret the result as an equilibrium in the competitive economy.

We proceed with the optimal plan. Equations (1.1) and (1.2) describe a control problem with k as the state variable and c as the control. The current-value Hamiltonian is

$$H(k_t, c_t, \lambda_t, t) = u(c_t) + \lambda_t \, [A_t f(k_t) - c_t],$$

where λ is the costate. In some sense A is also a state variable, but since its evolution through time is exogenous we indicate its time-dependence simply by introducing t into the list of arguments for H. Necessary conditions for an optimum include

(3.1a) $\quad u'(c_t) = \lambda_t$

(3.1b) $\quad \dot{\lambda}_t = [\rho - A_t f'(k_t)] \, \lambda_t$

(3.1c) $\quad \dot{k}_t = A_t f(k_t) - c_t.$

The solution of these equations, together with the transversality condition and the initial value k_0, completely determine the optimal plan.

The properties of the plan depend on the rate of technical progress. We consider first the case in which A is constant, which allows us to describe the qualitative features of the solution with the use of a two-dimensional diagram. Equations (3.1) define a dynamic system in λ and k with tendencies for motion illustrated in figure 1.1. The set of points for which $\dot{\lambda} = 0$ is a vertical line in (k,λ)-space defined by $Af'(k^*) = \rho$. (The Inada conditions guarantee that the solution $\lambda = 0$ is infeasible.) To the right of this line λ is increasing, to the left decreasing. Similarly, the condition $\dot{k} = 0$ defines a downward-sloping line, with k increasing to the right, decreasing to the left. These directions of motion are indicated by arrows in the diagram. The Inada conditions guarantee that the two curves, $\dot{\lambda} = 0$ and $\dot{k} = 0$, cross at nonnegative values k^* and y^*. These values clearly define a stationary point of the system, so any path satisfying the initial condition that converges to this point is a solution. Since the dynamic equations are continuous, we can postulate a unique curve, labeled SS in figure 1.1, in which the dynamics of λ and k combine to lead exactly to (k^*,λ^*). Any point on this curve therefore satisfies the necessary conditions for an optimum. By choosing the point on this curve where $k = k_0$, we satisfy the initial condition as well and obtain the unique solution to the control problem.

The model has a number of implications that can be compared with data from actual economies. In the steady state it implies:

1. Growth rates of output and the capital stock are zero. In particular, the growth rate does not depend on ρ, A, or any other parameters governing preferences or technology.
2. Higher values of A, and lower values of ρ, imply higher <u>levels</u> of y and k.

The point is that without technical change there is no sustained growth. Changes in parameters and government policies are merely <u>level effects</u>. Outside the steady state the model predicts:

3. Monotonic adjustment of capital and output to their steady state levels.

This involves temporary nonzero rates of growth, but not the sustained growth we have observed in the Western world over the last two hundred years.

Exercises to 1.3

1. For the neoclassical growth model verify that $H^*(k,\lambda,t)$ is concave in k and therefore that the solution to the first-order conditions is a maximum.
2. Starting from an initial capital stock below the long-run equilibrium path, describe the dynamic behavior of the real interest rate (marginal product of capital) and the share of investment in NNP ($\dot{k}/y$).
3. For the growth model with $u(c) = \log c$ and $f(k) = k^\alpha$, $0 < \alpha < 1$, find an analytic solution for the path of the capital stock when $A_t = 1$.
4. Consider the growth model with $u(c) = \log c$ and $y = f(k) = k^\alpha$, $\alpha > 1$. Derive the optimal path and compare it to the neoclassical growth model. Comment on any problems that might arise in supporting the optimum as a competitive equilibrium.
5. Describe the effect of a tax which takes a constant fraction, τ, of output and throws it away on (a) the steady state values of y and k and (b) their rates of growth.

1.4 Optimal Growth with Exogenous Technical Change

The absence of growth in the previous section drives us to contemplate models with productivity growth of some sort. In the neoclassical tradition by far the simplest theoretical device is to postulate exogenous increases in the productivity parameter, A. We do exactly that in a specialized version of the model. We would like the model to permit paths in which output, capital, and consumption grow at the same constant exponential rate, so we introduce a number of special features. We assume technical change and utility take the form

(4.1) $\quad A_t = A_0 e^{at}$

(4.2) $\quad u(c) = [c^{1-\gamma}-1]/(1-\gamma).$

The homothetic preferences inherent in (4.2) are necessary for constant growth rates of consumption to coexist with a constant real interest rate. We also assume Cobb-Douglas technology:

(4.3) $\quad y_t = A_t k_t^{\alpha} n_t^{1-\alpha},$

where n_t is labor inputs. As before we will set $n_t = 1$.

The first-order conditions for the optimal plan then include

(4.4a) $\quad c_t^{-\gamma} = \lambda_t$

(4.4b) $\quad \dot{\lambda}_t = (\rho - A_t \alpha k_t^{\alpha-1})\lambda_t$

(4.4c) $\quad \dot{k}_t = A_t k_t^{\alpha} - c_t.$

We derive the optimal plan by guessing the existence of a "balanced growth path" in which y, k, and c grow at the same constant exponential rate, g, verifying that this path satisfies the first-order conditions for an optimum, and proving convergence to this path from any initial capital stock. We start with the production function, which implies

$$\dot{y}/y = a + \alpha\dot{k}/k.$$

If both y and k grow at rate g, then

$$g = a/(1-\alpha).$$

We leave it to the reader to verify that there exist initial values k_0, A_0, c_0, and λ_0 such that k and c grow at rate $g = a/(1-\alpha)$, A at rate a, and λ at rate $-\alpha g$, and such that the laws of motion (4.4) are satisfied. We have thus extended the first two predictions of the neoclassical growth model:

1'. Growth rates of capital and output are equal in the steady state and proportional to the rate of technical progress.

and

2'. This rate of growth does not depend on preferences or on the level of the productivity parameter, A.

Once more we see that p, γ, and A_0 are level effects: changes have no influence on the steady state growth rate.

We now derive the complete dynamics of an optimal path starting from any initial capital stock. Define the variables

$$k_t^* = e^{-gt}k_t$$

$$\lambda_t^* = e^{-\gamma g}\lambda_t$$

and note that equations (4.4) can be expressed as

$$(c_t^*)^{-\gamma} = \lambda_t^*$$

$$\dot{\lambda}_t^* = (\rho+\alpha g-\alpha A_0 k_t^{*\alpha-1})\lambda_t^*$$

$$\dot{k}^* = A_0 k_t^{*\alpha} - c_t^* - g k_t^*.$$

This can be analyzed using the methods of the previous section, with a similar outcome:

3'. The capital stock and output approach the balanced growth path monotonically, as illustrated in figure 1.3.

Exercises to 1.4

1. Write down the transversality condition for the problem of this section. Under what conditions is it satisfied?
2. Verify that the share of output going to capital, f'k/f, is constant with Cobb-Douglas technology even with productivity growth.

1.5 The Model and the Evidence

So how well does the model fit what we observe? To answer this question we need some idea of what the world looks like. The following are a collection of "stylized facts," which means that they seem to be good approximations in most cases. We need to keep in mind, however, that data for long time periods have a substantial amount of measurement error, as a quick look through Kuznets (1971) or Denison (1967) makes clear. This is especially true for underdeveloped countries, so many of these "facts" are based on developed economies. Still, they give us a general idea of what to look for in a model.

Five general facts about long-term development are:

1. The capital stock and output grow at roughly the same rate, with the ratio k/y approximately equal to 3.
2. The wage rate has grown at approximately the same rate, but
3. Labor's share has been almost constant.

One implication of 2 and 3 is that the production function is Cobb-Douglas.

4. There is no tendency for either levels or growth rates of per capita income in different countries to converge, although convergence may be a property of the most advanced countries.

In other words, growth rates themselves are quite persistent, and countries with low (high) rates of growth often continue to have low (high) rates of growth for long periods of time. A fifth fact, suggested by close examination of growth rates in advanced countries and casual comparisons between current growth rates and the standard of living in, say, Roman times is:

5. Growth rates are increasing

The neoclassical growth model with exogenous technical change bears up fairly well with facts 1 to 3 (as indeed it was designed to do). Consider as a first approximation the predictions of the model for isolated economies--that is, economies with completely immobile factors, including capital, and no trade. Then each economy will behave as outlined in the last section. Along a balanced growth path the capital-output ratio is constant, as required by the data, and with Cobb-Douglas technology the share of labor in total output is constant.

The final two facts are more of a problem. If all countries have the same technology, then we expect convergence of levels and growth rates of per capita output. Clearly this is not the case, so we might postulate different technologies. This fits the facts, but in a trivial way. Fact five is similar: we can "explain" it only by assuming that the rate of technical progress is accelerating.

We can inject additional realism into the model by allowing some factor mobility. We will see that this reinforces any differences between countries due to technology differences. Assume first that capital is perfectly mobile across countries, but labor is not. Then capital will be allocated to equate marginal products across countries. If one country has a greater rate of technical progress, then its capital stock will be increasing at a greater rate as well. This makes the differences in their per capita growth rates larger, since the unproductive country is also capital poor. With labor mobility the situation is even more extreme: no one will work in the less productive country so its national product is zero.

Exercises to 1.5

1. Read Denison (1967, chs. 17 and 20, or 1974, pp. 71-83) and discuss the problems of distinguishing empirically between increasing returns to scale and "advances in knowledge." What additional evidence might you bring to bear on the issue?

2. Consider a world of two countries with identical preferences,

$$U^i = \int_0^\infty e^{-\rho t} \log c_t^i \, dt, \quad i = 1, 2,$$

but different rates of technical change:

$$y_t^i = A_i e^{a_i t} (k_t^i)^\alpha, \quad i = 1, 2,$$

with $A_1 < A_2$ and $a_1 > a_2$. Compute the optimum of the social planner's problem

$$\max_{\{c^1, c^2\}} \lambda U^1 + (1-\lambda) U^2$$

subject to

$$c_t^1 + c_t^2 + \dot{k}_t \le y_t^1 + y_t^2,$$

the two technologies, the world capital constraint,

$$k_t^1 + k_t^2 \le k_t,$$

and nonnegativity constraints on k^1 and k^2.

What are the dynamics of output in each country with (a) no trade or factor mobility, (b) mobility of capital, (c) mobility of both capital and labor?

6. Neoclassical Growth in Perspective

We have seen that the neoclassical growth model can be made to fit the gross features of growth in real economies, but only by postulating exogenous rates of productivity growth that vary across countries. In principle we would like our theory to explain productivity as well. Some attempts to do just that are described in the next chapter.

1.7 A Guide to the Literature

The neoclassical growth model is a product of the 1950s and is usually associated with Solow (1956), although similar models were published by Tobin and Swan around the same time. The theory of optimal growth dates from Ransey (1928); the modern treatment stems from Cas (1965) and Koopmans (1966). Lucas, Prescott, and Stokey (1986) discuss the interpretation of the optimal plan as a competitive equilibrium; the theory dates back to Debreu (1954).

Evidence on growth rates and other features of national economies over long periods of time is available in numerous books and articles by Kuznets and Denison. A useful up-to-date source is the World Bank's annual World Development Report. Some of the stylized facts on growth are summarized by Kaldor (1957) and Solow (1970).

2. ENGINES OF GROWTH

2.1 Growth with Linear Technology

We have seen that the neoclassical growth model cannot explain persistent differences in levels and growth rates in per capita income across countries without resorting to exogenous differences in levels and rates of change of productivity. We devote most of this chapter to theories that attempt to explain these productivity differences of a deeper level, but it is useful to attack the problem indirectly by considering a seemingly minor variation of the neoclassical growth model. This variation will give us an idea of how to build growth rate effects into models, even if it does so in a purely mechanical fashion.

Most of the predictions of the neoclassical growth model stem from a single assumption: that the marginal product of capital is declining. To illustrate the importance of this assumption, and also the possibilities that open up once it is discarded, consider the growth model with preferences

$$(1.1) \qquad \int_0^\infty e^{-\rho t} \log c_t \, dt$$

and linear technology

$$(1.2) \qquad y_t = Ak_t = c_t + \dot{k}_t.$$

The technology is the key since capital is no longer subject to diminishing returns.

We solve the problem by optimal control. The current value Hamiltonian is

$$H(c,k,\lambda) = \log c + \lambda[Ak-c],$$

which yields the first-order conditions

(1.3a) $c^{-1} = \lambda$

(1.3b) $\dot{\lambda} = (\rho - A)\lambda$

(1.3c) $\dot{k} = Ak - c.$

The optimal solution involves (see exercise 1)

(1.4) $k_t = e^{(A-\rho)t}k_0$

so the capital stock and output grew exponentially at rate $A - \rho$.

The point is that we now have a model capable of generating persistent, nonzero rates of growth without resorting to exogenous technological change (although to be fair, our theory is still pretty artificial). We can also generate differences in levels of output from different initial capital stocks. This model, unlike the neoclassical growth model, shows no tendency toward convergence of levels of output.

Our task in the remaining sections is to explore various means of escaping from diminishing returns in some reproductive factor and thus build in some of the features of linear technology. Our reference earlier to "technological change" suggests that we get better at producing things, so most of our effort will be invested in theories of learning and knowledge transmission that formalize this process. There are plausible arguments that learning technologies are not subject to diminishing returns in the aggregate, so we will arrive at models with some formal similarities to the one we have just studied. A subsidiary theme involves decentralization of decision-making when production exhibits increasing returns to scale. Suppose, for example, that the linear technology (1.2) included a second factor, n say, whose quantity is constant:

$$y_t = Ak_t n_t^\alpha .$$

The optimal solution can be computed as before, with An^α taking the place of A, but with increasing returns this may not be supportable as a competitive equilibrium. In most of the models we study we get around this difficulty by postulating an external effect, so that the increasing returns apparent in the aggregate are not available to individual decision-makers. In some cases the competitive equilibrium differs from the optimal plan, but the former often retains many of the features of the latter.

Exercises to 2.1

1. Use (1.3a,b) to express the costate as

$$\lambda_t = e^{(\rho-A)t}\lambda_0$$

and capital accumulation as

$$\dot{k} = Ak - e^{(A-\rho)t}/\lambda_0 .$$

Verify that

$$k_t = \{k_0+[e^{-\rho t}-1]/\rho\lambda_0\}e^{At}$$

is a solution to (k) satisfying the initial condition $k_t = k_0$ at $t = 0$. Now apply the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$$

to obtain (1.4).

2. Draw a phase diagram for the optimal growth problem with linear technology and compare it to the neoclassical growth model.

3. Derive the optimal plan with linear technology [eq(1.2)] and utility function

$$\int_0^\infty e^{-\rho t} u(c_t)$$

where $u(c) = c^{1-\gamma}/(1-\gamma)$. Describe the impact of the preference parameter on the solution. Does a solution exist for any values of ρ, A, and γ?

4. Derive the optimal plan for the linear-technology economy when the government takes a fraction of the total output and throws it away. Compare your answer to that of question 5, section 1.3.

2.2 Learning by Doing

One of the most promising theories of productivity growth is learning-by-doing. We know that people get better at things as they accumulate experience, and the management science literature is filled with references to "the experience curve." Arrow (1962) notes that in practice such learning exhibits "sharply diminishing returns," so we will want to consider learning curves of the form

$$A_t = \ell(e_t),\ \ell' > 0,\ \ell'' < 0,$$

where e is a measure of "experience" and A is our usual productivity parameter.

The question is how experience is measured. A particularly simple formulation specifies experience as cumulative output, so if output from, say, one unit of labor is

$$y_t = \ell(e_t)$$

then

$$\dot{e}_t = y_t.$$

Since ℓ' is decreasing and bounded from below by zero we know ℓ' approaches some constant:

$$\lim_{e\to\infty} \ell'(e) = \ell'_\infty \geq 0$$

so

$$\dot{y}/y = \ell'(e) \to \ell'_\infty.$$

The rate of growth of output is therefore determined completely by the function ℓ. We could make this more realistic by introducing additional inputs in production, but the idea would be essentially the same.

Two problems with this approach as a theory of growth are the following. First, when l'_∞ is positive we have a fairly simplistic theory. We have simply replaced exogenous technical change with an exogenous learning curve. This is an important, since we can base ℓ on studies of microbehavior, but a small one. Second, the evidence suggests that learning curves have the property $l'_\infty = 0$, in which case the model predicts no sustained growth. Arrow (1962), for example, cites a study of airplane manufacturing that approximates learning by the curve

$$\ell(e) \propto e^{1/3},$$

where e here is the number of airplanes produced. Ghemawat (1985) reports that "literally thousands of studies have shown that production costs usually decline by 10-30 percent with each doubling of cumulated output," which implies a similar learning curve.

The trick is to embody learning in a good that can increase without bound, just as we think intuitively that knowledge has no bound. Arrow embodies learning in capital goods. A slightly more tractable formulation is due to Lucas (1985). Production now uses both capital and labor, and labor's usefulness is determined by an ability variable, h:

$$y_t = k_t^{\alpha}[h_t n_t]^{1-\alpha}.$$

We think of h as human capital, or knowledge. The driving force in the model is the behavior of h over time. Lucas uses

$$\dot{h}_t = \delta n_t h_t$$

so that hard work now (large n) makes us more productive later.

Exercises to 2.2

1. Derive the cost function underlying Ghemawat's statement that costs decline by (say) 20 percent for every doubling of cumulated output. What does the growth rate converge to in an economy with this learning technology?
2. Write down an experience curve that produces k% growth asymptotically.
3. Put distortion in Lucas's LBD model.
4. Describe a version of Lucas's LBD model with leisure.