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A Compound Note
(Or, a Note on Harry Johnson's Note
on the Theory of Transactions Demand for Cash)

by

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In a recent note Harry Johnson observes that in the Baumol-Tobin inventory model of the demand for money, the individual is assumed to maximize end-of-period wealth subject to a consumption constraint. Johnson argues that a better assumption is individual maximization of consumption subject to an end-of-period wealth constraint. He claims that this reversal of the role of objective and constraint functions gives rise to a different demand function for cash. In this note it is maintained that both assumptions are unduly restrictive. It is more consistent with standard consumer theory to assume the individual maximizes utility, a function of both consumption and end-of-period wealth, subject only to the constraint that his choice be attainable.

This note is divided into three parts. In the first part the generalized model is described in detail. By assuming extreme types of utility functions, both the Baumol-Tobin model and a corrected version of Johnson's model fall out as special cases. In the second part reasons for the difference in the Baumol-Tobin and Johnson money demand functions are explored. It is shown that Johnson's finding of a different demand for money for given consumption expenditures is incorrect. In the final section conditions are derived which utility functions must satisfy in order for standard money demand relationships to obtain.

I. A Broadened Framework for the Theory of Transactions Demand for Cash

Money inventory models can be interpreted as describing an efficient payments process in a multiperiod utility maximization setting. Defining a period to be the time elapsing between an individual's noninterest income receipts, these models determine a consumption -- end-of-period wealth frontier; i.e., the greatest attainable pairs of level rates of consumption over the period and stocks of wealth at the end of the period.[1/] The same frontier results whether consumption is maximized subject to an end-of-period wealth constraint or end-of-period wealth is maximized subject to a consumption constraint.[2/]

In a multiperiod model end-of-first-period wealth can be considered a proxy for future consumption. It might be assumed, therefore, that an individual's preferences can be represented by a numerical ordering over the consumption -- end-of-(first)-period wealth space. His objective under this assumption is to choose a consumption rate and end-of-period wealth stock from those which are mutually attainable in order to maximize his level of satisfaction. Hence, given initial conditions and market constraints, an inventory model generates the frontier to the individual's opportunity set in the consumption -- end-of-period wealth space. A preference ordering over this space permits a single point to be chosen on the frontier, the one which maximizes the individual's level of satisfaction (see Figure 1).

Average holdings of money per period can be associated with each point of the frontier. Thus, for given parameter values, a money demand function expresses the average quantity of money held at the satisfaction maximization point on the consumption -- end-of-period wealth frontier.

Let:

$W = f(c;\gamma)$ be the functional representation of the consumption -- end-of-period wealth frontier, where

W = stock of wealth at the end of the period ($W \geqq 0$)

c = rate of consumption over the period ($c \geqq 0$)

γ = vector of parameters.

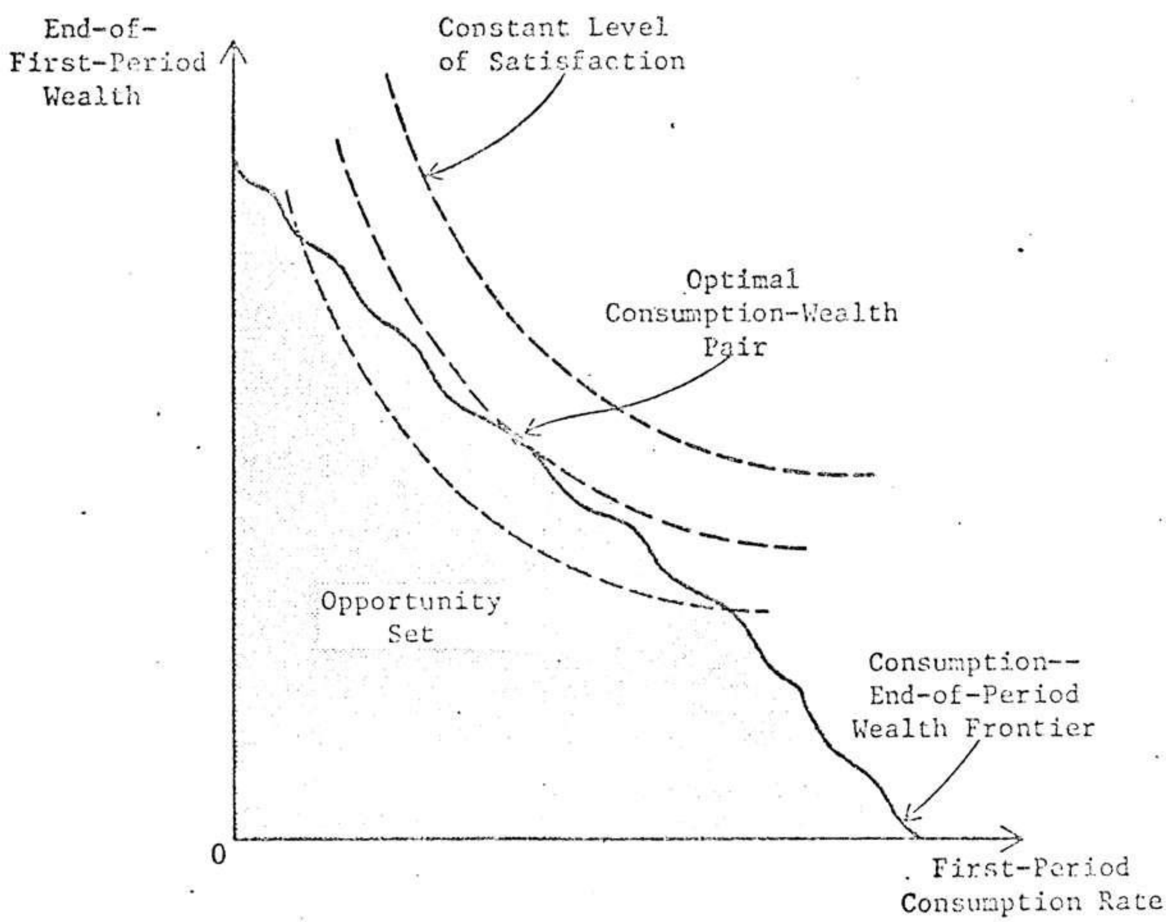
Let:

$U = U(c,W)$ be an indicator of the individual's level of satisfaction for every pair $\langle c, W \rangle \geq 0$[3/] and

$M = m(c,W;\gamma)$ be the average holdings of money per period for every pair $\langle c, W \rangle$ such that $W - f(c;\gamma) = 0$.

Figure 1

A Broadened Framework



The individual seeks to maximize $U = U(c,W)$ subject to $W - f(c;\gamma) = 0$. Maximization subject to the constraint makes it possible to express the maximizers $\hat{c}$ and $\hat{W}$ (in a neighborhood of γ) as:

$$\hat{c} = g(\gamma)$$

$$\hat{W} = h(\gamma), \text{ where } h(\gamma) \equiv f(g(\gamma);\gamma) \equiv f(\hat{c};\gamma).$$

The individual's demand for money is then:

$$\hat{M} = \hat{m}(\gamma), \text{ where } \hat{m}(\gamma) \equiv m(g(\gamma),h(\gamma);\gamma).$$

II. An Interpretation of the Baumol-Tobin and Johnson Approaches

The money demand function $\hat{m}(\gamma)$ will not be the same, in general, as the ones which result using the Baumol-Tobin and Johnson approaches. Suppose for $\gamma = \gamma_o$ the individual maximizes his satisfaction at point 'a' on the consumption -- end-of-period wealth frontier pictured in Figure 2. His average holdings of money at point 'a' is:

$$\hat{M} = m(c_a, W_a; \gamma_o).$$

Now suppose there is a change in γ_o to γ_1 causing an upward shift in the frontier (e.g., an increase in initial wealth):

$$f(c;\gamma_1) > f(c;\gamma_o). \quad \text{(See Figure 2.)}$$

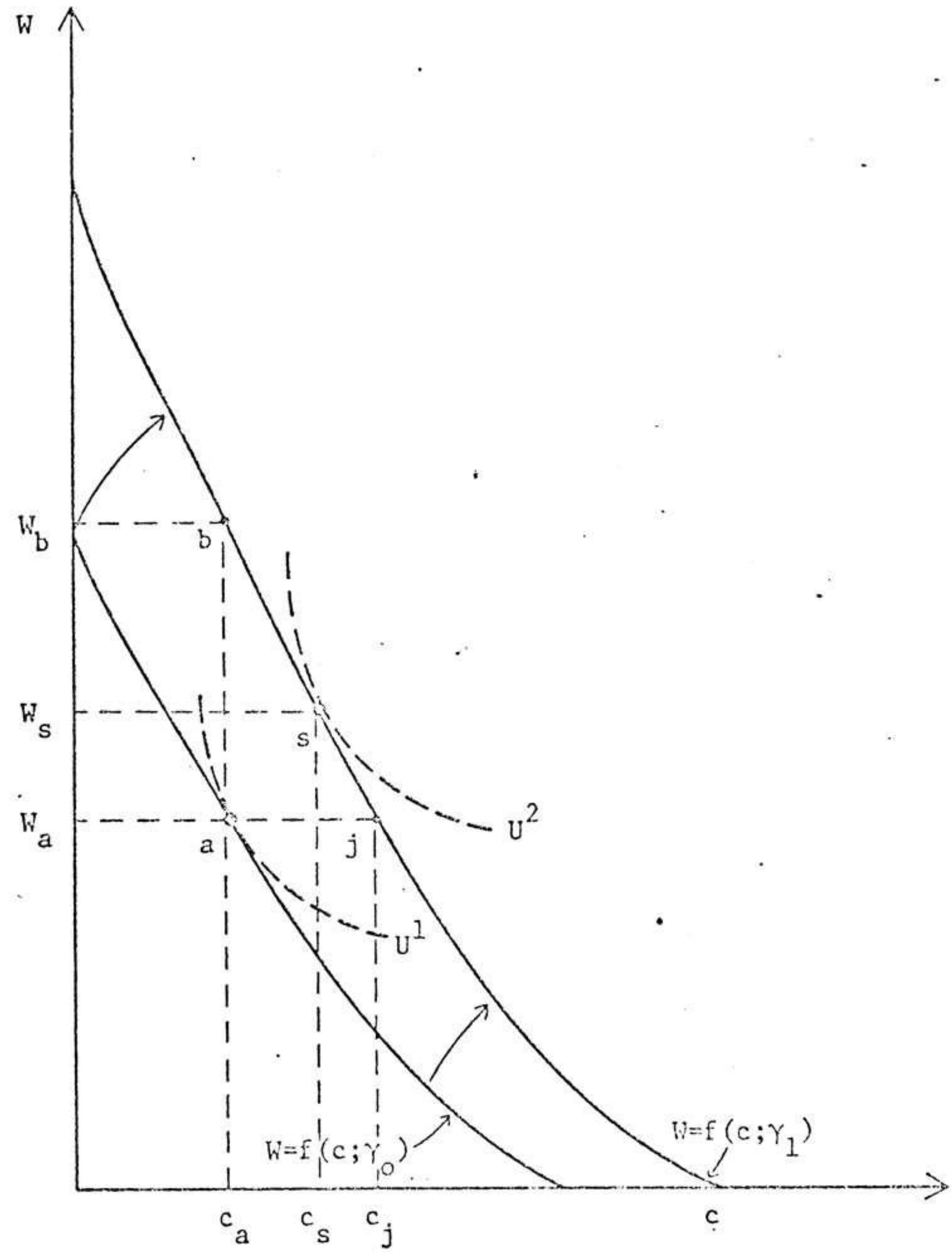
The Baumol-Tobin analysis assumes that the individual's consumption is fixed. According to this approach the individual's new equilibrium is at point 'b', and the change in average money holdings is:

$$(\Delta\hat{M})_{\text{Baumol-Tobin}} = m(c_a,W_b;\gamma_1) - m(c_a,W_a;\gamma_o)$$

The Johnson analysis assumes that the individual's end-of-period wealth is

Figure 2

An Interpretation
of the Baumol-Tobin and Johnson Approaches



fixed. According to this approach the individual's new equilibrium is at point 'j', and the change in average money holdings is:

$$(\Delta\hat{M})_{\text{Johnson}} = m(c_j, W_a; \gamma_1) - m(c_a, W_a; \gamma_o).$$

The Baumol-Tobin and Johnson approaches implicitly assume very special types of preference orderings. A more general satisfaction maximization approach, of which the Baumol-Tobin and Johnson approaches are special cases, indicates the new equilibrium is at point 's' and the change in average money holdings is:

$$(\Delta\hat{M})_{\substack{\text{satisfaction}\\ \text{maximization}}} = m(c_s, W_s; \gamma_1) - m(c_a, W_a; \gamma_o).$$

Hence, a change in parameter values causes the frontier to shift, and generally new equilibrium values result for both consumption over the period and wealth at the end of the period.

In summary, an individual's money demand function depends on his preferences between present and future consumption and, therefore, is partly subjective in nature. An inventory model, on the other hand, is an objective construct, and its logical end product is an efficiency locus: a consumption -- end-of-period wealth frontier. Money demand functions are not products of inventory models alone; they are products of inventory models augmented by individual preference relations. The function m, which relates average money holdings to points on the frontier, is independent of preferences and, hence, is not a money demand function.

This framework suggests that any difference in the Baumol-Tobin and Johnson money demand functions can be traced to a difference in assumed consumption-wealth preferences. It also suggests that for given parameter values Baumol-Tobin and Johnson should find the same frontier with identical money holdings at every point on that frontier. Johnson's claim to have found a different demand for money <u>for given consumption expenditures</u> cannot be valid. It is, in fact, due to a misspecification of his model (discussed in Part III).

III. Implications for the Transactions Demand for Cash from a Broadened Framework

Assuming zero proportional transaction costs, the consumption -- end-of-period wealth frontier using either the Baumol-Tobin or Johnson approach is

$$W = (1+r)M_o - c(1+r/2) - \sqrt{2brc} \equiv f(c;M_o,r,b)$$

where M_o = initial wealth assumed to be in the form of money,

r = simple rate of interest per period, and

b = fixed transaction cost in the bond market (i.e., cost is independent of transaction size).4/

This function is derived by maximizing $W(n)$ with respect to n, where

$$W(n) = (M_o-c)(1+r) + \frac{r}{n}(c - \frac{c}{n}) + \frac{r}{n}(c - \frac{2c}{n}) + \ldots$$

$$+ \frac{r}{n}(c-(\frac{n-1}{n})c) - nb$$

$$= (M_o-c)(1+r) + \frac{rc}{2}(\frac{n-1}{n}) - nb$$

and

n = number of transactions in bond market,

$1/n$ = time elapsing between bond transactions.

The maximizer n^* is given by

$n^* = \sqrt{\frac{rc}{2b}}$, and the frontier is simply $W = W(n^*)$.5/

The consumption -- end-of-period wealth frontier is a convex function:

$\frac{d^2f}{dc^2} = \frac{\sqrt{2brc}}{4c^2} > 0$. Thus, the assumption of fixed market transaction costs implies the individual's opportunity set is not convex. This indicates that there are increasing returns to scale in inventory management.

The individual seeks to maximize $U(c,W)$ subject to $W = f(c;M_o,r,b)$. More simply, he seeks to maximize $U[c,f(c;M_o,r,b)]$ with respect to c. Supposing an interior solution, the first and second order conditions for $\hat{c}$ to be a maximizer are:

(1^{st}) $U_1 + U_2f_1 = 0$ and

(2^{nd}) $U_{11} + U_{12}f_1 + U_{21}f_1 + U_{22}f_1^2 + U_2f_{11} < 0$, all partials evaluated at $c = \hat{c}$.

The first-order condition is that the frontier must be tangent to an indifference curve at $c = \hat{c}$. The second-order condition is that the indifference curve has a higher degree of convexity than the frontier at $c = \hat{c}$.

The individual's money demand function is $\hat{M} = \frac{\hat{c}}{2n^*} = \sqrt{\frac{b\hat{c}}{2r}}$. The elasticities of $\hat{M}$ with respect to the parameters can be expressed:

(i) $\varepsilon(\hat{M}/M_o) = \frac{1}{2}\varepsilon(\hat{c}/M_o)$,

(ii) $\varepsilon(\hat{M}/r) = \frac{1}{2}[\varepsilon(\hat{c}/r)-1]$, and

(iii) $\varepsilon(\hat{M}/b) = \frac{1}{2}[\varepsilon(\hat{c}/b)+1]$,

where $\varepsilon(x/y) = \left(\frac{\partial x}{\partial y}\right)\left(\frac{y}{x}\right)$.

The elasticity of money demand with respect to any parameter is a linear function of the elasticity of consumption demand with respect to the same parameter. The latter elasticity depends on the individual's utility function. Thus, we now consider properties the utility function must possess in order for the following standard money demand relationships to obtain:

(a) $\varepsilon(\hat{M}/M_o) > 0$,

(b) $\varepsilon(\hat{M}/r) < 0$, and

(c) $\varepsilon(\hat{M}/b) > 0$.

These properties, derived by application of maximization theory, are:

(a) $\varepsilon(\hat{M}/M_o) > 0 \Longleftrightarrow \varepsilon(\hat{c}/M_o) > 0$

$$\Longleftrightarrow U_2U_{12} - U_1U_{22} > 0,$$

(b) $\varepsilon(\hat{M}/r) < 0 \Longleftrightarrow \varepsilon(\hat{c}/r) < 1$

$$\Longleftrightarrow (\hat{W}-M_o - \frac{\hat{c}r}{2})[U_2U_{12}-U_1U_{22}] - \hat{c}[U_1U_{12}-U_2U_{11}] < (r/2)U_2^2,$$

and

(c) $\varepsilon(\hat{M}/b) > 0 \Longleftrightarrow \varepsilon(\hat{c}/b) > -1$

$$\Longleftrightarrow (1+r/2)[U_2U_{12}-U_1U_{22}] + [U_1U_{12}-U_2U_{11}] > \frac{\sqrt{2br\hat{c}}}{2\hat{c}^2} U_2^2,$$

all partials evaluated at the maximum $\langle\hat{c},\hat{W}\rangle$.

The conditions that c and W are not inferior goods (in the weak sense) are:

$U_2U_{12} - U_1U_{22} > 0$ and $U_1U_{12} - U_2U_{11} > 0$, respectively.[6/]

Property (a) states that if consumption is not an inferior good, average money holdings will increase when the initial money endowment increases. Property (b) suggests that if neither consumption nor end-of-period wealth are inferior goods, and $\hat{c} \geqq (\frac{r}{1+r})M_o$, average money holdings will decrease when the interest rate increases. Property (c) is not easy to decipher, but it can be shown that if $\frac{\partial \hat{W}}{\partial b} \leqq 0$, then $\varepsilon(\hat{c}/b) > -1$ and average money holdings will increase when the bond market transaction cost increases.

Footnotes

1/ $\langle x,y \rangle$ is a greatest pair in a set $X \times Y$ means there exists no pair $\langle x',y' \rangle$ in that set such that: $x' > x$ and $y' \geq y$ or $x' \geq x$ and $y' > y$.

2/ It is only necessary to show that the functional representation of the frontier found by maximizing end-of-period wealth 'W' subject to a consumption 'c' constraint is strictly monotonically decreasing; i.e., if $\langle c_1,\hat{W}_1 \rangle$, $\langle c_2,\hat{W}_2 \rangle$ are two points on the frontier and $c_1 < c_2$, then $\hat{W}_1 > \hat{W}_2$. (Both c and W are in real terms.)

The maximal W for given c is provided by an optimal market strategy, where a strategy consists of times of transactions and values of assets exchanged. Let s be an optimal strategy for c_2. A strategy s' will be constructed from s such that given $c_1 < c_2$, s' implies $W' > \hat{W}_2$. It then follows $\hat{W}_1 \geq W' > \hat{W}_2$.

Suppose, as Baumol-Tobin and Johnson do, that the commodity market transaction cost is zero and that the nominal rate of return on commodities is negative. These assumptions imply that real consumption expenditures under s are being made at a steady rate equal to the rate of consumption c_2. Under s' let all market transactions be the same as under s except let the rate of real consumption expenditures be lowered to c_1 and let the individual accumulate the difference in nominal expenditures in the form of money. Then under s', W' exceeds $\hat{W}_2$ by the accumulated difference in nominal consumption expenditures discounted by the end-of-period price level.

3/ The function U is assumed to be twice differentiable, and $U_1, U_2 > 0$; $U_{11} < 0$; $U_{22} < 0$; $U_{11}U_{22} - U_{12}^2 > 0$.

4/ Johnson's income constraint is in error, and this causes the frontier his model generates to differ from the Baumol-Tobin frontier. Assuming zero proportional transaction costs, average money holdings at a point on the Johnson frontier are given by:

$$M_J = b/2 + \sqrt{\frac{bc}{2r}}$$

(found by solving for M_o in Johnson's income constraint and substituting the expression into his formula $M = \sqrt{\frac{(2+r)bM_o}{4r}}$). The corresponding relation determined by the Baumol-Tobin model is:

$$M_{B-T} = \sqrt{\frac{bc}{2r}} .$$

The fact that Johnson's model generates a different formula for average money holdings implies it generates a different frontier also. He writes his income constraint as:

$$c = (M_o + r(\frac{M_o}{2} - M))(1 - \frac{b}{2M}).$$

Assuming the individual desires zero end-of-period wealth as Johnson does, his constraint should have been written:

$$c = M_o + \frac{r(\frac{c}{2} - M) - \frac{bc}{2M}}{1 + r} .$$

5/ Three technical notes are in order.
(a) Tobin's theorem on equal spacing of bond market transactions has been applied to derive $W(n)$.
(b) It is supposed that parameter values imply $\hat{n} > 1$ for all values of $c \geq 0$.
(c) The optimal number of bond market transactions is properly an integer, but $\hat{n}$ is treated in this text as a real number. The advantage of doing this is that it allows calculus tools to be applied to the solution routine. This procedure results in a smoothing of the consumption -- end-of-period wealth frontier. When n is restricted to the set of positive integers, the frontier is composed of a series of connected line segments. The projection of a segment on the consumption axis measures the change in consumption required to alter the optimal number of bond market transactions. When $\hat{n}$ is defined to be a real number, its optimal value changes continuously as the value of consumption changes. Since the set of real numbers not less than one contains the positive integers as a proper subset, the smoothed frontier can never be below the true frontier. Moreover the frontiers are tangent wherever n determined by calculus techniques is an integer. The relationship between the two frontiers implies they have the following properties in common:

i. Sign of slope
ii. Curvature
iii. Direction of shifts to parameter changes.

6/ See Henderson and Quandt [2], p. 27, for definition of inferior good.