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AN EQUILIBRIUM MODEL OF QUILTS
UNDER OPTIMAL CONTRACTING

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ABSTRACT

In this article we use the techniques developed in examining optimal contracting with imperfect information to build a simple equilibrium model of a labor market with imperfect information. We then use the model to examine the effects that imperfect information imposes on labor markets, particularly when compared with full information and noncontractual base lines. We demonstrate that quits increase in periods of high output, without postulating exogenous price rigidity.

More people quit their jobs when the economy is booming than when it is in a slump.^{1/} However, our explanations of this phenomenon are fairly unsatisfactory--indeed, hardly less ad hoc than the "animal spirits" rationalization (when times are good, workers feel bold enough to leave current employment; when times are bad, they prefer the security of the old job).

One economic rationalization appeals to fixed wage rates: If wages for current employees are sticky but wages for new hires are flexible, then we will observe increased demand for labor met by an increased mobility. Of course, we need a good explanation for the wage rigidity and why it should be less for new hires than for current employees.

A second, and theoretically more satisfying rationalization, hinges on search markets. Individuals are assumed not to know the alternative employment opportunities available to them, and must quit their jobs to engage in search in order to determine what is out there. From this starting point, various devices may be used to get the result that search is more valuable in boom times than in slumps.^{2/}

Unfortunately for the usefulness of these explanations, it would appear that search is not an important part of the employment process for job quitters, as opposed to job losers. Search explanations have always had to contend with the question of why it is more costly to search when employed than when unemployed; and it would appear that for job quitters this question is particularly relevant. Apparently most individuals who quit one job already have their next job offer in hand.^{3/}

However, this observation would appear to leave us in a bind: Most quits are voluntary changes by employees from one job to a different (presumably more productive) job. If the business cycle is an across-the-board variation in the demand for labor, why then should mobility relate at all to this demand change?

This paper demonstrates that such a link can indeed be forged, provided we include imperfect information, not as previously, on the part of the quitting employees, but on the part of the firms employing them. We construct an equilibrium model of contracting between workers and firms, where contracts act both to allocate labor and to reallocate risk. We emphasize that the contracts chosen by firms and workers are optimal: Parties are

free to agree to make wages as fixed or flexible as they desire, and indeed, contracts mimicking the spot market would be perfectly feasible, although we will see that such agreements would be inferior to others available.

The only limitation on the freedom of individuals to contract will be the informational restriction of the employers, which will place some incentive compatibility constraints on the contracts chosen. The particular informational restrictions are described below. They seem to us natural ones, although of course the reader must judge for him or herself.

The importance of these informational restrictions is demonstrated in the text by comparison of our contracting equilibrium to a contract equilibrium under full information. Under full information it turns out, strikingly enough, that real wages are rigid, but with expected number of quits independent of the business cycle.

However, in the imperfect information contracting equilibrium the expected number of quits rises with higher economy-wide average productivity, despite the fact that wages increase as well.

In deriving the market equilibrium below, we start from the optimization problem faced by each firm/worker pair in establishing the optimal contract between them. This individual optimization problem is a special case of the one described in Kahn [1984]. The interested reader is referred there for further details.^{4/}

The Structure

For simplicity we assume that all individuals' preferences can be described by the same expected utility function $u(y_i)$ where y_i is the total compensation i receives. The function u is increasing and strictly concave.^{5/}

The individual owns an indivisible unit of labor; he can use it only with one firm. The individual's productivity is stochastic. It is assumed to differ according to which firm employs him. (If it did not, there would never be a reason to switch jobs.) We assume firms are divided into N "industries." The individual's productivity draw is identical at all firms within an industry, but draws are independent across industries. Let the matrix $X = (x_{ij})$ where x_{ij} indicates i 's productivity in the j th industry. We let $F(\cdot)$ represent the distribution of x_{ij} and we assume $F(\cdot)$ is the integral of a density function which is positive and continuous on the interval $[a,b]$ and zero elsewhere.

We assume that the owners of the firms are risk neutral. A firm can hire as many or as few employees as it chooses. Although each employee may have different productivity and may receive different compensation, we will assume for simplicity that each individual's productivity is independent of the number of other individuals hired, and that x_{ij} is independent of $x_{i'j'}$ for $i \neq i'$. All these assumptions are designed to keep the problem as symmetric as possible; they can be relaxed without difficulty.

All firms and employees know the distribution of variables in the matrix X . In addition, each firm in industry j

observes the realization of all elements in column j , and each individual observes the realizations of all elements in row i . In other words, each employee knows his productivity at any job in the economy, and each firm knows how productive any individual would be if he joined. Thus we allow each agent in the economy access to a large amount of information--more, in fact, than is necessary for our results. We do so to keep the analysis simple and to have the results in a model as close to full information as possible.

The key limitation we will place on information is a natural one: We assume that a firm cannot observe how productive a worker would be if he were employed in a different industry; nor can the firm verify any employee's claim as to the compensation another industry might offer him.

Spot Markets and Full Insurance

Provided there are several firms within each industry, spot markets for labor may be simply described: In equilibrium the employee works for a firm in the industry where he is most productive, and receives compensation

$$v_i = \max_j x_{ij}$$

Suppose we initially assigned individuals randomly to firms. Then, because of the independence of all draws, we would expect in any realization of X that a proportion $(N-1)/N$ of the employees would find that they would be more productive elsewhere: That is to say, with a spot market there is no systematic variation in the number of "quits."

If there were opportunities for full insurance, the employee would contract at the beginning of time to deliver his pay v_i to the insuring company in return for a guaranteed amount equal to the expectation of v_i . The choice of employer would be the same as in the spot market; again, quits would not vary systematically with any other variable.

Incorporation of Imperfect Information Contracts

Now let us return to the imperfect information framework. Suppose we allow an employee to sign a contract with any one firm before realizations of X are observed by any agent. We assume contracts may be made explicitly contingent on any information either the firm or the employee can observe. However, if the information is observable by only one party to the contract, then the contract must be made incentive-compatible in order to ensure that the information is honestly revealed to the uninformed party.

How, it might be asked, could we expect any contracts to result in this framework? After all, a contract is a precommitment to work for the firm in certain circumstances--and by our assumptions, it is never the case that staying with the firm is more productive than leaving it.

The answer is that part of an individual's gain from signing up with an initial firm is the insurance it provides. This insurance benefit is so great that it may pay to make arrangements to stay with the firm in circumstances which would be inefficient in a full information world.

This point can be illustrated by taking an extreme case. Suppose the worker was completely unproductive at a firm no

matter what the state. Could there be any advantage in signing up with such a firm? The answer is yes: The worker would like complete insurance but this is impossible since he cannot convey to the firm the pay he will receive in the spot market. But there is a way he can partially provide the information simply by refraining from taking a job elsewhere. This (in)action is his signal that his productivity outside is sufficiently low and that the firm should pay the insurance. On the other hand, in situations in which outside jobs are very promising, he pays the firm for the privilege of taking an outside offer.

Since all firms and industries are identical ex ante, we do not need to bother with the choice of which firm the individual contracts with. Because we are assuming all agents' productivities are independent draws and that there is no interaction between various agents' productivity within a firm, the only information relevant for establishing the contract is the vector X_i , the i th row of the matrix X , containing individual i 's realized productivities at all firms.

Formally, a contract between worker i and a firm in industry j can be described by a pair of functions $y(X_i)$ and $l(X_i)$ representing the payment to the individual and the labor required of him for any realization of the variables X_i (thus l takes the values 0 or 1). However, given the incentive compatibility restrictions, compensation can be described more simply by a pair of functions $s(x_{ij})$ and $p(x_{ij})$ where s represents the payment an individual receives from the firm and p represents a penalty extracted should he quit the firm (so that the initial firm pays

him $s - p$ if he quits). In applications $s - p$ could be positive if, for example, pensions are vested.

Since the realization of x_{ij} can be observed by both employer and employee, we make both functions contingent upon this variable. Because of the incentive compatibility restrictions, an employer can do no better than to leave the quit decision in the hands of the employee. Given the realizations of X_i the individual simply chooses to leave the firm if the penalty for leaving is less than spot pay. Thus

$$l(X_i) = 1 \text{ if } W_i(X_i) - p(x_{ij}) > 0$$

$$l(X_i) = 0 \text{ if } W_i(X_i) - p(x_{ij}) < 0$$

where $W_i(X_i)$ is the compensation the individual receives if the realization of his productivity is X_i and he leaves the firm he has contracted with in order to sell his services elsewhere. Since l depends on X_i only through x_{ij} and W_i we will henceforth write l as a function of these two variables.

Equilibrium

We assume the following market structure: Before the realization of X is revealed, there is trading in labor contracts. As mentioned before, an agent can contract with at most one firm. After X is revealed, there is trading of labor on a spot market and then production and consumption take place. In effect this means that if an individual who has contracted to work for one firm chooses to work for another firm, he receives the (possibly negative) amount $s - p$ from the initial firm, and the

spot wage from the new firm. In the initial period, firms compete through their offers of contracts. Formally speaking, since firms and employees are ex ante identical, we may regard this competition as a Nash equilibrium among firms where the space of contracts is the correct strategy space.

In general such a strategy space might be complicated to examine, since the most profitable contract for one firm to offer might vary in hard-to-determine ways with the menu of competitors' contracts. These difficulties might be great if changes in one employer's offered contract were to induce complicated changes in $W_i(X_i)$. However, as noted before, our assumptions guarantee that in a spot market wage is always equal to productivity at the best firm. Thus we know that

$$(1) \quad W_i(X_i) = \max_j (x_{ij})$$

independent of any contract offered by any firm in the first period. Then equilibrium in the first period market for contracts is characterized by finding contracts $(s(x), p(x), l(x, w))$ which maximize employee utility, subject to two constraints:

- (a) that each firm makes nonnegative expected profits given the contracts it offers and
- (b) that the distribution of ex post offers on the spot market is described by (1).

Such contracts will be called "optimal" contracts. If optimal contracts yield expected utility which is greater than

$$E(W_i(X_i))$$

Then all employees will sign such contracts. Otherwise, none will, and trade will occur solely on the spot market.

Optimal Contracts

Let $G(z)$ be the distribution of the best offer outside of industry j . Then

$$G(z) = F^{N-1}(z)$$

and the joint distribution of (x, z) is

$$(2) \quad F(x)F^{N-1}(z)$$

Define

$$(3) \quad v = \max(x, z)$$

so that v is the best offer available on the spot market. We now wish to find the triple $\{s(\cdot), p(\cdot), \ell(\cdot)\}$ which maximizes

$$(4a) \quad Eu(s(x) + (v - p(x))(1 - \ell(x, v)))$$

subject to

$$(4b) \quad \begin{aligned} \ell(x, v) &= 1 \text{ if } v < p(x) \\ &= 0 \text{ if } v > p(x) \end{aligned}$$

and

$$(4c) \quad E\{-s(x) + x\ell(x, v) + p(x)(1 - \ell(x, v))\} \geq 0$$

(Expectations are taken with respect to the distribution described in (2).) We use v in the expression rather than z , since an employee could always leave to take a job at another firm in the same industry.

The appendix demonstrates that, under the distributional assumptions of the next section:

Theorem 1. An optimal contract exists.

Theorem 2. In the optimal contract, individuals have no incentive to quit to work for other firms in the same industry.

Theorem 3. In the optimal contract, the expected frequency of quits conditional on x is a nondecreasing function of x .

Theorem 4. In the optimal contract, employees remain with their initial firms whenever they are most productive at that firm. In addition they remain with their initial firms in some cases where their productivity elsewhere is higher.

In the optimal contract, pay is generally higher for workers who are ex post more productive, and penalties for leaving are greater. This holds despite the fact that the risk-averse employee would prefer to receive constant wages while employed, because such variation is necessary to encourage more valuable employees to leave only when they have particularly lucrative offers elsewhere.

In a two-period model the increasing penalty might be interpreted in the following way: Part of the employee's first-period compensation is deferred into a nonvested pension scheme,

whose value will be connected to the terminal salary the employee achieves. Thus the penalty for leaving prematurely is increasing in second-period productivity.

We can show the optimal contract dominates the spot market solution by a simple variational argument. Start with a contract which mimics the spot market ($s(x) = x$; $p(x) = 0$). Then increasing p and s for any x can improve expected utility while leaving profits unchanged.

Theorem 4 states that there is insufficient mobility relative to a spot market solution (or relative to a full-information insurance solution). But expected mobility is higher when draws of x are low.

It should be noted that the problem we describe breaks down as the number of independent industries becomes large. For then v approaches b with certainty. If there is no variation in the best spot offer there is no uncertainty in the problem and both spot and contract outcomes reduce to full insurance.

However, not too much weight should be placed on this fact, since we use the term "industry" mainly for concreteness of exposition. What we need is first, variability of outside offers--certainly a reasonable assumption in itself--and second, that the firm at which the worker is most productive not be unique, so that the competitive wage equals the individual's productivity (even this second requirement is probably not crucial, but it allows us to avoid considering the bidding games that would otherwise arise, given our informational assumptions). The assumptions beyond that are simply to retain sufficient symmetry to keep the problem simple.

Quits and Output

Suppose each year represents an independent realization of the matrix X , and in response, the economy described above generates resultant values for pay and job choices. We now return to our initial question: is there any correlation between quits and output?

Let us begin by assuming there is only one individual in the economy. His output q depends on the draws of x and v as follows:

$$\begin{aligned} q(x,z) &= x && \text{if } p(x) > z \\ &= z && \text{if } p(x) < z \end{aligned}$$

(we can use z rather than v in the description since we know that $q(x,z) = z$ only if $z = v$). If $z > p(x)$, then there was a quit; if $z < p(x)$, there was no quit. Thus the odds of a quit conditional on $q(x,z) = q$ are^{6/}

$$(5) \quad \frac{F(p^{-1}(q))g(q)}{f(q)G(p(q))}$$

Suppose the economy were such that quits were efficient. Then the above formula would hold with $p(x) = x$ or quit probabilities equal to

$$\frac{F(q)g(q)}{f(q)G(q)} = N$$

Thus the probability of a quit would be independent of q . However, if quits are not efficient, this constancy does not hold.

To examine (5) more carefully, we must make a specific distributional assumption. We assume $F(x)$ is uniformly distributed. Then (5) becomes

$$\frac{p^{-1}(q)}{q} \left[\frac{q}{p(q)} \right]^N$$

The appendix demonstrates that $\frac{p^{-1}(q)}{q}$ and $\frac{q}{p(q)}$ are both increasing functions of q in the case of the uniform distribution. Thus the expected frequency of quits increases with the observed level of individual output.

Given the independence of q across individuals, it is straightforward to prove the following:

Corollary: The total output of the economy and the total number of quits are positively correlated.

Summary and Conclusions

We have shown that in an economy with optimal employment contracting but imperfect information about employees' outside offers there can be a positive correlation between aggregate output and the quit rate, where this correlation cannot naturally be obtained either under full information or with spot markets.

The following is an intuitive interpretation of this result. In this economy, wages perform two functions: First, they are the medium of an insurance scheme provided by the contracting firm, and second, they serve as a signal directing the employee to the most productive job available. These two functions conflict: for insurance purposes, a wage rate should not vary with productivity; for efficient signalling, it should vary one-for-one. The optimal contract is a compromise between these two goals: Under the contract, people do tend to move on to higher productivity jobs should those jobs become available, but they do not move as often as they would under perfect informa-

tion. Income variability is moderated by the contract, but not as much as it would be with full-information insurance.

As a result, we find in this economy some of the effects of a fixed-price contract, without having to start with the arbitrary assumption that prices must be fixed. Because the penalty for quitting does not vary sufficiently for efficient employment, individuals are more likely to quit and take a spot-market job when outside productivity is high and to stay when outside productivity is low. Again, we must emphasize that these results would not have held had employment been at the full-information efficient level.

The major limitations in this analysis stem from the restrictions on the functional forms we have used. No restrictions have been placed on the utility function, but we have limited ourselves to risk-neutral firms. A more desirable formulation would allow the possibility that owners are risk averse (as, for example in Grossman and Hart [1983]) or, equivalently, that liability is limited (as in Kahn and Scheinkman [1984]). The form of distribution we have considered for productivity shocks is extremely restrictive. It would be worthwhile both to allow more general functional forms for individual shocks and to drop the assumption of independence across individuals, allowing for correlated aggregate fluctuations. Of secondary importance, but still desirable, would be an extension of the model to handle production functions with diminishing returns to employment.

The other set of extensions worth considering would vary the set of permissible contracts. For instance, in contrast to

the article by Grossman, Hart, and Maskin, our analysis does not assume aggregate variables are observable in sufficient time to make contracts depend on them. In contrast to Harris and Holmstrom [1982], we have placed no restrictions on the penalties that can be extracted for a quit. In contrast to Prescott and Townsend, we have not included randomized strategies in the contract. Each of these extensions will be worth pursuing; however, it does not appear that any of them would reverse our basic result, since none of them lead to full-information efficiency.

Appendix

We will proceed through a description of several alternative formulations of the optimal contract problem and show their interrelations. Recall that the realization of each x_{ij} is assumed drawn from an interval $[a,b]$ and that the draws are independent. Also, recall that X_i is the vector (x_{i1}, \dots, x_{iN}) .

Problem A:

Find

$$\{y(X_i), \ell(X_i)\}: [a,b]^N \rightarrow \mathbb{R} \times \{0,1\}$$

to maximize

$$E\{u(y(X_i) + (1-\ell(X_i))W_i(X_i))\}$$

subject to

$$u(y(X_i) + (1-\ell(X_i))W_i(X_i)) \geq u(y(X_i') + (1-\ell(X_i'))W_i(X_i))$$

for any X_i, X_i' such that their j^{th} components are identical, and

subject to

$$E\{\ell(X_i)x_{ij} - y(X_i)\} \geq 0$$

(The search for functions y and ℓ must be restricted, of course, to those integrable with respect to the distribution of X_i .)

Comment: This is the problem in its basic form. It can be simplified as follows:

Problem B:

Find

$$\{s(x), p(x), \ell(x, v)\}: [a, b]^2 \rightarrow \mathbb{R}^2 \times \{0, 1\}$$

to maximize

$$E\{U^*(s(x), p(x), v)\}$$

where

$$U^*(s, p, v) \equiv \max_{\ell \in \{0, 1\}} U(s, p, v, \ell)$$

and

$$U(s, p, v, \ell) \equiv u(s + (1 - \ell)(v - p))$$

subject to the following restrictions:

$$(B1) \quad \ell(x, v) \in \arg \max_{\{0, 1\}} U(s(x), p(x), v, \ell)$$

and

$$(B2) \quad E\{\ell(x, v)x - s(x) + p(x)(1 - \ell(x, v))\} \geq 0.$$

Comments: Here the vector X_i of productivity of individual i in all industries has been reduced to two sufficient statistics: x (shorthand for x_{ij}) the productivity of the individual at the contracting firm and $v \equiv W_i(X_i)$ the spot wage (by our assumptions equal to $\max_j x_{ij}$). A straightforward application of the revelation principle will demonstrate the equivalence of these two problems. The incentive compatibility restriction is buried in the restriction on $\ell(x, v)$. Again, s , p , and ℓ must be assumed

integrable with respect to the joint distribution of x and v , which distribution in turn can be derived from the distribution of X_i .

Our first result is that without loss of generality we can place bounds on the range of $p(x)$ in problem B:

Lemma: Let $\{s(x), p(x), \ell(x, v)\}$ solve problem B. Consider any functions $\tilde{s}(x)$, $\tilde{p}(x)$ such that

$$\begin{aligned}\tilde{p}(x) &> b \text{ if } p(x) > b \\ &\leq a \text{ if } p(x) \leq a \\ &= p(x) \text{ otherwise}\end{aligned}$$

$$\begin{aligned}\tilde{s}(x) &= s(x) - p(x) + \tilde{p}(x) \text{ if } p(x) \leq a \\ &= s(x) \text{ otherwise}\end{aligned}$$

Then $\{\tilde{s}(x), \tilde{p}(x), \ell(x, v)\}$ also solves problem B.

In other words, penalties greater than the highest possible outside pay, or lower than the lowest possible outside pay are irrelevant.

Without loss of generality therefore we can consider problem B modified by the additional restriction:

$$(B3) \quad p(x) \in [a, b] \text{ for all } x$$

In our model $v \geq x$ for all realizations of (x, v) . That is, the spot wage is at least as great as productivity in the contracting firm. This is the natural assumption to make given the structure we have developed. However, it is somewhat difficult to work with, in part because the distributions of x and v are not independent.

We therefore wish to add an extra restriction to the contract, which we can again demonstrate causes no welfare loss. The restriction we wish to add is to require the individual to remain with the contracting firm in all cases where his productivity there is at least as great as his productivity elsewhere:

$$(B4) \quad \ell(x,v) = 1 \text{ for all } v \leq x$$

Lemma: If a feasible contract for problem B does not satisfy (B4) then there is a feasible contract satisfying (B4) for which expected utility is at least as great.

Proof: Consider the following modified contract

$$\tilde{s}(x) = \max (s(x), s(x) - p(x) + x)$$

$$\tilde{p}(x) = \max (x, p(x))$$

$$\tilde{\ell}(x,v) \in \arg \max_{\{0,1\}} U(\tilde{s}(x), \tilde{p}(x), v, \ell)$$

$$= 1 \text{ if } U(\tilde{s}(x), \tilde{p}(x), v, 0) = U(\tilde{s}(x), \tilde{p}(x), v, 1)$$

This modified contract leaves utility unchanged if $p(x) \geq x$. On the other hand if $p(x) < x$ utility may increase since

$$\begin{aligned} & \tilde{s}(x) + (v - \tilde{p}(x))(1 - \ell) \\ &= s(x) + (x - p(x))\ell + (v - p(x))(1 - \ell) \\ &\geq u(s(x), p(x), v, \ell) \end{aligned}$$

Note also that in the new contract, since $\tilde{p}(x) \geq x$, we have

$$\tilde{\ell}(x,v) = 1 \text{ if } x \geq v$$

So that (B4) is satisfied.

Finally, profits are

$$-\tilde{s}(x) + x\tilde{\ell}(x,v) + \tilde{p}(x)(1-\tilde{\ell}(x,v))$$

If $p(x) \geq x$ profits are unchanged.

If $p(x) < x$ profits equal

$$-(s(x)-p(x)+x) + x = -s(x) + p(x)$$

In the old contract profits were

$$-s(x) + p(x) + [x-p(x)]\ell(x,v)$$

But by the assumptions on the joint distribution of x and v , we know that $v \geq x$, so that

$$v > p(x)$$

and $\ell(x,v)$ must have been 0. Therefore, profits are unchanged and so the new contract is feasible if the old one was.

Armed with this result we can further modify the problem to simplify the joint distribution of x and v . Suppose the spot wage were

$$z = \max_{j' \neq j} x_{ij},$$

In other words, we now assume it is impossible to quit the firm and work for another firm in the same industry. Note that

$$(B5) \quad v = \max(x, z)$$

Also note that the distributions of z and x , unlike v and x , are independent. Thus we have a modified problem:

Problem C

Find $\{s(x), p(x), \ell(x, z)\}$: $[a, b]^2 \rightarrow \mathbb{R}^2 \times \{0, 1\}$ to maximize

$$E\{U^*(s(x), p(x), z)\}$$

where the function $U^*(\)$ is defined as in problem B, subject to

$$(C1) \quad \ell(x, z) \in \arg \max_{\{0, 1\}} U(s(x), p(x), z, \ell)$$

$$(C2) \quad E\{\ell(x, z), x - s(x) + p(x)(1 - \ell(x, z))\} \geq 0$$

and

$$(C3) \quad a \leq p(x) \leq b$$

Note therefore that the problem is identical to problem B with z substituted for v . Again the following lemma restricts consideration to solutions for which

$$(C4) \quad \ell(x, z) = 1 \text{ for all } z \leq x$$

(It will later be demonstrated that optimal solutions indeed satisfy this restriction.)

Lemma: Restrict attention to contracts $\{s(\cdot), p(\cdot), \ell(\cdot, \cdot)\}$ which satisfy (B4) for problem B (or (C4) for problem C). Within this set contracts yield identical profits and utilities for every realization of X_1 , when v , x , and z are related by (B5).

Proof: Payoffs are identical if

$$l(x, z) = l(x, \max(x, z)) \text{ for all } x, z.$$

For this, (C4) is sufficient.

Therefore we will look for the optimum for problem C and check to see that it satisfies (C4). If it does, we know that an identical contract is optimal for problem B.

As noted in the text, the joint distribution of (x, z) is

$$F(x)G(z)$$

where $G(z) = (F(z))^{N-1}$

Since we are assuming $F(\cdot)$ is nonatomic, the probability that

$$p(x) = v$$

is negligible and we can therefore further simplify the problem:

Problem D

Find $\{s(x), p(x)\}$

to maximize

$$\iint u(s(x) + \max[0, z - p(x)]) dG(z) dF(x)$$

subject to

$$\int p(x) - s(x) dF(x) + \int G(p(x))(x - p(x)) dF(x) \geq 0$$

$$a \leq p(x) \leq b$$

Comment: Given any penalty $p(x)$, the employee leaves if $z > p(x)$ and stays if $z < p(x)$, and the borderline case $z = p(x)$ can be ignored. Given x , we can regard a contract as paying the employer $p(x) - s(x)$ if the employee quits, and $p(x) - s(x) + x - p(x)$ if he stays, which he does with probability $G(p(x))$.

For ease of notation we will henceforth define:

$$\pi(x,p,s) \equiv p - s + G(p)(x-p)$$

That is, π is the expected profits given payment s , penalty p and conditional on productivity x . Expected profits (unconditional) are

$$\int \pi(x,p,s) dF(x)$$

Problem E

Find $\{s(x), p(x)\}$

to maximize

$$\begin{aligned} & \int \int u(s(x) + \max[0, z - p(x)]) dG(z) dF(x) \\ & + k \int \pi(x, p(x), s(x)) dF(x) \end{aligned}$$

Lemma: If for some $k > 0$, the functions $\{s(x), p(x)\}$ are a solution to problem E and $\int \pi(x, p(x), s(x)) dF(x) = 0$ then $\{s(x), p(x)\}$ is a solution to problem D.

Proof: Suppose not. Then there exists $\tilde{s}(x), \tilde{p}(x)$ such that expected utility is greater and expected profits are greater than or equal to zero. Without loss of generality we can assume that

expected profits are equal to zero (If they were not, increase $\tilde{s}(x)$ until they are; expected utility continues to increase). But then $\{\tilde{s}(x), \tilde{p}(x)\}$ dominates $\{s(x), p(x)\}$ for problem E, contradicting our assumption.

Lemma: $\{s(x), p(x)\}$ solves E iff for almost every x $(s(x), p(x))$ solves problem F:

Problem F

Find (s, p) to maximize

$$\int u(s + \max[0, z - p]) dG(z) + k\pi(x, p, s)$$

Lemma: There exists a solution to problem F for every x , and for every k in the interval $K = \{k \mid k = u'(y), y \in R\}$.

Recall now that we are assuming $g(z) > 0$ and continuous for all $z \in [a, b]$. Then necessary conditions for an interior maximum to problem F are that

$$(F1) \quad \int u'(s + \max[0, z - p]) dG(z) - k = 0$$

$$(F2) \quad - \int_p^b u'(s + z - p) dG(z) + k(1 - G(p)) - kg(p)(p - x) = 0$$

For the maximum to problem F to occur at $p = b$, it is necessary that $x = b$. For the maximum at $p = a$ it is necessary that $x = a$.

It is immediate that:

Theorem: If for some k there exist functions $\{s(x), p(x)\}$ such that

(a) for all $x \in (a,b)$, $(s(x),p(x))$ is the unique solution to the pair of equations (F1-F2).

$$(b) \int \pi(x,p(x),s(x))dF(x) = 0$$

Then $\{s(\cdot),p(\cdot)\}$ is the optimum contract.

Note also that (F1) and (F2) yield

$$(F3) \quad u'(s)G(p) - kG(p) - kg(p)(p-x) = 0$$

We will use (F1) and (F3) for the implicit definition of (s,p) as functions of (x,k) . Note from (F1) and (F3) it is immediate that for any solution $p \geq x$. Note also that (F1) may be used to define

$$s(p,k): \quad R \times K \rightarrow R$$

a function which is nondecreasing in p and decreasing in k .

The final step is to demonstrate that for the density

$$G(z) \equiv gz^{N-1}$$

there exists $\{s(\cdot),p(\cdot)\}$ satisfying conditions (a) and (b) of the above theorem. We start by demonstrating that for any k for which the pair (F1), (F3) have a solution, that solution is unique.

In this case, (F3) can be written as

$$(u'(s(p,k))-k)p - (N-1)k(p-x) = 0$$

or

$$x = \frac{p[Nk - u'(s(p,k))]}{(N-1)k}$$

a strictly increasing, hence invertible function of p .

Thus there is at most one value of p (and s) solving the equation, but existence of a maximizer guarantees there is a solution as long as $k \in K$, and $x \in (a,b)$. Note also that

Lemma: x/p increases with x .

We may further rewrite (F3) as

$$(F4) \quad k \left[\frac{x}{p} - \frac{N}{N-1} \right] + \frac{u'(s(p,k))}{N-1} = 0$$

Given $x \in (a,b)$ and $k \in K$ we know there is a unique p satisfying (F4). Moreover, by the implicit function theorem, $p(k)$ is continuous for all (x,k) since

$$-\frac{kx}{p^2} + \frac{u''}{N-1} \frac{\partial s}{\partial p} < 0$$

Since expected profits are a continuous function of p and s and p and s are continuous functions of k , we know that if we can find a k such that expected profits are positive, and one such that expected profits are negative, then here exists a intermediate k satisfying condition (b) and we are done.

We therefore look for k such that

$$\pi(x, p(x,k), s(x,k)) \geq 0 \text{ for all } x \in (a,b)$$

$$\pi(x, p(x,k), s(x,k)) \geq \pi(x, p(x,k), s(b,k))$$

$$= x + [1-G(p)] [p-x] - u'^{-1}(k)$$

$$\geq \min_{\substack{x \in [a,b] \\ p \in [x,b]}} (x + [1-G(p)] [p-x]) - u'^{-1}(k)$$

$$\geq a - u'^{-1}(k)$$

Therefore for $k > u'(a)$ profits are guaranteed positive. A similar analysis will demonstrate that for $k < u'(3b-a)$ profits are negative.

Footnotes

1/The contemporaneous correlation between quits and GNP across the business cycle is .893 (Prescott et al. [1983]).

2/Various forms of risk aversion or moral hazard have been incorporated: See Jovanovic [1984], or Mortensen [1978] and [1983].

3/Quits are cyclical, but unemployment among quitters is steady, and much too small to allow for significant periods of unemployment among quitters (I am grateful to Robert Topel for this information).

4/Alternatively we could build an equilibrium model from the model of Ito [1984] to achieve similar results. For another example of the use of macroeconomic equilibrium models of contracting with imperfect information see Grossman, Hart, and Maskin [1983].

5/As the reader will see, it is easy (and natural) to extend the model so that the worker's utility can depend on non-monetary characteristics of his job.

6/This analysis can also be done conditioning on v rather than q . The results will be the same.

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