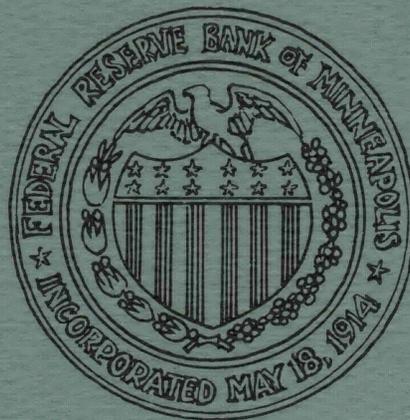
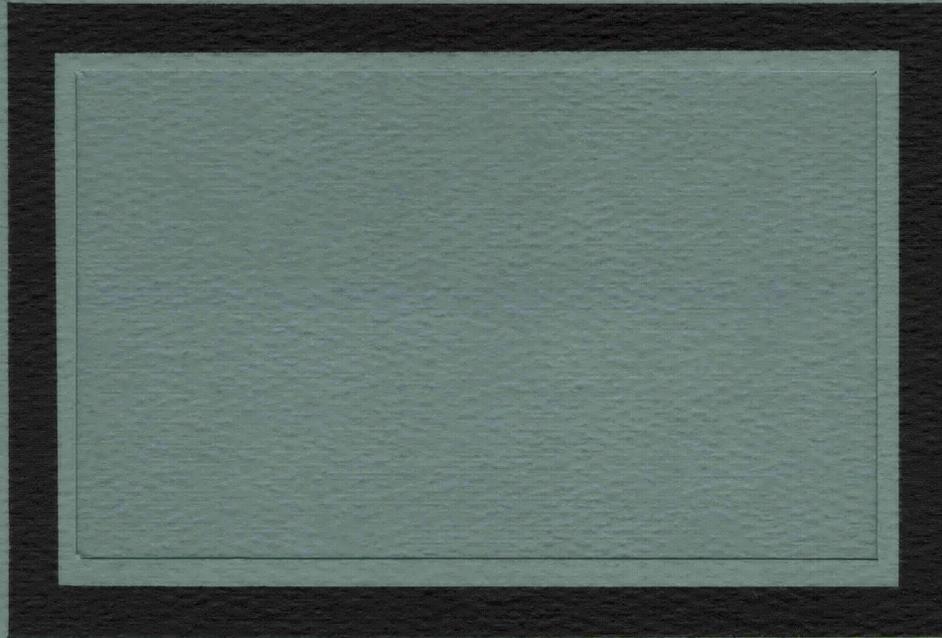


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Notes on the Investment Schedule

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### Notes on the Investment Schedule

As a comparison of the "Keynesian" model with Tobin's Dynamic Aggregative Model reveals, whether or not it is assumed that there exists a market in stocks of capital at each moment has drastic theoretical implications, particularly about the potency of fiscal policy as a device for inducing short-run movements in output and employment. The structure of the Keynesian model depends sensitively on ruling out a market in existing stocks of capital and instead positing a demand schedule on the part of firms for a finite rate of addition per unit time to their capital stocks. That element of the Keynesian model is perhaps its most essential piece, ruling out as it does the movements of capital across firms and industries that thwart fiscal policy in Tobin's model; at the same time, the investment schedule is the weakest part of the Keynesian model from a theoretical viewpoint, being defended (at least until recently) on a very ad hoc basis.

These notes describe the most successful attempt to rationalize the Keynesian investment schedule, a line of work due to Eisner and Strotz, Lucas, Gould, and Treadway.\* The key to the theory is the assumption that there are costs associated with adjusting the capital stock at a rapid rate per unit of time, and that these costs increase rapidly with the absolute rate of investment, so rapidly that the firm

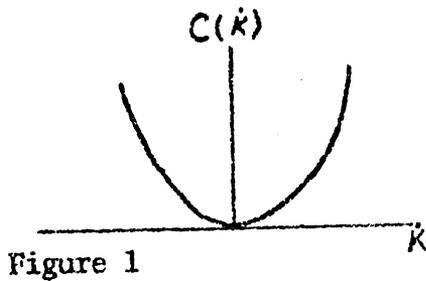
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\* R. Eisner and R. H. Strotz, "Determinants of Business Investment," Research Study Two in Impacts of Monetary Policy, (Englewood Cliffs, N. J.: Prentice Hall, 1963); R. E. Lucas, "Adjustment Costs and the Theory of Supply," Journal of Political Economy, 75 (1967), 321-334; J. P. Gould, "Adjustment Costs in the Theory of Investment of the Firm," Review of Economic Studies, 35 (1969) 47-56; A. B. Treadway, "On Rational Entrepreneurial Behavior and the Demand for Investment," Review of Economic Studies, 36 (1969), 227-240.

never attempts to achieve a jump in its capital stock at any moment. These adjustment costs occur at a rate per unit of time (measured in capital goods per unit of time) described by the twice differentiable function  $C(\dot{K})$ , which obeys

$$C(\dot{K}) \geq 0 \text{ as } \dot{K} \geq 0,$$

$$C''(\dot{K}) > 0, C(0) = 0.$$



Costs of adjusting the capital stock are nonnegative and increase at an increasing rate with the absolute value of investment.

The firm's discounted net cash flow net of costs of adjustment is defined to be

$$f(N(t), K(t), \dot{K}(t), t) = e^{-rt} [p(t)F(K(t), N(t)) - w(t)N(t) - J(t)\delta K(t) - J(t)\dot{K}(t) - J(t)C(\dot{K})]$$

where  $J(t)$  is the price of capital goods at time  $t$ , and  $r$  is the instantaneous interest rate, assumed constant over  $[0, T]$ . We continue to assume that  $F(K, N)$  is linearly homogeneous in  $K$  and  $N$ . The firm chooses paths of  $N$  and  $K$  over time to maximize its present value over the time interval  $[0, T]$ , which is

$$PV = \int_0^T f(N(t), K(t), \dot{K}(t), t) dt + S(T)K(T)e^{-rt}$$

where  $S(T)K(T)$  is the scrap value, if any, of the capital stock at time  $T$ . We think of  $T$  as being in the very distant future. The firm operates in competitive markets for output and labor, being able to rent all the

labor it desires at the wage  $w$ , and to sell all the output it wants at the price  $p$ . The firm starts out with capital stock  $K(0)$ .

Among the necessary conditions for an extremum for present value are the "Euler equations"

$$\frac{\partial f}{\partial N} = 0 \quad t \in [0, T]$$

$$\frac{\partial f}{\partial K} - \frac{d}{dt} \frac{\partial f}{\partial \dot{K}} = 0 \quad t \in [0, T].$$

(A heuristic explanation of these Euler equations is contained in the appendix to these notes.) Evaluating these derivatives, we have

$$\frac{\partial f}{\partial N} = e^{-rt} [p(t) \frac{\partial F}{\partial N} - w(t)]$$

$$\frac{\partial f}{\partial K} = e^{-rt} [p(t) \frac{\partial F}{\partial K} - J(t)\delta]$$

$$\frac{\partial f}{\partial \dot{K}} = -e^{-rt} [J(t) + J(t)C'(\dot{K})]$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial f}{\partial \dot{K}} &= -e^{-rt} [J(t) + \dot{J}(t)C'(\dot{K}) + J(t)C''(\dot{K})\dot{K}] \\ &+ re^{-rt} [J(t) + J(t)C'(\dot{K})]. \end{aligned}$$

So for our problem the Euler equations become

$$(1) \quad p(t) \frac{\partial F}{\partial N} - w(t) = 0 \quad \text{or} \quad \frac{\partial F}{\partial N} = \frac{w(t)}{p(t)}$$

and

$$(2) \quad p \frac{\partial F}{\partial K} - J\delta - rJ + \dot{J} - (rJ - \dot{J})C'(\dot{K}) + JC''(\dot{K})\dot{K} = 0$$

Equation (1) requires that the marginal product of labor equal the real wage at each moment, an equation that determines the labor-capital ratio at each moment (since  $F(N, K)$  is linearly homogeneous). Equation (2) is

a differential equation that determines the (finite) rate of growth of the capital stock at each moment. To simplify the problem, we now assume that firms expect the prices  $J(t)$ ,  $p(t)$ , and  $w(t)$  to grow over time at the same constant rate per unit time  $\pi$ , over the entire horizon of our problem. This makes  $\dot{J}/J = \dot{p}/p = \dot{w}/w = \pi$  for all  $t$ , and leaves relative prices and wages constant over time. Furthermore, assume for simplicity that the cost-of-change function is quadratic, so that

$$C(\dot{K}) = \frac{\gamma}{2} \dot{K}^2 \quad \gamma > 0.$$

On this assumption, equation (2) becomes

$$P \frac{\partial F}{\partial K} - J\delta - rJ + \dot{J} - (rJ - \dot{J})\gamma\dot{K} + J\gamma\ddot{K} = 0.$$

Dividing by  $J$  and solving for  $\ddot{K}$  gives

$$(3) \quad \ddot{K} = \frac{1}{\gamma} [r + \delta - \frac{\dot{J}}{J} - \frac{P}{J} F_K] + (r - \frac{\dot{J}}{J})\dot{K}.$$

On our assumptions, since all relative prices are constant over time, and since  $r$  and  $\dot{J}/J$  are constant over time, the above equation is a fixed coefficient, linear differential equation in  $\dot{K}$ :

$$(4) \quad \frac{d\dot{K}}{dt} = A + B\dot{K}$$

$$\text{where } A = \frac{1}{\gamma} [r + \delta - \pi - \frac{P}{J} F_K]$$

$$B = (r - \pi) > 0.$$

Since  $w/p$  is constant over time, so is  $N/K$ , making  $F_K$  and therefore  $A$  constant over time. The differential equation (4) has the solution

$$(5) \quad \dot{K} = \alpha e^{Bt} - \frac{A}{B}$$

where  $\alpha$  is a constant chosen to insure that an initial condition or terminal condition is satisfied. If  $\alpha = 0$ , then  $\dot{K} = -A/B$  for all  $t$ .

$w(t) = e^{\pi t} w(0)$ , and  $J(t) = e^{\pi t} J(0)$ . Noting that  $B = (r - \pi)$ , we have

$$f(\quad) = e^{-Bt} D \left[ -\frac{A}{B} t + \frac{\alpha}{B} e^{Bt} + K(0) - \frac{\alpha}{B} \right] - e^{-Bt} J(0) \left( \alpha e^{Bt} - \frac{A}{B} \right)$$

$$- e^{-Bt} \left( \alpha^2 e^{2Bt} - 2\alpha e^{Bt} + \left(\frac{A}{B}\right)^2 \right) \frac{\gamma}{2} J(0)$$

$$f(N(t), K(t), \dot{K}(t), t) = D \left[ -\frac{A}{B} t e^{-Bt} + \frac{\alpha}{B} + K(0) e^{-Bt} \right]$$

$$- J(0) \left\{ \alpha - \frac{A}{B} e^{-Bt} \right\}$$

$$- \frac{\gamma}{2} \left\{ \alpha e^{Bt} - 2\alpha + \left(\frac{A}{B}\right)^2 e^{-Bt} \right\} J(0).$$

As  $t \rightarrow \infty$ , the first term in braces approaches  $\alpha/B$ ; the second term in braces approaches  $\alpha$ , while the third term in braces approaches  $\infty$  unless  $\alpha = 0$ , in which case it approaches zero. These calculations imply that as  $t$  becomes large, the discounted net cash flow  $f(\quad)$  becomes a larger and larger negative number (since the last term in braces is multiplied by  $\gamma/2$ ), unless  $\alpha = 0$ . This occurs because the rate of investment is increasing approximately exponentially, causing costs of adjustment to rise at an even faster exponential rate. These costs of adjustment become so large eventually that they swamp the firm's revenue, and lead to large negative net returns. For large enough  $T$ , that will make present value a very large negative number. Clearly, such paths are not optimal ones for the firm to follow, even though they satisfy the Euler equations. To rule out such paths, the condition  $\alpha = 0$  must be met, implying that the pertinent solution of our differential equation (4) is

$$\dot{K} = -\frac{A}{B}$$

or

$$(7) \quad \dot{K} = \frac{1}{\gamma} \left[ \frac{P_F K - (r + \delta - \pi)}{r - \pi} \right],$$

Notice that in the context of a one-sector model,  $p = J$ , so that (7) is a version of our Keynesian investment schedule

$$\dot{K} = I(q-1) \quad I' > 0$$

where  $q = (F_K - (r+\delta-\pi))/(r-\pi)$ .

Derivations of the Keynesian investment schedule along the lines sketched above seem to provide the most satisfactory theoretical foundations yet laid down for that schedule. The theory obviously depends critically on the assumption that costs of adjustment increase at an increasing rate with the absolute value of investment ( $C'' > 0$ ). The arbitrary nature of that critical assumption explains why some economists are uneasy with the notion of the investment schedule.

Appendix: Heuristic Explanation of the Euler Equations

We are interested in choosing time paths of  $x(t)$  and  $y(t)$ ,  $t \in [0, T]$  to obtain the extremum of the functional

$$(1) \quad J(x, y, \dot{y}, t) = \int_0^T f(x(t), y(t), \dot{y}(t)) dt.$$

Among the necessary conditions for  $J$  to obtain an extremum are the Euler equations:

$$\frac{\partial f}{\partial x(t)} = 0 \quad t \in [0, T]$$

$$\frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial f}{\partial \dot{y}} = 0 \quad t \in [0, T].$$

To motivate these equations, we consider the following discrete time approximation to (1):

$$(2) \quad \tilde{J} = \sum_{t=0}^T \epsilon f(x(t), y(t), \frac{y(t+\epsilon) - y(t)}{\epsilon})$$

where  $t = 0, \epsilon, 2\epsilon, \dots, T - \epsilon, T$ . Notice that the limit of the sum in equation (2) as  $\epsilon$  approaches zero is the integral in equation (1). We propose to study the first-order conditions for obtaining an extremum of  $\tilde{J}$  as  $\epsilon$  approaches zero. It is illuminating to write out several terms of  $\tilde{J}$  explicitly:

$$\begin{aligned} \tilde{J} = & \epsilon f(x(0), y(0), \frac{y(\epsilon) - y(0)}{\epsilon}) + \epsilon f(x(\epsilon), y(\epsilon), \frac{y(2\epsilon) - y(\epsilon)}{\epsilon}) \\ & + \dots + \epsilon f(x(n), y(n), \frac{y(n+\epsilon) - y(n)}{\epsilon}) + \dots \end{aligned}$$

Differentiating  $\tilde{J}$  with respect to  $x(t)$ ,  $t = 0, \epsilon, \dots, T$ , and setting the derivatives to zero we obtain

$$(3) \quad \frac{\partial f}{\partial x(t)} = 0, \quad t = 0, \epsilon, \dots, T.$$

Setting the partial derivative of  $\tilde{J}$  with respect to  $y(t+\epsilon)$  equal to zero for  $t = 0, \dots, T-\epsilon$ , we have

$$\begin{aligned} \frac{\partial \tilde{J}}{\partial y(t+\epsilon)} &= \epsilon \frac{\partial f(x(t), y(t), \frac{y(t+\epsilon)-y(t)}{\epsilon})}{\partial (\frac{y(t+\epsilon)-y(t)}{\epsilon})} \cdot \frac{1}{\epsilon} \\ &+ \epsilon \frac{\partial f(x(t+\epsilon), y(t+\epsilon), \frac{y(t+2\epsilon)-y(t+\epsilon)}{\epsilon})}{\partial y(t+\epsilon)} \\ &- \epsilon \frac{\partial f(x(t+\epsilon), y(t+\epsilon), \frac{y(t+2\epsilon)-y(t+\epsilon)}{\epsilon})}{\partial (\frac{y(t+2\epsilon)-y(t+\epsilon)}{\epsilon})} \cdot \frac{1}{\epsilon} = 0 \end{aligned}$$

Dividing by  $\epsilon$  and rearranging, we obtain

$$\begin{aligned} &\frac{\partial f(x(t+\epsilon), y(t+\epsilon), \frac{y(t+2\epsilon)-y(t+\epsilon)}{\epsilon})}{\partial y(t+\epsilon)} \\ &- \frac{1}{\epsilon} \left\{ \frac{\partial f(x(t+\epsilon), y(t+\epsilon), \frac{y(t+2\epsilon)-y(t+\epsilon)}{\epsilon})}{\partial (\frac{y(t+2\epsilon)-y(t+\epsilon)}{\epsilon})} \right. \\ &\left. - \frac{\partial f(x(t), y(t), \frac{y(t+\epsilon)-y(t)}{\epsilon})}{\partial (\frac{y(t+\epsilon)-y(t)}{\epsilon})} \right\} = 0. \end{aligned}$$

Taking limits as  $\epsilon$  goes to zero we have

$$(4) \quad \frac{\partial f}{\partial y(t)} - \frac{d}{dt} \frac{\partial f}{\partial \dot{y}(t)} = 0 \quad \text{for } t \in [0, T].$$

Equations (3) and (4) are the "Euler equations" associated with our continuous-time extremum problem. It is not surprising that the limits of the discrete time marginal conditions coincide with continuous time Euler equations, since the limit of our discrete time  $\tilde{J}$  equals the continuous time  $J$ .

