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THE LINEAR OPTIMAL REGULATOR PROBLEM
WITH PERIODIC COEFFICIENTS

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The Linear Optimal Regulator Problem with Periodic Coefficients

Let $R_s(t)$ be $n \times n$ symmetric negative semidefinite matrices ($R_s(t) \leq 0$, for all index functions $s(\cdot)$ and for all t); $Q_s(t)$ be $m \times m$ symmetric negative definite matrices ($Q_s(t) < 0$, for all index functions $s(\cdot)$ and for all t) with $m \leq n$; $A_s(t)$ be $n \times n$ matrices; $B_s(t)$ be $n \times m$ matrices; x_t be $n \times 1$ vectors; v_t be $m \times 1$ vectors; and let the initial vector x_{t_0} and an $n \times n$ symmetric negative semidefinite terminal matrix P_{t_1} be given. Finally, let ξ_t be $n \times 1$ vector white noises^{1/} with $E(\xi_t \xi_t^T) = \Psi_t$, where the Ψ_t are positive definite $n \times n$ matrices, and let E_t denote linear least squares projection onto information available at time t . Then the problem of maximizing

$$(1) \quad J(x_{t_0}) = E_{t_0} \left[\sum_{t=t_0}^{t_1-1} \{x_t^T R_s(t) x_t + v_t^T Q_s(t) v_t\} + x_{t_1}^T P_{t_1} x_{t_1} \right]$$

subject to a given x_{t_0} and

$$(2) \quad x_{t+1} = A_s(t)x_t + B_s(t)v_t + \xi_{t+1},$$

over choices of the $m \times n$ matrices $F_{t_0}, F_{t_0+1}, \dots, F_{t_1-1}$ used to set v_t according to

$$(3) \quad v_t = -F_t x_t,$$

is known as the optimal linear regulator problem. A wide variety of economic problems can be posed as linear optimal regulator problems.

Optimal linear regulator problems have been extensively studied and much is known about solving them [see, for example, Bertsekas or Kwakernaak and Sivan]. Most frequently studied are the general time-varying coefficients case, when $s(t) = t$, and the time-invariant case, when $s(t) = \bar{s}$, a constant. In the former case, the solution consists of one sequence of formulas for calculating the matrices $F_{t_0}, F_{t_0+1}, \dots, F_{t_1-1}$ as functions of the matrices $P_{t_1}, Q_{t_0}, Q_{t_0+1}, \dots, Q_{t_1-1}, R_{t_0}, R_{t_0+1}, \dots, R_{t_1-1}, A_{t_0}, A_{t_0+1}, \dots, A_{t_1-1}$, and $B_{t_0}, B_{t_0+1}, \dots, B_{t_1-1}$ and another sequence of formulas for calculating the maximized value of the problem. These formulas also work in the time-invariant case, of course, but the invariance of the coefficients in that case allows answers to additional questions concerning the limiting behavior of the system and its maximizing solution as $t_0 \rightarrow -\infty$.

The purpose of this paper is to show that many of the questions that can be answered about the limiting behavior of time-invariant systems can also be answered for time-varying systems with periodic coefficients. These systems, where $s(t)$ is a periodic sequence of integer period p , or $s(t+p) = s(t)$ for all t , include time-invariant systems as a special case ($p=1$). Many limiting properties of time-invariant systems can be generalized to the periodic case, where the matrices F_t which solve the problem, instead of settling down to a fixed value, settle down to a periodic sequence of values (if they settle down at all).

The first section of the paper reviews the solution of the time-varying linear optimal regulator problem and then shows that, when its coefficients are periodic, the same solution is obtained by embedding the time-varying problem in a higher dimensional time-invariant problem. The second section makes use of the higher dimensional time-invariant problem to establish certain limiting properties of the system and its maximizing solution.

I. Converting the periodic-coefficient linear optimal regulator problem to a time-invariant problem.

We first derive the solution to the finite-horizon problem introduced in equations (1) - (3) of the introduction. As Bertsekas (1976) shows, this problem can be solved by the recursive algorithm known as dynamic programming (DP).

According to the DP algorithm, we begin at the t_1-1 stage, where we seek a $v_{t_1-1} = -F_{t_1-1}x_{t_1-1}$ to maximize

$$(4) \quad J(x_{t_1-1}) = E_{t_1-1} \left[x_{t_1-1}^T R_s(t_1-1) x_{t_1-1} + v_{t_1-1}^T Q_s(t_1-1) v_{t_1-1} \right. \\ \left. + (A_s(t_1-1)x_{t_1-1} + B_s(t_1-1)v_{t_1-1} + \xi_{t_1})^T P_{t_1} (A_s(t_1-1)x_{t_1-1} \right. \\ \left. + B_s(t_1-1)v_{t_1-1} + \xi_{t_1}) \right],$$

subject to a given x_{t_1-1} . Differentiating with respect to v_{t_1-1} gives 2/

$$0 = 2 Q_s(t_1-1) v_{t_1-1} + 2 B_s^T(t_1-1) P_{t_1} A_s(t_1-1) x_{t_1-1} \\ + 2 B_s^T(t_1-1) P_{t_1} B_s(t_1-1) v_{t_1-1},$$

or

$$(4) \quad v_{t_1-1} = -F_{t_1-1} x_{t_1-1},$$

where

$$(5) \quad F_{t_1-1} = [Q_s(t_1-1) + B_s^T(t_1-1) P_{t_1} B_s(t_1-1)]^{-1} B_s^T(t_1-1) P_{t_1} A_s(t_1-1).$$

Equations (4) and (5) show that the optimal value of the problem starting at t_1-1 with initial value x_{t_1-1} is

$$J^*(x_{t_1-1}) = x_{t_1-1}^T R_s(t_1-1) x_{t_1-1} + x_{t_1-1}^T F_{t_1-1}^T Q_s(t_1-1) F_{t_1-1} x_{t_1-1} \\ + (A_s(t_1-1) x_{t_1-1} - B_s(t_1-1) F_{t_1-1} x_{t_1-1})^T P_{t_1} (A_s(t_1-1) x_{t_1-1} \\ - B_s(t_1-1) F_{t_1-1} x_{t_1-1}) + E_{t_1-1} \xi_{t_1}^T P_{t_1} \xi_{t_1},$$

or

$$(6) \quad J^*(x_{t_1-1}) = x_{t_1-1}^T P_{t_1-1} x_{t_1-1} + E_{t_1-1} \xi_{t_1}^T P_{t_1} \xi_{t_1},$$

where

$$(7) \quad P_{t_1-1} = [R_s(t_1-1) + A_s^T(t_1-1) \{P_{t_1} - P_{t_1} B_s(t_1-1) (Q_s(t_1-1) + B_s^T(t_1-1) P_{t_1} B_s(t_1-1))^{-1} B_s^T(t_1-1) P_{t_1}\} A_s(t_1-1)].$$

The matrix P_{t_1-1} is also symmetric and negative semidefinite [Bertsekas, p.72], so the next step in the DP algorithm, which is to find a $v_{t_1-2} = -F_{t_1-2} x_{t_1-2}$ that maximizes

$$(8) \quad J(x_{t_1-2}) = E_{t_1-2} [x_{t_1-2}^T R_s(t_1-2) x_{t_1-2} + v_{t_1-2}^T Q_s(t_1-2) v_{t_1-2} + (A_s(t_1-2) x_{t_1-2} + B_s(t_1-2) v_{t_1-2} + \xi_{t_1-1})^T P_{t_1-1} (A_s(t_1-2) x_{t_1-2} + B_s(t_1-2) v_{t_1-2} + \xi_{t_1-1}) + E_{t_1-1} \xi_{t_1}^T P_{t_1} \xi_{t_1}],$$

has the same form as the t_1-1 maximization except for the additional term $E_{t_1-1} \xi_{t_1}^T P_{t_1} \xi_{t_1}$. Since this additional term does not involve v_{t_1-2} in any way, it does not affect the choice of the maximizing F_{t_1-2} . Thus for the t_1-2 stage and, by working backwards in the same manner, for stages t_1-3, t_1-4, \dots, t_0 , we have that the optimizing choices for F_{t_1-k} (and hence for v_{t_1-k}) and the optimized values of $J(x_{t_1-k})$ are given by, for $k = 1, 2, \dots, t_1 - t_0$,

$$(9) \quad v_{t_1-k} = -F_{t_1-k} x_{t_1-k},$$

and

$$(10) \quad J^*(x_{t_1-k}) = x_{t_1-k}^T P_{t_1-k} x_{t_1-k} \\ + E_{t_1-k} \sum_{h=0}^{k-1} \xi_{t_1-k+h+1}^T P_{t_1-k+h+1} \xi_{t_1-k+h+1},$$

where

$$(11) \quad P_{t_1-k} = [R_s(t_1-k) + A_s^T(t-k) \{P_{t_1-k+1} - P_{t_1-k+1} B_s(t_1-k) \\ (Q_s(t_1-k) + B_s^T(t_1-k) P_{t_1-k+1} B_s(t_1-k))^{-1} B_s^T(t_1-k) P_{t_1-k+1}\} \\ A_s(t_1-k)]$$

and

$$(12) \quad F_{t_1-k} = (Q_s(t_1-k) + B_s^T(t_1-k) P_{t_1-k+1} B_s(t_1-k))^{-1} B_s^T(t_1-k) \\ P_{t_1-k+1} A_s(t_1-k).$$

The final term in equation (10) is derived from repeated application of the law of iterated linear projections, which says that $E_t (E_{t+1}\epsilon) = E_t\epsilon$, for any random variable ϵ . Equation (11) is known as the matrix

Riccati equation. For a more complete derivation of (9) - (12) in the general finite-horizon, time-varying coefficients case, the reader is invited to check Bertsekas [1976, Chapters 2 and 3].

Equations (9) - (12) were derived without specifying the index function $s(t)$, so they solve the general time-varying coefficients problem and the special cases of time-invariant and periodic coefficients as well. In particular, if we now specify $s(t)$ to be periodic of integer period p , so that $s(t+p) = s(t)$ for all t , equations (9) - (12) solve the finite-horizon, periodic-coefficients linear optimal regulator problem.

Identical solutions to the periodic-coefficients problem can be obtained by embedding it in the higher dimensional, essentially time-invariant problem given by choosing $u_t = -L_t y_t$ to maximize

$$(13) \quad \tilde{J}(y_{t_0}) = E_{t_0} \left[\sum_{t=t_0}^{t_1-1} \{y_t^T R y_t + u_t^T Q u_t\} + y_{t_1}^T K_{t_1} y_{t_1} \right]$$

subject to

$$(14) \quad y_{t+1} = A y_t + B u_t + e_{s(t+1)} \xi_{t+1},$$

where, in terms of the vectors and matrices previously defined and the notation $\text{diag} (Z_0, Z_1, \dots, Z_{p-1})$ for a diagonal (or block diagonal) matrix with diagonal elements Z_0, Z_1, \dots, Z_{p-1} ,

$$(15) \quad Q = \text{diag} [Q_s(0), Q_s(1), \dots, Q_s(p-1)],$$

a $pn \times pn$ symmetric negative definite matrix;

$$(16) \quad R = \text{diag} [R_s(0), R_s(1), \dots, R_s(p-1)],$$

a $pn \times pn$ symmetric negative semidefinite matrix;

$$(17) \quad K_{t_1} = \text{diag} [K_{t_1}^0, K_{t_1}^1, \dots, K_{t_1}^{p-1}],$$

with $K_{t_1}^i = P_{t_1}$, for $i=0, 1, 2, \dots, p-1$, so that K_{t_1} is a $pn \times pn$ symmetric negative semidefinite matrix;

$$(18) \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & A_s(p-1) \\ A_s(0) & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & A_s(1) & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & A_s(2) & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & & \cdot & \cdot \\ \cdot & \cdot & & & & & \cdot & \cdot \\ \cdot & \cdot & & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & A_s(p-2) & 0 \end{bmatrix}$$

a $pn \times pn$ matrix;

$$(19) \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & B_s(p-1) \\ B_s(0) & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & B_s(1) & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & B_s(2) & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & B_s(p-2) & 0 \end{bmatrix}$$

a $pn \times pm$ matrix;

$$(20) \quad e_s(t) = [O_n^0, O_n^1, \dots, O_n^{i-1}, I_n^i, O_n^{i+1}, \dots, O_n^{p-1}]^T$$

with O_n^j an $n \times n$ zero matrix in the j^{th} block of the $pn \times n$ matrix $e_s(t)$, for $j \neq i$, and I_n^i an $n \times n$ identity matrix in the i^{th} block of $e_s(t)$, where $i = \text{mod}_p(t)$. (That is, $i \in (0, 1, \dots, p-1)$ and $t = \ell p + i$ for some integer ℓ .) The structure of the $pn \times 1$ vectors y_t and the $pm \times 1$ vectors u_t will be explained in more detail below and can be ignored for now. The $n \times 1$ vector white noises ξ_t are as previously defined.

Using formulas (9) - (12) to derive the solution to the problem posed in (13) - (14) gives

$$(21) \quad \tilde{J}^*(y_{t_1-k}) = y_{t_1-k}^T K_{t_1-k} y_{t_1-k} + E_{t_1-k} \sum_{h=0}^{k-1} \xi_{t_1-k+h+1}^T e_s^T(t_{1-k+h+1}) K_{t_1-k+h+1} e_s(t_{1-k+h+1}) \xi_{t_1-k+h+1}$$

and

$$(22) \quad u_{t_1-k} = -L_{t_1-k} y_{t_1-k},$$

where

$$(23) \quad K_{t_1-k} = [R + A^T \{K_{t_1-k+1} - K_{t_1-k+1} B(Q + B^T K_{t_1-k+1} B)^{-1} B^T K_{t_1-k+1}\} A]$$

and

$$(24) \quad L_{t_1-k} = (Q + B^T K_{t_1-k+1} B)^{-1} B^T K_{t_1-k+1} A,$$

for $k = 1, 2, \dots, t_1 - t_0$, with K_{t_1} given by (17).

To see how (21) - (24) can match up with (9) - (12), we need to calculate more detailed expressions for L_{t_1-k} and K_{t_1-k} . Recalling the structure of K_{t_1} given by (17), we shall see that K_{t_1-1} has a similar structure. We have from (23) that

$$(25) \quad K_{t_1-1} = [R + A^T \{K_{t_1} - K_{t_1} B(Q + B^T K_{t_1} B)^{-1} B^T K_{t_1}\} A].$$

Note that

$$(26) \quad B^T K_{t_1} = \begin{bmatrix} 0 & B_s^T(0) & 0 & \cdot & \cdot & 0 \\ 0 & 0 & B_s^T(1) & \cdot & \cdot & 0 \\ 0 & 0 & 0 & B_s^T(2) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & B_s^T(p-2) & \cdot \\ B_s^T(p-1) & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} K_{t_1}^0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & K_{t_1}^1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & K_{t_1}^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & K_{t_1}^3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & K_{t_1}^{p-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (B_s^T(0)K_{t_1}^1) & 0 & \cdot & \cdot & 0 \\ 0 & 0 & (B_s^T(1)K_{t_1}^2) & \cdot & \cdot & 0 \\ 0 & 0 & 0 & (B_s^T(2)K_{t_1}^3) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & (B_s^T(p-2)K_{t_1}^{p-1}) & \cdot \\ (B_s^T(p-1)K_{t_1}^0) & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

so that^{3/}

$$(27) \quad B^T K_{t_1} B = \text{diag} [(B_{s(0)}^T K_{t_1}^1 B_{s(0)}), (B_{s(1)}^T K_{t_1}^2 B_{s(1)}), \\ \dots, (B_{s(p-1)}^T K_{t_1}^0 B_{s(p-1)})],$$

and

$$(28) \quad (Q + B^T K_{t_1} B)^{-1} = \text{diag} [(Q_{s(0)} + B_{s(0)}^T K_{t_1}^1 B_{s(0)})^{-1}, (Q_{s(1)} + B_{s(1)}^T \\ K_{t_1}^2 B_{s(1)})^{-1}, \dots, (Q_{s(p-1)} + B_{s(p-1)}^T K_{t_1}^0 B_{s(p-1)})^{-1}],$$

where the existence of these inverses is assured because of the negative definiteness of Q and $Q_{s(0)}, Q_{s(1)}, \dots, Q_{s(p-1)}$. With $(Q + B^T K_{t_1} B)^{-1}$ diagonal, we can obtain, through calculations very similar to those above, that

$$(29) \quad K_{t_1} B (Q + B^T K_{t_1} B)^{-1} B^T K_{t_1} = \text{diag} \{ [K_{t_1}^0 B_{s(p-1)} (Q_{s(p-1)} \\ + B_{s(p-1)}^T K_{t_1}^0 B_{s(p-1)})^{-1} B_{s(p-1)}^T], [K_{t_1}^1 B_{s(0)} (Q_{s(0)} \\ + B_{s(0)}^T K_{t_1}^1 B_{s(0)})^{-1} B_{s(0)}^T], \dots, [K_{t_1}^{p-1} B_{s(p-2)} (Q_{s(p-2)} \\ + B_{s(p-2)}^T K_{t_1}^{p-1} B_{s(p-2)})^{-1} B_{s(p-2)}^T] \},$$

and thus that

$$\begin{aligned}
 (30) \quad K_{t_1-1} = & \text{diag} \{ [R_{s(0)} + A_{s(0)}^T \{ K_{t_1}^1 + K_{t_1}^1 B_{s(0)} \} (Q_{s(0)} \\
 & + B_{s(0)}^T K_{t_1}^1 B_{s(0)})^{-1} B_{s(0)}^T K_{t_1}^1 \} A_{s(0)}], [R_{s(1)} + A_{s(1)}^T \{ K_{t_1}^2 \\
 & + K_{t_1}^2 B_{s(1)} \} (Q_{s(1)} + B_{s(1)}^T K_{t_1}^2 B_{s(1)})^{-1} B_{s(1)}^T K_{t_1}^2 \} A_{s(1)}], \\
 & \dots, [R_{s(p-1)} + A_{s(p-1)}^T \{ K_{t_1}^0 + K_{t_1}^0 B_{s(p-1)} \} (Q_{s(p-1)} \\
 & + B_{s(p-1)}^T K_{t_1}^0 B_{s(p-1)})^{-1} B_{s(p-1)}^T K_{t_1}^0 \} A_{s(p-1)}] \}.
 \end{aligned}$$

Note that K_{t_1-1} , like K_{t_1} , is a $pn \times pn$ symmetric negative semidefinite matrix, since each of its diagonal blocks is an $n \times n$ symmetric negative semidefinite matrix (Bertsekas, p. 72). Furthermore, the i^{th} diagonal element of K_{t_1-1} , where $i = \text{mod}_p(t_1-1)$, is the same as the expression for P_{t_1-1} derived for the periodic-coefficients case (equation (7) or (11), with $s(t+p) = s(t)$, for all t).

To derive L_{t_1-1} , recall equations (24) and (28) and note that

$$\begin{aligned}
 (31) \quad B_{t_1}^T K_{t_1} A = & \text{diag} [B_{s(0)}^T K_{t_1}^1 A_{s(0)}, B_{s(1)}^T K_{t_1}^2 A_{s(1)}, \dots, \\
 & B_{s(p-1)}^T K_{t_1}^0 A_{s(p-1)}],
 \end{aligned}$$

so that

$$\begin{aligned}
 (32) \quad L_{t_1-1} = & \text{diag} [(Q_{s(0)} + B_{s(0)}^T K_{t_1}^1 B_{s(0)})^{-1} B_{s(0)}^T K_{t_1}^1 A_{s(0)}, \\
 & (Q_{s(1)} + B_{s(1)}^T K_{t_1}^2 B_{s(1)})^{-1} B_{s(1)}^T K_{t_1}^2 A_{s(1)}, \dots, (Q_{s(p-1)} \\
 & + B_{s(p-1)}^T K_{t_1}^0 B_{s(p-1)})^{-1} B_{s(p-1)}^T K_{t_1}^0 A_{s(p-1)}].
 \end{aligned}$$

Again, for $i = \text{mod}_p(t_1-1)$, the i^{th} block on the diagonal of L_{t_1-1} is identical to the expressions derived for F_{t_1-1} for the periodic-coefficients case (equations (5) or (12), with $s(t+p) = s(t)$, for all t).

From equation (23), the formula for K_{t_1-2} is

$$(33) \quad K_{t_1-2} = [R + A^T \{K_{t_1-1} - K_{t_1-1} B(Q + B^T K_{t_1-1} B)^{-1} B^T K_{t_1-1}\} A].$$

If we let $K_{t_1-1} = \text{diag}[K_{t_1-1}^0, K_{t_1-1}^1, \dots, K_{t_1-1}^{p-1}]$, where $K_{t_1-1}^i$ equals the i^{th} block on the diagonal of the expression on the right side of equation (30), then equation (33) has the same form as equation (25). By calculations analogous to those leading up to (30),

$$(34) \quad K_{t_1-2} = \text{diag} \{ [R_{s(0)} + A_{s(0)}^T \{K_{t_1-1}^1 + K_{t_1-1}^1 B_{s(0)} (Q_{s(0)} + B_{s(0)}^T K_{t_1-1}^1 B_{s(0)})^{-1} B_{s(0)}^T K_{t_1-1}^1\} A_{s(0)}], [R_{s(1)} + A_{s(1)}^T \{K_{t_1-1}^2 + K_{t_1-1}^2 B_{s(1)} (Q_{s(1)} + B_{s(1)}^T K_{t_1-1}^2 B_{s(1)})^{-1} B_{s(1)}^T K_{t_1-1}^2\} A_{s(1)}], \dots, [R_{s(p-1)} + A_{s(p-1)}^T \{K_{t_1-1}^0 + K_{t_1-1}^0 B_{s(p-1)} (Q_{s(p-1)} + B_{s(p-1)}^T K_{t_1-1}^0 B_{s(p-1)})^{-1} B_{s(p-1)}^T K_{t_1-1}^0\} A_{s(p-1)}] \},$$

and, working backward in the same manner for $k = 1, 2, \dots, t_1 - t_0$,

$$\begin{aligned}
 (35) \quad K_{t_1-k} = & \text{diag} \{ [R_{s(0)} + A_{s(0)}^T \{ K_{t_1-k+1}^1 + K_{t_1-k+1}^1 B_{s(0)} \} (Q_{s(0)} \\
 & + B_{s(0)}^T K_{t_1-k+1}^1 B_{s(0)})^{-1} B_{s(0)}^T K_{t_1-k+1}^1 \} A_{s(0)}], [R_{s(1)} + A_{s(1)}^T \{ K_{t_1-k+1}^2 \\
 & + K_{t_1-k+1}^2 B_{s(1)} \} (Q_{s(1)} + B_{s(1)}^T K_{t_1-k+1}^2 B_{s(1)})^{-1} B_{s(1)}^T K_{t_1-k+1}^2 \} A_{s(1)}], \\
 & \dots, [R_{s(p-1)} + A_{s(p-1)}^T \{ K_{t_1-k+1}^0 + K_{t_1-k+1}^0 B_{s(p-1)} \} (Q_{s(p-1)} \\
 & + B_{s(p-1)}^T K_{t_1-k+1}^0 B_{s(p-1)})^{-1} B_{s(p-1)}^T K_{t_1-k+1}^0 \} A_{s(p-1)}] \},
 \end{aligned}$$

where the $n \times n$ matrices (which we will denote by $K_{t_1-k}^i$, for $i=0,1,2, \dots, p-1$) on the diagonal of K_{t_1-k} , for $k=1,2, \dots, t_1-t_0$, are each symmetric and negative definite.

The next important step in matching the solutions (21) - (24) of the higher dimensional problem with the solutions (9) - (12) of the periodic-coefficients problem is to show that, for $i = \text{mod}_p(t_1-k)$, the i^{th} block of K_{t_1-k} in (35) is identical to the expression for P_{t_1-k} derived in equation (11) for the periodic-coefficients case. Because $K_{t_1} = \text{diag} [P_{t_1}, P_{t_1}, \dots, P_{t_1}]$, it was easy to show that, for $i = \text{mod}_p(t_1-1)$, the i^{th} block of K_{t_1-1} was identical to P_{t_1-1} . Now, if $i = \text{mod}_p(t_1-1) \neq 0$, then $i-1 = \text{mod}_p(t_1-2)$ and we want to find P_{t_1-2} in the $i-1$ block of K_{t_1-2} . If $i = \text{mod}_p(t_1-1) = 0$, then we want P_{t_1-2} to be in the $p-1$ block. Equation

(34) shows that this is just what happens in the calculation of K_{t_1-2} . P_{t_1-1} , which appeared as $K_{t_1-1}^i$ in K_{t_1-1} , appears in the expression for $K_{t_1-2}^{i-1}$ in K_{t_1-2} (where we let $i-1 = p-1$ if $i=0$). This fact, along with $s(t_1-2)=s(i-1)$, shows that $K_{t_1-2}^{i-1} = P_{t_1-2}$. Similarly, $K_{t_1-k}^i = P_{t_1-k}$, where $i = \text{mod}_p(t_1-k)$ and $K_{t_1-k}^i$ denotes the i^{th} block on the diagonal of K_{t_1-k} .

We can now easily calculate L_{t_1-k} , $k=1,2,\dots,t_1-t_0$, and show where F_{t_1-k} is located on its diagonal. From equation (24) and the diagonal structure of K_{t_1-k+1} ,

$$(36) \quad L_{t_1-k} = \text{diag}[(Q_{s(0)} + B_{s(0)}^T K_{t_1-k+1}^1 B_{s(0)})^{-1} B_{s(0)}^T K_{t_1-k+1}^1 A_{s(0)}, \\ (Q_{s(1)} + B_{s(1)}^T K_{t_1-k+1}^2 B_{s(1)})^{-1} B_{s(1)}^T K_{t_1-k+1}^2 A_{s(1)}, \dots, \\ (Q_{s(p-1)} + B_{s(p-1)}^T K_{t_1-k+1}^0 B_{s(p-1)})^{-1} B_{s(p-1)}^T K_{t_1-k+1}^0 A_{s(p-1)}].$$

Since $K_{t_1-k+1}^{i+1} = P_{t_1-k+1}$, for $i+1 = \text{mod}_p(t_1-k+1)$, as discussed above, it follows immediately from (36) and (12) that $L_{t_1-k}^i = F_{t_1-k}$ (using $i+1=0$ if $i=p-1$).

To summarize the results so far, we can say that equations (21) - (24) solve the problem posed in (13) and (14), and that for $k=1,2,\dots,t_1-t_0$, K_{t_1-k} and L_{t_1-k} are diagonal matrices whose i^{th} blocks along the diagonal, for $i = \text{mod}_p(t_1-k)$, are given by P_{t_1-k} and F_{t_1-k} , respectively.

We now begin to complete the correspondence between the periodic-coefficient problem and the higher dimensional, time-invariant problem by specifying the structure of y_{t_0} , the given initial condition. Let

$$(37) \quad y_{t_0} = [(y_{t_0}^0)^T, (y_{t_0}^1)^T, \dots, (y_{t_0}^{p-1})^T]^T,$$

where

$$(38) \quad y_{t_0}^i = \begin{cases} x_{t_0}, & \text{if } i \equiv \text{mod}_p(t_0) \\ \text{an } n \times 1 \text{ matrix of zeros,} & \text{otherwise.} \end{cases}$$

Then $\tilde{J}^*(y_{t_0})$, as given by equation (21), is the same as $J^*(x_{t_0})$, as given by equation (10), for

$$(39) \quad e_s^T(t_1 - k + h + 1) K_{t_1 - k + h + 1} e_s(t_1 - k + h + 1) = K_{t_1 - k + h + 1}^j = P_{t_1 - k + h + 1},$$

where $j \equiv \text{mod}_p(t_1 - k + h + 1)$, and

$$(40) \quad y_{t_0}^T K_{t_0} y_{t_0} = x_{t_0}^T K_{t_0}^i x_{t_0} = x_{t_0}^T P_{t_0} x_{t_0},$$

where $i \equiv \text{mod}_p(t_0)$. That is, the two problems have the same maximized value.

The optimal controls, v_t and u_t , for $t = t_0, t_0 + 1, \dots, t_1 - 1$, also match up, as do the state vectors x_t and y_t . Recall that y_{t_0} is $pn \times 1$, with $p-1$ $n \times 1$ blocks of zeros and the $n \times 1$ block x_{t_0} occupying the i^{th}

position, where $i = \text{mod}_p(t_0)$. From this fact and the fact that $L_{t_0}^i = F_{t_0}$, we get

$$\begin{aligned}
 (41) \quad u_{t_0} &= -L_{t_0} y_{t_0} = [0_{1 \times m}^0, 0_{1 \times m}^1, \dots, 0_{1 \times m}^{i-1}, (-F_{t_0} x_{t_0})^T, 0_{1 \times m}^{i+1}, \\
 &\quad \dots, 0_{1 \times m}^{p-1}]^T \\
 &= [0_{1 \times m}^0, 0_{1 \times m}^1, \dots, v_{t_0}^T, \dots, 0_{1 \times m}^{p-1}]^T,
 \end{aligned}$$

where the $1 \times m$ zero vectors $0_{1 \times m}^j$ occupy all blocks but the i^{th} . That is, u_{t_0} is a $pm \times 1$ vector partitioned into p $m \times 1$ blocks, all of which are zero except the i^{th} , which equals v_{t_0} , where $i = \text{mod}_p(t_0)$.

With x_{t_0} occupying the i^{th} block of y_{t_0} and v_{t_0} occupying the i^{th} block of u_{t_0} , the other blocks being zero, for $i = \text{mod}_p(t_0)$, we can show that x_{t_0+1} and v_{t_0+1} occupy the $i+1$ blocks of y_{t_0+1} and u_{t_0+1} , respectively, where we let $i+1=0$ when $i=p-1$. To see this, note that

$$(42) \quad Ay_t = \begin{bmatrix} 0 & 0 & 0 & \cdot & A_{s(p-1)} \\ A_{s(0)} & 0 & 0 & \cdot & 0 \\ 0 & A_{s(1)} & 0 & \cdot & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 0 & \cdot & A_{s(p-2)} & 0 \end{bmatrix} \begin{bmatrix} y_t^0 \\ y_t^1 \\ y_t^2 \\ \cdot \\ \cdot \\ \cdot \\ y_t^{p-1} \end{bmatrix} = \begin{bmatrix} A_{s(p-1)} y_t^{p-1} \\ A_{s(0)} y_t^0 \\ A_{s(1)} y_t^1 \\ \cdot \\ \cdot \\ \cdot \\ A_{s(p-2)} y_t^{p-2} \end{bmatrix}$$

and

$$(43) \quad Bu_t = \begin{bmatrix} 0 & 0 & 0 & \cdot & B_{s(p-1)} \\ B_{s(0)} & 0 & 0 & \cdot & 0 \\ 0 & B_{s(1)} & 0 & \cdot & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 0 & \cdot & B_{s(p-2)} & 0 \end{bmatrix} \begin{bmatrix} u_t^0 \\ u_t^1 \\ u_t^2 \\ \cdot \\ \cdot \\ \cdot \\ u_t^{p-1} \end{bmatrix} = \begin{bmatrix} B_{s(p-1)} u_t^{p-1} \\ B_{s(0)} u_t^0 \\ B_{s(1)} u_t^1 \\ \cdot \\ \cdot \\ \cdot \\ B_{s(p-2)} u_t^{p-2} \end{bmatrix} .$$

Equations (42) and (43), the structures of y_{t_0} and u_{t_0} described above, and equation (20) thus imply that

$$(44) \quad y_{t_0+1} = Ay_{t_0} + Bu_{t_0} + e_s(t_0+1) \xi_{t_0+1}$$

$$= A \begin{bmatrix} 0_{nx1}^0 \\ \cdot \\ \cdot \\ 0_{nx1}^{i-1} \\ x_{t_0} \\ 0_{nx1}^{i+1} \\ \cdot \\ \cdot \\ \cdot \\ 0_{nx1}^{p-1} \end{bmatrix} + B \begin{bmatrix} 0_{mx1}^0 \\ \cdot \\ \cdot \\ 0_{mx1}^{i-1} \\ v_{t_0} \\ 0_{mx1}^{i+1} \\ \cdot \\ \cdot \\ \cdot \\ 0_{mx1}^{p-1} \end{bmatrix} + \begin{bmatrix} 0_{nxn}^0 \xi_{t_0+1} \\ \cdot \\ \cdot \\ \cdot \\ 0_{nxn}^i \xi_{t_0+1} \\ I_{nxn} \xi_{t_0+1} \\ \cdot \\ 0_{nxn}^{i+2} \xi_{t_0+1} \\ \cdot \\ \cdot \\ 0_{nxn}^{p-1} \xi_{t_0+1} \end{bmatrix} = \begin{bmatrix} 0_{nx1}^0 \\ \cdot \\ \cdot \\ 0_{nx1}^i \\ 0_{nx1}^{i+1} \\ \cdot \\ \cdot \\ \cdot \\ 0_{nx1}^{i+2} \\ \cdot \\ \cdot \\ 0_{nx1}^{p-1} \end{bmatrix} = A_s(i)x_{t_0} + B_s(i)v_{t_0} + \xi_{t_0+1}$$

$$= [(0_{nx1}^0)^T, \dots, (0_{nx1}^i)^T, (x_{t_0+1})^T, (0_{nx1}^{i+2})^T, \dots, (0_{nx1}^{p-1})^T]^T.$$

If $i \bmod_p(t_0) \neq p-1$, this shows that x_{t_0+1} does occupy the $i+1$ block of y_{t_0+1} , with the other blocks zero. If $i \bmod_p(t_0) = p-1$, a similar demon-

stration can be made (letting $i+1=0$ in this case). With x_{t_0+1} in the $i+1$ block of y_{t_0+1} , the rest of y_{t_0+1} being zero, and with F_{t_0+1} occupying the $i+1$ block along the diagonal of L_{t_0+1} , we have immediately that v_{t_0+1} occupies the $i+1$ block of u_{t_0+1} , the other blocks being zero.

By similar reasoning, we can show that x_{t_1-k} lies in the $\text{mod}_p(t_1-k)$ block of y_{t_1-k} , the rest of y_{t_1-k} being zero, and that v_{t_1-k} occupies the $\text{mod}_p(t_1-k)$ block of u_{t_1-k} , the rest of u_{t_1-k} being zero. We have already shown that P_{t_1-k} occupies the $\text{mod}_p(t_1-k)$ block along the diagonal of K_{t_1-k} , that F_{t_1-k} lies in the $\text{mod}_p(t_1-k)$ block along the diagonal of L_{t_1-k} , and that $\tilde{J}(y_{t_0}) = J(x_{t_0})$. Thus, for the finite-horizon, periodic-coefficient linear optimal regulator problem, all of the information in the solution equations (9) - (12) can also be obtained from the solution equations (21) - (24) of the higher dimensional, time-invariant problem posed by equations (13) - (14).

II. Sufficient conditions for convergence of the linear optimal regulator with periodic coefficients.

In a sense, this section of the paper is superfluous. Having already shown how to convert periodic-coefficient linear optimal regulator problems into time-invariant problems, we could simply refer to standard references on time-invariant problems [for example, Bertsekas or Kwakernaak and Sivan] to discover sufficient conditions for non-explosive behavior of the state vector and convergence of the value of the problem and the feedback rules as the horizon goes to infinity. However, it should be both convenient and interesting to review some of the main theorems here and investigate the special forms they take for problems of the form described in equations (13) - (20). We will review the main convergence theorems and concepts of controllability, controllability canonical form, reconstructability, detectability, stability, and stabilizability. By specializing these concepts and convergence theorems to the case of the problem posed in equations (13) - (20), we are actually generalizing them to the class of periodic-coefficient linear optimal regulator problems.

A. Review of Standard Convergence Results

Consider the linear system $x(t+1) = Ax(t) + Bv(t)$, where $x(t)$ is an $n \times 1$ state vector, $v(t)$ is an $m \times 1$ control vector ($m < n$), A is an $n \times n$ matrix of constants, and B is an $n \times m$ matrix of constants. This

system is said to be completely controllable (or just controllable, for our purposes) if the system can be moved from zero at any initial time t_0 to any terminal state $x_1 \in R^n$ within finite time $(t_1 - t_0)$. It turns out that the system is controllable if and only if it can be transferred from any initial state $x_0 \in R^n$ at any initial time t_0 to any terminal state $x_1 \in R^n$ within a finite time $t_1 - t_0$ [Kwakernaak and Sivan, p.54].

To check for controllability, we make use of the Cayley-Hamilton theorem [Noble, p.372], which implies that $A^i = \sum_{j=0}^{n-1} g_j^i A^j$, for $i = n, n+1, n+2, \dots$. That is, the n^{th} and higher integer powers of an $n \times n$ square matrix are linear combinations of the matrices $[I, A, \dots, A^{n-1}]$. This in turn implies

Theorem 1: The n -dimensional linear time-invariant system $x(t+1) = Ax(t) + Bv(t)$ is completely controllable if and only if the controllability matrix $P = (B, AB, A^2B, \dots, A^{n-1}B)$ is of full row rank n ; that is, only if the column vectors of P span R^n [Kwakernaak and Sivan, pp.459-60]. We also say that the matrices A and B are a controllable pair if $(B, AB, A^2B, \dots, A^{n-1}B)$ is of full row rank.

Many systems that arise naturally from economic models are not completely controllable. For that reason, we introduce the concept of the controllable subspace of the linear time-invariant system $x(t+1) = Ax(t) + Bv(t)$, which is the linear subspace consisting of the states that can be reached from the zero state in finite time. It follows easily from the proof of Theorem 1 that [Kwakernaak and Sivan, p.58]

Theorem 2: The controllable subspace of the n-dimensional linear time-invariant system $x(t+1) = Ax(t)+Bv(t)$ is the linear subspace spanned by the column vectors of the controllability matrix $P = [B, AB, A^2B, \dots, A^{n-1}B]$.

We shall now transform the n-dimensional system $x(t+1) = Ax(t) + Bv(t)$ to a more revealing form. Let the dimension of the controllable subspace = rank of the controllability matrix = r, where $r \leq n$. Choose a basis for the controllability subspace consisting of the $n \times 1$ vectors f_1, f_2, \dots, f_r , and choose $n \times 1$ vectors $f_{r+1}, f_{r+2}, \dots, f_n$ so that f_1, f_2, \dots, f_n is a basis for R^n . Form the nonsingular transformation matrix $T = (T_1, T_2)$, where $T_1 = (f_1, f_2, \dots, f_r)$ and $T_2 = (f_{r+1}, f_{r+2}, \dots, f_n)$. Let $\tilde{x}(t) = T^{-1}x(t)$, so that $T\tilde{x}(t) = x(t)$. The system can thus be written $T\tilde{x}(t+1) = AT\tilde{x}(t) + Bv(t)$, or, premultiplying by T^{-1} , $\tilde{x}(t+1) = \tilde{A}\tilde{x}(t) + \tilde{B}v(t)$, where $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$. Partitioning T^{-1} as $T^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, where U_1 is $r \times n$ and U_2 is $(n-r) \times n$, it can be shown

[Kwakernaak and Sivan, pp.461-62] that

Theorem 3: The transformed system described above has the form

$$(45) \quad \begin{bmatrix} \tilde{x}_1(t+1) \\ \tilde{x}_2(t+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} v(t),$$

where the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable, and $\tilde{A}_{11} = U_1 A T_1$, $\tilde{A}_{12} = U_1 A T_2$, $\tilde{A}_{22} = U_2 A T_2$, $\tilde{B}_1 = U_1 B$, $\tilde{x}_1(t)$ is an $r \times 1$ vector, and $\tilde{x}_2(t)$ is an $(n-r) \times 1$ vector.

Expression (45) is called the controllability canonical form of the system $x(t+1) = Ax(t) + Bv(t)$. The controllability matrix of this system takes the form

$$\tilde{P} = [\tilde{B}, \tilde{A}\tilde{B}, \dots, \tilde{A}^{n-1}\tilde{B}] = \begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \dots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

so that, in effect, any point in its controllable subspace can be reached by the completely controllable system $\tilde{x}_1(t+1) =$

$\tilde{A}_{11}\tilde{x}_1(t) + \tilde{B}_1 v(t)$. In fact, given any $n \times 1$ vector $x' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$, and any $r \times 1$ vector x''_1 , the system (45) can be moved from the initial state

$\tilde{x}(t_0) = x'$ to the terminal state $\tilde{x}(t_1) = x'' = \begin{bmatrix} x''_1 \\ x''_2 \end{bmatrix}$, where $x''_2 = \tilde{A}_{22}^{t_1-t_0} x'_2$ is $(n-r) \times 1$, within finite time $t_1 - t_0$.

Since the matrix \tilde{A}_{22} alone determines the movement of the elements of the state vector corresponding to the noncontrollable subspace, its properties are very important to the limiting behavior of the system. We say that an $n \times n$ matrix A and the homogeneous linear system $x(t+1) = Ax(t)$ are stable if for any $x(t_0)$ in R^n , $\lim_{j \rightarrow \infty} x(t_0 + j) = 0$. The following theorem is well-known [Kwakernaak and Sivan, p.28]:

Theorem 4: The matrix A and the homogeneous linear system $x(t+1) = Ax(t)$ are stable if and only if the eigenvalues of A are strictly less than unity in modulus.

We now combine the notions of controllability and stability by calling the pair (A,B) and the system $x(t+1) = Ax(t)+Bv(t)$ stabilizable if the matrix \tilde{A}_{22} in the controllability canonical form of the system is stable.^{4/ 5/} If we consider a stabilizable system in controllability canonical form (e.g., equations (45) with \tilde{A}_{22} stable), then, for any initial state vector $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ at time t_0 and any $r \times 1$ vector x''_1 , the system can be driven to $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix}$, where x''_2 is an $(n-r) \times 1$ vector, within finite time t_1-t_0 , and the system can be driven arbitrarily close to $\begin{bmatrix} x_1'' \\ 0 \end{bmatrix}$ as $t_1 \rightarrow \infty$.

The concepts of reconstructability and detectability are normally introduced when the controller of the system cannot directly observe the state matrix $x(t)$. We are not considering such problems here, but we will need these concepts. It will be sufficient for our purposes to say that the pair (A,B) is reconstructable if the pair (A^T, B^T) is controllable, and that the pair (A,B) is detectable if the pair (A^T, B^T) is stabilizable.

We now present theorems on the convergence of feedback rules and maximized values for linear optimal regulator problems. First con-

sider the finite-horizon nonstochastic linear optimal regulator problem whose law of motion is written in controllability canonical form as

$$(46) \quad \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} v(t) \quad ,$$

where (A_{11}, B_1) is controllable and A_{22} is a stable matrix, and whose criterion function

$$(47) \quad \sum_{t=t_0}^{t_1-1} \left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + v(t)^T Q v(t) \right. \\ \left. + \begin{bmatrix} x_1(t_1) \\ x_2(t_1) \end{bmatrix}^T \begin{bmatrix} P_{11}(t_1) & P_{12}(t_1) \\ P_{21}(t_1) & P_{22}(t_1) \end{bmatrix} \begin{bmatrix} x_1(t_1) \\ x_2(t_1) \end{bmatrix} \right\}$$

is maximized over choices of

$$(48) \quad v(t) = -[F_1(t) \quad F_2(t)] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad ,$$

where $x_1(t)$ is $r \times 1$, $x_2(t)$ is $(n-r) \times 1$, $v(t)$ is $m \times 1$, r is the dimension of the controllable subspace, and all other matrices are of conformable dimensions. For convenient reference let's also refer to this problem as one of maximizing

$$\sum_{t=t_0}^{t_1-1} [x(t)^T R x(t) + v(t)^T Q v(t)] + x(t_1)^T P(t_1) x(t_1)$$

subject to

$$x(t+1) = Ax(t) + Bv(t)$$

over choices of $v(t) = -F(t)x(t)$, where $x(t)$, $P(t_1)$, A , B , and $F(t)$ denote the corresponding vectors and matrices in equations (46) - (48). If we assume that R and $P(t_1)$ are symmetric and negative semidefinite and that Q is symmetric and negative definite, then for a given $x(t_0)$, the solution to the problem is given by equations (9) - (12) (with $s(t) = \bar{s}$, a constant).

Now consider the problem as we let the horizon, $t_1 - t_0$, go to infinity by driving t_0 to $-\infty$. As Sargent [pp. V-9, V-10] notes (using slightly different notation), "We would find the following two characteristics desirable. First, as we drive $t_0 \rightarrow -\infty$, we would like P_{t_0} [$P(t_0)$ in our notation] to converge to a constant matrix P which is independent of the given terminal matrix P_{t_1} [i.e., $P(t_1)$]. This is a desirable feature because it implies ... that the sequence of optimal control laws $\{F_{t_0}\}$ also converges to a constant as $t_0 \rightarrow -\infty$. This has the practical implication that the feedback law $\{F_t\}$ that solves the infinite horizon problem is time invariant, so that $F_t = F$ for all t , and that the resulting closed loop system

$$x_{t+1} = (A - BF)x_t \dots$$

is time invariant. Our second desideratum is, given that the closed loop system is time invariant, that it be stable. This requires that the matrix $(A-BF)$ be stable, that is, have eigenvalues with moduli less than unity."

A number of theorems on the convergence of $P(t)$ and the stability of the closed loop system are available for this problem; the antecedents of the following theorem are fairly weak:

Theorem 5: In the optimal linear regulator problem described in equation's (46) - (48), assume that (A,B) is stabilizable and that $P_{11}(t_1)$ and R_{11} are negative semidefinite. Express R_{11} as $R_{11} = -G^T G$, and assume that (A_{11},G) is detectable. The matrices R_{12} , R_{22} , $P_{12}(t_1)$ and $P_{22}(t_1)$ are unrestricted; let $R_{21} = R_{12}$ and $P_{21}(t_1) = P_{12}(t_1)$. Then

(a) Iterations on the matrix Riccati equation (11) converge to a unique matrix P^* which is independent of $P(t_1)$, and $P_{11}^* = \lim_{t_0 \rightarrow -\infty} P_{11}(t_0)$ is negative semidefinite.

(b) The optimal feedback rule $F(t_0)$ converges to the stationary rule F^* as $t_0 \rightarrow -\infty$, where $F_1^* = \lim_{t_0 \rightarrow -\infty} F_1(t_0)$ is independent of R_{12} and R_{22} while $F_2^* = \lim_{t_0 \rightarrow -\infty} F_2(t_0)$ is independent of R_{22} .

(c) The optimal stationary closed loop system matrix $(A-BF^*)$ is stable.

Proof: See Sargent (pp. V-29, V-30).

For stochastic systems, a similar result holds [Sargent, pp. V-30, V-31].

Theorem 6: With the matrices $R, Q, A, B, P(t_1), x(t)$, and $v(t)$ as described in equations (46) - (48) and $\xi(t)$ an $n \times 1$ vector white noise with $E\xi(t)\xi(t)^T = \Psi(t)$, a positive semidefinite matrix, assume that R_{11} and $P(t_1)$ are negative semidefinite; Q is negative definite; (A, B) is stabilizable; and (A_{11}, G) is detectable, where $R_{11} = -G^T G$. Then, for the problem of maximizing

$$\lim_{t_0 \rightarrow -\infty} \left(\frac{1}{t_1 - t_0} \right) E_{t_0} \left\{ \sum_{t=t_0}^{t_1-1} [x(t)^T R x(t) + v(t)^T Q v(t)] + x(t_1)^T P(t_1) x(t_1) \right\}$$

subject to $x(t+1) = Ax(t) + Bv(t) + \xi(t+1)$ over choices of $v(t) = -F(t)x(t)$,

- (a) Iterations on the matrix Riccati equation (11) converge to a unique matrix P^* which is independent of $P(t_1)$.
- (b) The optimal feedback rule $F(t_0)$ converges to the stationary rule F^* as $t \rightarrow -\infty$.
- (c) If $\Psi(t) = \Psi$ for all t , the maximal value of the criterion function is $\text{tr}[P\Psi]$, where $\text{tr}[\cdot]$ denotes the trace of a matrix.

B. Convergence Results for Periodic-Coefficient Problems.

Now we shall elaborate some of the special forms the conditions and theorems just reviewed take for the system described in equations (13) - (20). It should suffice to merely illustrate these special forms for the case $p=4$; that is, for a quarterly cycle in the coefficient matrices. Then we have

$$(49) \quad A = \begin{bmatrix} 0 & 0 & 0 & A_3 \\ A_0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{bmatrix}$$

$$(50) \quad B = \begin{bmatrix} 0 & 0 & 0 & B_3 \\ B_0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 \\ 0 & 0 & B_2 & 0 \end{bmatrix}$$

$$(51) \quad Q = \text{diag } [Q_0, Q_1, Q_2, Q_3]$$

$$(52) \quad R = \text{diag } [R_0, R_1, R_2, R_3]$$

$$(53) \quad K(t_1) = \text{diag } [K_{t_1}^0, K_{t_1}^1, K_{t_1}^2, K_{t_1}^3],$$

where $K_{t_1}^i = P_{t_1}^i$, $i=0,1,2,3$.

The controllability matrix P has the form

$$(54) \quad P = [B, AB, A^2B, \dots, A^{4n-1}B],$$

a $4n \times 16nm$ matrix, where

$$(55) \quad \text{for } \text{mod}_4(q)=0, \quad A^{qB} = \begin{bmatrix} 0 & 0 & 0 & \overline{A}_3^{q^*} B_3 \\ \overline{A}_0^{q^*} B_0 & 0 & 0 & 0 \\ 0 & \overline{A}_1^{q^*} B_1 & 0 & 0 \\ 0 & 0 & \overline{A}_2^{q^*} B_2 & 0 \end{bmatrix}$$

$$(56) \quad \text{for } \text{mod}_4(q)=1, \quad A^{qB} = \begin{bmatrix} 0 & 0 & \overline{A}_3^{q^*} A_3 B_2 & 0 \\ 0 & 0 & 0 & \overline{A}_0^{q^*} A_0 B_3 \\ \overline{A}_1^{q^*} A_1 B_0 & 0 & 0 & 0 \\ 0 & \overline{A}_2^{q^*} A_2 B_1 & 0 & 0 \end{bmatrix}$$

$$(57) \quad \text{for } \text{mod}_4(q)=2, \quad A^{qB} = \begin{bmatrix} 0 & \overline{A}_3^{q^*} A_3 A_2 B_1 & 0 & 0 \\ 0 & 0 & \overline{A}_0^{q^*} A_0 A_3 B_2 & 0 \\ 0 & 0 & 0 & \overline{A}_1^{q^*} A_1 A_0 B_3 \\ \overline{A}_2^{q^*} A_2 A_1 B_0 & 0 & 0 & 0 \end{bmatrix}$$

$$(58) \quad \text{for } \text{mod}_4(q)=3, \quad A^{qB} = \begin{bmatrix} \overline{A}_3^{q^*} A_3 A_2 A_1 B_0 & 0 & 0 & 0 \\ 0 & \overline{A}_0^{q^*} A_0 A_3 A_2 B_1 & 0 & 0 \\ 0 & 0 & \overline{A}_1^{q^*} A_1 A_0 A_3 B_2 & 0 \\ 0 & 0 & 0 & \overline{A}_2^{q^*} A_2 A_1 A_0 B_3 \end{bmatrix}$$

with $q^* = [(q - \text{mod}_4(q))/4]$ and

$$(59) \quad \bar{A}_0 = A_0 A_3 A_2 A_1$$

$$(60) \quad \bar{A}_1 = A_1 A_0 A_3 A_2$$

$$(61) \quad \bar{A}_2 = A_2 A_1 A_0 A_3$$

$$(62) \quad \bar{A}_3 = A_3 A_2 A_1 A_0.$$

The controllability of the pair (A,B) , via Theorem 1, depends on the rank of P in (54). From Theorem 2, the controllable subspace of the system, which we denote by C , is the linear subspace spanned by the column vectors of P .

We will now construct a controllability canonical form by choosing from a particular class of transformations. To identify this class of transformations, first note that the space R^{4n} is the direct sum of four orthogonal linear subspaces, $R^{4n} = S_1 \oplus S_2 \oplus S_3 \oplus S_0$, where S_3 is spanned by the column vectors of $[I, 0, 0, 0]^T$, S_0 is spanned by the column vectors of $[0, I, 0, 0]^T$, S_1 is spanned by the column vectors of $[0, 0, I, 0]^T$, and S_2 is spanned by the column vectors of $[0, 0, 0, I]^T$ where 0 is an $n \times n$ matrix of zeros and I is an $n \times n$ identity matrix. Then the controllability subspace can be decomposed in similar fashion as $C = C_1 \oplus C_2 \oplus C_3 \oplus C_0$, where $C_1 = C \cap S_1$, $C_2 = C \cap S_2$, $C_3 = C \cap S_3$, and $C_0 = C \cap S_0$. This decomposition of C is useful because of the nature of P , where each column has all zeros in at least three out of the four subspaces S_0, S_1, S_2, S_3 . That is, for any column vector w_i of P such that for some $k \in \{0,1,2,3\}$ the projection of w_i on S_k is not zero, and

any column vector w_j of P such that for some $l \in \{0,1,2,3\}$, $l \neq k$, the projection of w_j on S_l is not zero, then $w_i \neq w_j$ and w_i and w_j are orthogonal ($w_i^T w_j = 0$).

Because of this structure, we can identify separate controllability matrices for S_1, S_2, S_3 , and S_4 . Let

$$(63) \quad P_3 = [B_3, A_3 B_2, A_3 A_2 B_1, A_3 A_2 A_1 B_0, \bar{A}_3 B_3, \bar{A}_3 A_3 B_2, \bar{A}_3 A_3 A_2 B_1, \\ \bar{A}_3 A_3 A_2 A_1 B_0, \dots, \bar{A}_3^{n-1} B_3, \bar{A}_3^{n-1} A_3 B_2, \bar{A}_3^{n-1} A_3 A_2 B_1, \\ \bar{A}_3^{n-1} A_3 A_2 A_1 B_0];$$

$$(64) \quad P_0 = [B_0, A_0 B_3, A_0 A_3 B_2, A_0 A_3 A_2 B_1, \bar{A}_0 B_0, \bar{A}_0 A_0 B_3, \bar{A}_0 A_0 A_3 B_2, \\ \bar{A}_0 A_0 A_3 A_2 B_1, \dots, \bar{A}_0^{n-1} B_0, \bar{A}_0^{n-1} A_0 B_3, \bar{A}_0^{n-1} A_0 A_3 B_2, \\ \bar{A}_0^{n-1} A_0 A_3 A_2 B_1];$$

$$(65) \quad P_1 = [B_1, A_1 B_0, A_1 A_0 B_3, A_1 A_0 A_3 B_2, \bar{A}_1 B_1, \bar{A}_1 A_1 B_0, \bar{A}_1 A_1 A_0 B_3, \\ \bar{A}_1 A_1 A_0 A_3 B_2, \dots, \bar{A}_1^{n-1} B_1, \bar{A}_1^{n-1} A_1 B_0, \bar{A}_1^{n-1} A_1 A_0 B_3, \\ \bar{A}_1^{n-1} A_1 A_0 A_3 B_2];$$

$$(66) \quad P_2 = [B_2, A_2 B_1, A_2 A_1 B_0, A_2 A_1 A_0 B_3, \bar{A}_2 B_2, \bar{A}_2 A_2 B_1, \bar{A}_2 A_2 A_1 B_0, \\ \bar{A}_2 A_2 A_1 A_0 B_3, \dots, \bar{A}_2^{n-1} B_2, \bar{A}_2^{n-1} A_2 B_1, \bar{A}_2^{n-1} A_2 A_1 B_0, \\ \bar{A}_2^{n-1} A_2 A_1 A_0 B_3];$$

where $P_0, P_1, P_2,$ and P_3 are each $n \times 4n$. Then C_3 is the linear subspace spanned by the column vectors of $[P_3^T, 0, 0, 0]^T$; C_0 is the linear subspace spanned by the column vectors of $[0, P_0^T, 0, 0]^T$; C_1 is the linear subspace spanned by the column vectors of $[0, 0, P_1^T, 0]^T$; and C_2 is the linear subspace spanned by the column vectors $[0, 0, 0, P_2^T]^T$, where the zero matrices here have dimensions $4n \times n$.

We now exploit the decomposition of C in choosing a transformation for the system. Suppose C_i has dimension r_i , for $i=0,1,2,3$, so that $\text{rank } P_i = r_i$. For $i=0,1,2,3$, choose vectors $f_0^i, f_1^i, \dots, f_{r_i}^i$ that span the space spanned by the columns of P_i and let $T_1^i = [f_0^i, f_1^i, \dots, f_{r_i}^i]$. Choose vectors $f_{r_i+1}^i, f_{r_i+2}^i, \dots, f_n^i$ so that $f_0^i, f_1^i, \dots, f_n^i$ span R^n , and let $T_2^i = [f_{r_i+1}^i, f_{r_i+2}^i, \dots, f_n^i]$. Then for $i = 0,1,2,3$, T_1^i is $n \times r_i$ and T_2^i is $n \times (n-r_i)$. Now form

$$(67) \quad T = \begin{bmatrix} T_1^0 & 0 & 0 & 0 & T_2^0 & 0 & 0 & 0 \\ 0 & T_1^1 & 0 & 0 & 0 & T_2^1 & 0 & 0 \\ 0 & 0 & T_1^2 & 0 & 0 & 0 & T_2^2 & 0 \\ 0 & 0 & 0 & T_1^3 & 0 & 0 & 0 & T_2^3 \end{bmatrix} = [T_1, T_2],$$

where $T_1 = \text{diag} (T_1^0, T_1^1, T_1^2, T_1^3)$ and $T_2 = \text{diag} (T_2^0, T_2^1, T_2^2, T_2^3)$.

The rank of T is $4n$, so it has an inverse . Letting $U = T^{-1}$, we have that

$$(68) \quad U = \begin{bmatrix} U_1^0 & 0 & 0 & 0 \\ 0 & U_1^1 & 0 & 0 \\ 0 & 0 & U_1^2 & 0 \\ 0 & 0 & 0 & U_1^3 \\ U_2^0 & 0 & 0 & 0 \\ 0 & U_2^1 & 0 & 0 \\ 0 & 0 & U_2^2 & 0 \\ 0 & 0 & 0 & U_2^3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

where $U_1 = \text{diag} [U_1^0, U_1^1, U_1^2, U_1^3]$ and $U_2 = [U_2^0, U_2^1, U_2^2, U_2^3]$, so that,

$$(69) \quad T^{-1}T = UT = \begin{bmatrix} U_1^0 T_1^0 & 0 & 0 & 0 & U_1^0 T_2^0 & 0 & 0 & 0 \\ 0 & U_1^1 T_1^1 & 0 & 0 & 0 & U_1^1 T_2^1 & 0 & 0 \\ 0 & 0 & U_1^2 T_1^2 & 0 & 0 & 0 & U_1^2 T_2^2 & 0 \\ 0 & 0 & 0 & U_1^3 T_1^3 & 0 & 0 & 0 & U_1^3 T_2^3 \\ U_2^0 T_1^0 & 0 & 0 & 0 & U_2^0 T_2^0 & 0 & 0 & 0 \\ 0 & U_2^1 T_1^1 & 0 & 0 & 0 & U_2^1 T_2^1 & 0 & 0 \\ 0 & 0 & U_2^2 T_1^2 & 0 & 0 & 0 & U_2^2 T_2^2 & 0 \\ 0 & 0 & 0 & U_2^3 T_1^3 & 0 & 0 & 0 & U_2^3 T_2^3 \end{bmatrix}.$$

This implies that $I_{r_i} = U_1^i T_1^i$, $I_{(n-r_i)} = U_2^i T_2^i$, $O' = U_2^i T_1^i$, and $O'' = U_1^i T_2^i$, for $i=0,1,2,3$, where I_j denotes the $j \times j$ identity matrix and O' and O'' are zero matrices of dimensions $(n-r_i) \times r_i$ and $r_i \times (n-r_i)$, respectively.

The transformation matrix T and its inverse U allow us to transform the problem to controllability canonical form. Recall that the state vector is $y(t) = [(y^0(t))^T, (y^1(t))^T, (y^2(t))^T, (y^3(t))^T]^T$, where $y^i(t) = \begin{cases} x^i(t), & \text{if } i \equiv \text{mod}_4(t) \\ \text{an } n \times 1 \text{ matrix of zeros,} & \text{otherwise} \end{cases}$. Let $\tilde{y}(t) = T^{-1}y(t)$ and rewrite the problem as one of maximizing over choices of $u(t) = L(t)\tilde{y}(t)$ the criterion

$$(70) \quad J(\tilde{y}(t_0)) = E_{t_0} \left[\sum_{t=t_0}^{t_1-1} \{ \tilde{y}(t)^T \tilde{R} \tilde{y}(t) + u(t)^T Q u(t) \} + \tilde{y}(t_1)^T \tilde{K}(t_1) \tilde{y}(t_1) \right]$$

subject to

$$(71) \quad \tilde{y}(t+1) = \tilde{A} \tilde{y}(t) + \tilde{B}u(t) + T^{-1}e_{s(t+1)}\xi(t+1),$$

where

$$(72) \quad \tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 0 & 0 & U_{13}^0 A_{31} T_1^3 & 0 & 0 & 0 & U_{13}^0 A_{32} T_2^3 \\ U_{10}^1 A_{01} T_1^0 & 0 & 0 & 0 & U_{10}^1 A_{02} T_2^0 & 0 & 0 & 0 \\ 0 & U_{11}^2 A_{11} T_1^1 & 0 & 0 & 0 & U_{11}^2 A_{12} T_2^1 & 0 & 0 \\ 0 & 0 & U_{12}^3 A_{21} T_1^2 & 0 & 0 & 0 & U_{12}^3 A_{22} T_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{23}^0 A_{32} T_2^3 \\ 0 & 0 & 0 & 0 & U_{20}^1 A_{02} T_2^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U_{21}^2 A_{12} T_2^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{22}^3 A_{22} T_2^2 & 0 \end{bmatrix}$$

$$\text{or } \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \text{ with}$$

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0 & 0 & U_{13}^0 A_{31} T_1^3 \\ U_{10}^1 A_{01} T_1^0 & 0 & 0 & 0 \\ 0 & U_{11}^2 A_{11} T_1^1 & 0 & 0 \\ 0 & 0 & U_{12}^3 A_{21} T_1^2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & \tilde{A}_{11}^3 \\ \tilde{A}_{11}^0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{11}^1 & 0 & 0 \\ 0 & 0 & \tilde{A}_{11}^2 & 0 \end{bmatrix}$$

$$\tilde{A}_{12} = \begin{bmatrix} 0 & 0 & 0 & U_{13}^0 A_3 T_2^3 \\ U_{10}^1 A_0 T_2^0 & 0 & 0 & 0 \\ 0 & U_{11}^2 A_1 T_2^1 & 0 & 0 \\ 0 & 0 & U_{12}^3 A_2 T_2^2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & \tilde{A}_{12}^3 \\ \tilde{A}_{12}^0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{12}^1 & 0 & 0 \\ 0 & 0 & \tilde{A}_{12}^2 & 0 \end{bmatrix}$$

$$\tilde{A}_{22} = \begin{bmatrix} 0 & 0 & 0 & U_{23}^0 A_3 T_2^3 \\ U_{20}^1 A_0 T_2^0 & 0 & 0 & 0 \\ 0 & U_{21}^2 A_1 T_2^1 & 0 & 0 \\ 0 & 0 & U_{22}^3 A_2 T_2^2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & \tilde{A}_{22}^3 \\ \tilde{A}_{22}^0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{22}^1 & 0 & 0 \\ 0 & 0 & \tilde{A}_{22}^2 & 0 \end{bmatrix}$$

(73) $\tilde{B} = T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$, with

$$\tilde{B}_1 = \begin{bmatrix} 0 & 0 & 0 & U_{13}^0 B_3 \\ U_{10}^1 B_0 & 0 & 0 & 0 \\ 0 & U_{11}^2 B_1 & 0 & 0 \\ 0 & 0 & U_{12}^3 B_2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 & \tilde{B}_1^3 \\ \tilde{B}_1^0 & 0 & 0 & 0 \\ 0 & \tilde{B}_1^1 & 0 & 0 \\ 0 & 0 & \tilde{B}_1^2 & 0 \end{bmatrix}$$

$$(74) \quad \tilde{R} = T^T R T = \begin{bmatrix} \tilde{R}_{11}^0 & 0 & 0 & 0 & \tilde{R}_{12}^0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{11}^1 & 0 & 0 & 0 & \tilde{R}_{12}^1 & 0 & 0 \\ 0 & 0 & \tilde{R}_{11}^2 & 0 & 0 & 0 & \tilde{R}_{12}^2 & 0 \\ 0 & 0 & 0 & \tilde{R}_{11}^3 & 0 & 0 & 0 & \tilde{R}_{12}^3 \\ \tilde{R}_{21}^0 & 0 & 0 & 0 & \tilde{R}_{22}^0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{21}^1 & 0 & 0 & 0 & \tilde{R}_{22}^1 & 0 & 0 \\ 0 & 0 & \tilde{R}_{21}^2 & 0 & 0 & 0 & \tilde{R}_{22}^2 & 0 \\ 0 & 0 & 0 & \tilde{R}_{21}^3 & 0 & 0 & 0 & \tilde{R}_{22}^3 \end{bmatrix},$$

where $\tilde{R}_{jk}^i = (T_j^i)^T R_i T_k^i$, for $j, k = 1, 2$ and $i = 0, 1, 2, 3$,

$$\text{or} \quad \tilde{R} = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{21} & \tilde{R}_{22} \end{bmatrix} \quad \text{with}$$

$$\tilde{R}_{11} = \begin{bmatrix} T_1^{0T} R_0 T_1^0 & 0 & 0 & 0 \\ 0 & T_1^1 T R_1 T_1^1 & 0 & 0 \\ 0 & 0 & T_1^2 T R_2 T_1^2 & 0 \\ 0 & 0 & 0 & T_1^3 T R_3 T_1^3 \end{bmatrix} \equiv \begin{bmatrix} \tilde{R}_{11}^0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{11}^1 & 0 & 0 \\ 0 & 0 & \tilde{R}_{11}^2 & 0 \\ 0 & 0 & 0 & \tilde{R}_{11}^3 \end{bmatrix}$$

$$\tilde{R}_{22} = \begin{bmatrix} T_2^{0T} R_0 T_2^0 & 0 & 0 & 0 \\ 0 & T_2^1 T R_1 T_2^1 & 0 & 0 \\ 0 & 0 & T_2^2 T R_2 T_2^2 & 0 \\ 0 & 0 & 0 & T_2^3 T R_3 T_2^3 \end{bmatrix} \equiv \begin{bmatrix} \tilde{R}_{22}^0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{22}^1 & 0 & 0 \\ 0 & 0 & \tilde{R}_{22}^2 & 0 \\ 0 & 0 & 0 & \tilde{R}_{22}^3 \end{bmatrix}$$

$$\tilde{R}_{12} = \begin{bmatrix} T_1^{0T} R_0 T_2^0 & 0 & 0 & 0 \\ 0 & T_1^1 T R_1 T_2^1 & 0 & 0 \\ 0 & 0 & T_1^2 T R_2 T_2^2 & 0 \\ 0 & 0 & 0 & T_1^3 T R_3 T_2^3 \end{bmatrix} \equiv \begin{bmatrix} \tilde{R}_{12}^0 & 0 & 0 & 0 \\ 0 & \tilde{R}_{12}^1 & 0 & 0 \\ 0 & 0 & \tilde{R}_{12}^2 & 0 \\ 0 & 0 & 0 & \tilde{R}_{12}^3 \end{bmatrix}$$

and $\tilde{R}_{21} = \text{diag} [\tilde{R}_{12}^{0T}, \tilde{R}_{12}^{1T}, \tilde{R}_{12}^{2T}, \tilde{R}_{12}^{3T}] \equiv \text{diag} [\tilde{R}_{21}^0, \tilde{R}_{21}^1, \tilde{R}_{21}^2, \tilde{R}_{21}^3]$.

$\tilde{K}(t_1) = T^T K(t_1) T$ has a structure analogous to \tilde{R} . Note that if R is symmetric and negative definite, then so is \tilde{R} [Noble, p.393], and similarly for $K(t_1)$ and $\tilde{K}(t_1)$. We know that the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable.

The stabilizability of the system depends on the stability of \tilde{A}_{22} . The stabilizability of \tilde{A}_{22} can be checked directly, but we can also develop sufficient conditions which may be more convenient to compute. To this end, note that

Theorem 7: The matrix \tilde{A}_{22} is stable provided at least one of the matrices $\bar{A}_3 \equiv (\tilde{A}_{22}^3 \ \tilde{A}_{22}^2 \ \tilde{A}_{22}^1 \ \tilde{A}_{22}^0)$, $\bar{A}_0 \equiv (\tilde{A}_{22}^0 \ \tilde{A}_{22}^3 \ \tilde{A}_{22}^2 \ \tilde{A}_{22}^1)$, $\bar{A}_1 \equiv (\tilde{A}_{22}^1 \ \tilde{A}_{22}^0 \ \tilde{A}_{22}^3 \ \tilde{A}_{22}^2)$, and $\bar{A}_2 \equiv (\tilde{A}_{22}^2 \ \tilde{A}_{22}^1 \ \tilde{A}_{22}^0 \ \tilde{A}_{22}^3)$ is stable.

Proof: Let λ be any eigenvalue of \tilde{A}_{22} and partition its eigenvector x into 4 nx1 blocks x_0, x_1, x_2, x_3 , as $x = (x_0^T, x_1^T, x_2^T, x_3^T)^T$. Then $\tilde{A}_{22}x = \lambda x$, or

$$\begin{bmatrix} 0 & 0 & 0 & \tilde{A}_{22}^3 \\ \tilde{A}_{22}^0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{22}^1 & 0 & 0 \\ 0 & 0 & \tilde{A}_{22}^2 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{22}^3 x_3 \\ \tilde{A}_{22}^0 x_0 \\ \tilde{A}_{22}^1 x_1 \\ \tilde{A}_{22}^2 x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_0 \\ \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}.$$

By definition $x \neq 0$, so that either $x_0 \neq 0$, $x_1 \neq 0$, $x_2 \neq 0$, or $x_3 \neq 0$. Since the nonzero eigenvalues of \tilde{A}_{22} determine its stability, assume $\lambda \neq 0$.

Now suppose $x_3 = 0$. Then $\tilde{A}_{22}^3 x_3 = 0 = \lambda x_0$, so $x_0 = 0$. This in turn implies $\tilde{A}_{22}^0 x_0 = 0 = \lambda x_1$, so $x_1 = 0$, which in turn implies $x_2 = 0$.

Thus $x_3 = 0$ implies $x = 0$, which cannot be, so $x_3 \neq 0$. By similar reasoning, $x_0 \neq 0$, $x_1 \neq 0$, $x_2 \neq 0$.

Note that $(\tilde{A}_{22})^4 = \text{diag}(\bar{A}_3, \bar{A}_0, \bar{A}_1, \bar{A}_2)$. Also note that $(\tilde{A}_{22})^4 x = (\tilde{A}_{22})^3 \tilde{A}_{22} x = \lambda (\tilde{A}_{22})^3 x = \lambda^2 (\tilde{A}_{22})^2 x = \lambda^3 \tilde{A}_{22} x = \lambda^4 x$, so that λ^4 is an eigenvalue of $(\tilde{A}_{22})^4$, still with eigenvector x . Hence,

$$(\tilde{A}_{22})^4 x = \begin{bmatrix} \bar{A}_3 x_0 \\ \bar{A}_0 x_1 \\ \bar{A}_1 x_2 \\ \bar{A}_2 x_3 \end{bmatrix} = \begin{bmatrix} \lambda^4 x_0 \\ \lambda^4 x_1 \\ \lambda^4 x_2 \\ \lambda^4 x_3 \end{bmatrix} .$$

This implies that if λ is a nonzero eigenvalue of \tilde{A}_{22} , then λ^4 is an eigenvalue of $\bar{A}_3, \bar{A}_0, \bar{A}_1$, and \bar{A}_2 . If any of these four matrices is stable, then $|\lambda| < 1$ and \tilde{A}_{22} is stable. In other words, stability of either $\bar{A}_3, \bar{A}_0, \bar{A}_1$, or \bar{A}_2 is sufficient for the stability of \tilde{A}_{22} .

We have now shown a convenient controllability canonical form for the linear optimal regulator problem described in equations (13) - (20) and discussed ways of checking if the system is stabilizable. Then, provided $\tilde{R}_{11}(t_1)$ and \tilde{R}_{11} are negative semidefinite, the system converges as described in theorems 5 and 6 provided only that $(\tilde{A}_{11}, \tilde{G})$ is detectable, where $\tilde{R}_{11} = -\tilde{G}^T \tilde{G}$. The detectability of $(\tilde{A}_{11}, \tilde{G})$ can be checked by determining, by the methods described above, if $(\tilde{A}_{11}^T, \tilde{G}^T)$ is stabilizable.

If the sufficient conditions for convergence are satisfied, we know that the value of the problem (for a given initial vector) and the matrix $\tilde{K}(t_0)$ converge as $t_0 \rightarrow -\infty$. In addition, the feedback matrix $L(t_0)$ converges to L , the steady-state feedback matrix. From the first half of this chapter we know that L is of the form $\text{diag}(L^0, L^1, L^2, L^3)$, where

$$L^i = \lim_{t_0 \rightarrow -\infty} L_{t_0}^i, \text{ with } L_{t_0}^i \text{ defined according to equation (36). The } i^{\text{th}}$$

block along the diagonal of L is itself a steady-state feedback rule, a rule which is used (i.e., multiplies the nonzero components of $\tilde{y}(t)$) in every fourth period when $\text{mod}_4(t) = i$ (more generally, every p periods when $\text{mod}_p(t) = i$).

If we regard the periodicity of the coefficients in the system defined by equations (1) - (3) with $s(t+p) = s(t)$ as arising from a seasonal variation in technology, then each of the steady-state feedback matrices L^i on the diagonal of L can be thought of as a season-specific decision rule. That is, the fact that the four $m \times n$ feedback rules L^0, L^1, L^2, L^3 on the diagonal of L need not be identical means that, in the steady state of the original n -dimensional, periodic-coefficients, linear optimal regulator problem (1) - (3), a given value of the state vector $x(t)$ may require (in order to maximize the value function) different responses (i.e., settings of the control vector) in different seasons.

The feedback matrices, or decision rules, L^0, L^1, L^2 , and L^3 may differ not only in their "intercepts" but also in any of the "slope"

coefficients. This means that one common method for accounting for seasonality in econometrically estimated decision rules--allowing only the intercept term to vary with the seasons--is an overly restrictive specification in many cases. A more general approach is to allow all the coefficients of each decision rule to vary with the seasons, but subject to cross-equation restrictions derived from the optimization problem. Such cross-equations restrictions are satisfied by feedback matrices calculated as above, but the iterative technique used to calculate them is not a convenient means of actually deriving the restrictions. A technique for deriving these and other cross-equations restrictions on coefficients in decision rules for a certain class of linear-quadratic maximization problems has been presented by Todd.

Footnotes

1/That is,

$$E \xi_t = 0, \text{ for all } t,$$

$$E \xi_t \xi_s^T = 0, \text{ for } t \neq s,$$

$$E \xi_t \xi_t^T = \Psi_t, \text{ with } \Psi_t > 0,$$

and $E x_t \xi_s^T = 0, \text{ for } t < s.$

2/The second-order conditions for maximization are satisfied because $Q_s(t_1-1)$ is negative semidefinite and $R_s(t_1-1)$ is negative definite.

3/The key is to notice that $B^T \text{diag} [Z_0, Z_1, \dots, Z_{p-1}] B = \text{diag} [B_{s(0)}^T Z_1 B_{s(0)}, B_{s(1)}^T Z_2 B_{s(1)}, \dots, B_{s(p-1)}^T Z_0 B_{s(p-1)}],$

but that

$$\begin{aligned} & B \text{diag} [Z_0, Z_1, \dots, Z_{p-1}] B^T \\ &= \text{diag} [B_{s(p-1)} Z_{p-1} B_{s(p-1)}, B_{s(0)} Z_0 B_{s(0)}, \dots, \\ & B_{s(p-2)} Z_{p-2} B_{s(p-2)}]. \end{aligned}$$

The multiplications with A and A^T are analogous.

4/This definition is normally stated as a theorem, following a definition of stabilizability that states that the unstable subspace of the system is contained in its controllable subspace, but these concepts are not needed here. They are presented and used in Kwakernaak and Sivan, however [pp. 62-63].

5/The stabilizability of (A,B) is independent of the choice of transformation matrix T , for it can be shown that the controllable subspace of the system and the eigenvalues \tilde{A}_{22} do not depend on the choice of the columns f_1, f_2, \dots, f_n of T [Sargent, p. 15].

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