

by John Kareken, Thomas Muench,  
Thomas Supel, and Neil Wallace

**DETERMINING THE  
OPTIMUM MONETARY  
INSTRUMENT VARIABLE**

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## INTRODUCTION

For some time monetary economists and officials have been debating how central banks ought to operate. Should the Federal Reserve, for example, seek to control one or another of the monetary aggregates? And if so, which one? Or should it control some interest rate or rates?

We do not know how the Federal Reserve, or for that matter any other central bank, ought to operate. We do, though, know what seems to us a not unreasonable way of deciding; a way, that is, of determining the optimum monetary instrument variable. And in this paper we explain or, better, illustrate our way.

The central bank that is certain about the economic structure constraining it or does not care about the variance of policy outcomes can, with complete indifference, use any possible instrument variable. It is difficult, however, to imagine any central bank being certain or not caring about the variance of policy outcomes. The presumption must therefore be that most if not all central banks have a true choice to make: namely, which of all possible instrument variables to use.<sup>1</sup> And what we would have central banks do is decide by maximizing their respective expected utilities; or in other words, by comparing the maximum expected utilities associated with all the various possible instrument variables. What in effect we would have the Federal Reserve do is calculate alternative opportunity loci, there being one such for each possible instrument variable, and then, having specified values for its target variables, determine which of these loci or constraints allows it to achieve the greatest ex-

pected utility. We would have the Federal Reserve do this not once but, since the cost is not much, at the beginning of every policy period.

Nor is it impractical to suggest this. However complex the underlying economic structure, opportunity loci can be calculated. In Section I, we use a very simple economic structure. But we do so only because our purpose there is to explain our way of determining the optimum monetary instrument variable; and it is convenient in explaining to use a simple structure. In Sections II and III, wherein we derive actual Treasury bill rate and demand deposit loci, we use the Federal Reserve-MIT-University of Pennsylvania economic structure, which is very complex.<sup>2</sup>

We do then provide some numbers or experimental findings. We would caution, however, against paying much attention to them. They are not, we think, even suggestive of how the Federal Reserve ought to operate. We decided to include them in the paper only because they show that our way of determining the optimum monetary instrument variable is practical.

But our way is practical or feasible only for myopic central banks, for those concerned only about current-period developments or, by way of approximation, willing to pretend that they are. It is no accident that in Sections I and II we take utility as depending simply on current-period nominal gross national product and in Section III as depending on current-period real GNP and the current-period change in the price level. Had we taken utility as depending on future-period values as well, we would not have been able to go further; we would not have been able to show the practicability or feasibility of calculating and comparing the maximum expected utilities associated with the various possible instrument variables.

NOTE.—Messrs. Kareken, Muench, and Wallace, all of the Economics Department, University of Minnesota, are Consultants to the Federal Reserve Bank of Minneapolis; Mr. Supel is Senior Economist at that Bank. They would like to thank Arthur Rolnick for his very considerable help, and the Minneapolis Bank for its financial support.

<sup>1</sup> See W. Poole, "Optimal Choice of Monetary Policy Instruments in a Simple Stochastic Macro Model," *Quarterly Journal of Economics*, (May 1970), and J. Kareken, "The Optimal Monetary Instrument

Variable," *Journal of Money, Credit, and Banking*, (Aug. 1970).

<sup>2</sup> Hereinafter, we shall refer to the FR structure. There is, we understand, a new version. If so, we used an old version, the one described in part by F. de Leeuw and E. Gramlich in "The Federal Reserve-MIT Econometric Model," *Federal Reserve Bulletin*, (Jan. 1968).

It is not known what policies are optimal for a central bank that is uncertain about the true values of structural parameters and whose concern extends into the future.

We might have proposed comparing the expected utilities of arbitrary rather than optimal policies. But which ones? Or we might have proposed that variances of structural parameters be ignored. It seemed to us, however, that uncertainty about parameters is an important fact of life and that we ought therefore to take utility as depending only on current-period values of target variables.

Some readers might want to object that the Treasury bill rate and the stock of demand deposits are not possible Federal Reserve instrument variables. We believe, though, that the Federal Reserve if it wanted to could determine the bill rate exactly. It would only have to announce a price for bills. And is coming quite close to some preassigned value for, say, the 3-month average of demand deposits impossible? We think not. But it does not really matter if we have been inept in selecting possible Federal Reserve instrument variables. Our way might be used for choosing between (or among) other possible instrument variables.

## I. QUADRATIC UTILITY AND A SIMPLE ECONOMIC STRUCTURE

Let the monetary authority's utility function be

$$U = -(Y - \bar{Y})^2$$

where  $\bar{Y}$  is nominal current-quarter GNP and  $\bar{Y}$  is the desired or target value of  $Y$ . Then

$$EU = -VY - (EY - \bar{Y})^2$$

where  $E$  stands for expected value and  $V$  for variance. Iso-expected utility contours are parabolas, symmetric about  $EY = \bar{Y}$ , in the positive quadrant of the  $(EY, VY)$  plane. The relevant opportunity loci, or constraints subject to which  $EU$  is maximized, are therefore all attainable combinations of  $EY$  and  $VY$ .

Let the economic structure be

$$(1) \quad Y = s_0 + s_1 r + e_1$$

and

$$(2) \quad m = s_2 + s_3 Y + s_4 r + e_2$$

Equation 1 describes nominal aggregate demand as a function of the interest rate,  $r$ , and equation 2 the condition for equality between the actual stock of demand deposits,  $m$ , and the desired stock. The monetary authority is uncertain about the values of the parameters,  $s_0, s_1, \dots, s_4$  and about the values of the disturbances  $e_1$  and  $e_2$ .

If  $r$  is used as the instrument variable, the reduced-form equation for  $Y$  is equation 1. If  $m$  is used as the instrument variable, it is

$$(3) \quad Y = s_6 + s_6 m + e_3$$

where

$$s_3 = \frac{s_4 s_0 - s_2 s_1}{s_4 + s_1 s_3}$$

$$s_6 = \frac{s_1}{s_4 + s_1 s_3}$$

and

$$e_3 = \frac{s_1 e_1 - s_2 e_2}{s_4 + s_1 s_3}$$

From these reduced forms, the two loci can be obtained. To illustrate, from equation 1,

$$(4) \quad EY(r) = Es_0 + rEs_1 + Ee_1$$

and

$$(5) \quad VY(r) = V(s_0 + e_1) + r^2 Vs_1 + 2rC(s_0 + e_1, s_1)$$

where  $C$  stands for covariance. Solving equation 4 for  $r$  and substituting the result into equation 5 gives the  $r$ -locus

$$(6) \quad VY(r) = c_0 + c_1 EY(r) + c_2 [EY(r)]^2$$

where

$$c_0 = V(s_0 + e_1) + \frac{Vs_1[E(s_0 + e_1)]^2}{(Es_1)^2} - \frac{2E(s_0 + e_1)C(s_0 + e_1, s_1)}{Es_1}$$

$$c_1 = \frac{2C(s_0 + e_1, s_1)}{Es_1} - \frac{2Vs_1E(s_0 + e_1)}{(Es_1)^2}$$

$$c_2 = Vs_1/(Es_1)^2$$

Equation 6 gives all combinations of  $EY$  and  $VY$  attainable when  $r$  is used as the instrument variable.

The opportunity locus for  $m$ , the  $m$ -locus, is obtained in the same way as the  $r$ -locus was, but from equation 3.

As we show in Section II, traditional or classical estimation of equations 1 and 2 provides the basic information needed to determine numerical values for the coefficients of the  $r$ -locus (that is, for  $c_0$ ,  $c_1$ , and  $c_2$ ) and for the coefficients of the  $m$ -locus. And with numerical opportunity loci, the monetary authority can determine its optimum instrument variable. All it has to do is specify a target value for  $Y$ .

It is worth pausing briefly here to consider what it means to determine numerical opportunity loci by traditional estimation of the economic structure. Each variance of possible outcomes of  $Y$ , for example  $VY(r)$ , combines true randomness in the economy and uncertainty about the values of structural parameters. Indeed,  $VY(r)$ , like  $VY(m)$ , is a forecast variance; that is to say, a variance of forecast  $Y$  around "true" or actual  $Y$ . To be sure, the randomness of "true"  $Y$  is entirely attributable to  $e_1$  and  $e_2$ . But the monetary authority, in making its instrument variable choice, must also be influenced by how certain it is about parameter values. Suppose that when  $m$  is used as the instrument variable,  $Y$  is partly determined by some parameter the value of which is extremely uncertain; and when  $r$  is used as the instrument value,  $Y$  is not determined even in part by this parameter. If at all averse to risk, the monetary authority should then, *ceteris paribus*, use  $r$  as its instrument variable.

## II. QUADRATIC UTILITY AND A COMPLEX ECONOMIC STRUCTURE

The PR economic structure is, as we have said, very complex. There are many behavioral equations, some of which are nonlinear. It can be written

$$F_i(x, z, r, \alpha_i, e_i) = 0 \quad (i = 1, 2, \dots, K)$$

where  $x$  is a vector of the current values of endogenous variables,  $K$  in number;  $z$  is a vector of contemporaneous, nonpolicy exogenous variables;  $r$  is the rate on 3-month Treasury bills;  $\alpha_i$  is a vector of parameters; and  $e_i$  is a disturbance. If all nonlinear terms in  $x$  and  $r$  are approximated by first-order Taylor expansions, then the structure can be written

$$(7) \quad Ax = Br + C$$

where  $A$  is a  $K \times K$  matrix with elements  $a_{ij}$ ;  $B$  is a  $K \times 1$  matrix with elements  $b_{ij}$ ; and  $C$  is a  $K \times 1$  matrix with elements  $c_i$ . Also,

$$a_{ij} = f_{ij}(x^0, z, r^0, \alpha_i, e_i)$$

$$b_{ij} = g_{ij}(x^0, z, r^0, \alpha_i, e_i)$$

and

$$c_i = h_i(x^0, z, r^0, \alpha_i, e_i)$$

where  $x^0$  and  $r^0$  are the values of  $x$  and  $r$  used in making the model linear. It follows that

$$(8) \quad x_1 = Y = d_{11}r + d_{10}$$

and

$$(9) \quad x_2 = m = d_{21}r + d_{20}$$

where  $d_{j1} = A_j^{-1} B$ ,  $A_j^{-1}$  being the  $j$ th row of  $A^{-1}$ , and  $d_{j0} = A_j^{-1} C$ . Then

$$Y(r) = d_{11}r + d_{10}$$

and

$$Y(m) = \begin{bmatrix} d_{11} \\ d_{21} \end{bmatrix} m - \begin{bmatrix} d_{20}d_{11} - d_{10}d_{21} \\ d_{21} \end{bmatrix} = D_{11}m + D_{10}$$

So what is required are estimates of the first two moments of the vectors  $(d_{11}, d_{10})$  and  $(D_{11}, D_{10})$ . But since the  $d$ 's and  $D$ 's are complicated functions of the underlying random variables—the parameters  $\alpha_i$ , the disturbances  $e_i$ , and the contemporaneous values of non-instrument exogenous variables  $z_i$ —their distributions cannot be derived analytically from the distributions of the underlying random variables.<sup>3</sup> It is possible, though, to sample from the distributions of the underlying random variables, insert the sampled values into equation 7, and solve for values of the reduced-form coefficients, the  $d$ 's and the  $D$ 's. By repeated sampling, a set of values of the  $d$ 's and the  $D$ 's is built up, from which moments can be estimated and numerical opportunity loci derived.<sup>4</sup>

It is also possible to proceed differently. Relevant opportunity loci can be determined point by point from a nonlinear structure. For each of a set of values of  $r$  and each of a set of values of  $m$ , a sample of values of  $Y$  is generated and estimates of the first two moments are calculated. We decided against proceeding this way in part because of the cost. A great many simulations would have been required:  $2(nxp)$  simulations, in fact, in order get  $p$  points on each locus, using  $n$  observations on  $Y$  for each point.

<sup>3</sup> Even for very simple economic structures, such as that of Section II, it is difficult if not impossible to derive the distributions of the reduced-form coefficients as functions of the moments of the structural parameters. To determine the numerical loci implied by the structure of Section II, it would therefore also be necessary to sample from the joint distributions of the structural parameters that are consistent with statistical estimation.

<sup>4</sup> Why derive numerical opportunity loci rather than calculate expected utilities? It is just that to calculate expected utilities,  $\bar{Y}$ , the desired or target value of  $Y$ , must be known or assumed. But having derived numerical loci, one may find dominance in a neighborhood of some reasonable value for  $\bar{Y}$ —that one variance is smaller than the other at every value of  $EY$  in the neighborhood. Clearly, deriving numerical loci is for outsiders.

**DISTRIBUTIONS OF PARAMETERS, DISTURBANCES, AND EXOGENOUS VARIABLES.** We assumed that the mean of each parameter in  $F_i$  is equal to the corresponding estimate, that the variance-covariance matrix of a set of parameters is equal to a constant times the variance-covariance matrix of the corresponding estimators, and that the variance of the disturbance in  $F_i$  is equal to a constant times the corresponding residual variance.<sup>5</sup>

Sample values of  $\alpha_i$ , the vector of parameters in the  $i$ th equation, not the original structure, were generated jointly according to the matrix equation

$$\alpha_i = \bar{\alpha}_i + R_i v$$

where  $\bar{\alpha}_i$  is the vector of point estimates of  $\alpha_i$ ,  $R_i$  is a matrix such that  $R_i R_i'$  equals the estimated variance-covariance matrix of  $\bar{\alpha}_i$ , and  $v$  is a vector of random variables chosen independently of one another from a normal distribution with mean zero and variance one truncated at plus and minus two. The disturbance for the  $i$ th equation was generated according to

$$e_i = \hat{\sigma}_i v$$

where  $\hat{\sigma}_i$  is the estimated residual standard error for the  $i$ th equation and  $v$  is a single independent drawing from the same truncated normal. It follows that the expected value of  $\alpha_i$  is  $\bar{\alpha}_i$ , that the variance-covariance matrix of  $\alpha_i$  is 0.77 times the variance-covariance matrix for  $\bar{\alpha}_i$ , that the mean of  $e_i$  is zero, and that its variance is  $0.77 \hat{\sigma}_i^2$ . (The constant is 0.77 because we inadvertently failed to recognize that the variance of the truncated normal is 0.77 and not unity.)

We chose a truncated distribution for  $v$  because many of the equations of the FR structure are in a form inconsistent with an un-

<sup>5</sup> Thus, the data requirements for each estimated equation are the point estimates of the coefficients, the point estimates of the residual variance, and the inverse of the relevant cross-product matrix of the independent variables. The coefficient and residual variance estimates were available, but the cross-product matrices had to be re-estimated.

limited range for the disturbance. For example, several of the estimated equations for interest rates are linear, so that disturbances from a distribution with unlimited range could produce negative interest rates. Also, we did a certain amount of linearization and thereby changed some estimated equations which originally had forms that constrained the dependent variables to proper ranges.

There are quite a few noninstrument exogenous variables in the FR structure that can be treated as random. These include population, Federal Government expenditures, and exports. We assumed that these variables are generated by second-order autoregressive schemes,

$$z_{t,i} = \beta_{0i} + \beta_{1i}z_{t-1,i} + \beta_{2i}z_{t-2,i} + u_{t,i}$$

The  $\beta$ 's were taken as fixed and equal to the estimated coefficients from an ordinary least squares regression of  $z_i$  on two lagged values of itself over the period 1952-Q1 to 1968-Q4. (It was an oversight that we did not also take the  $\beta$ 's as random.) The disturbance,  $u_{t,i}$ , was treated as random with mean zero and variance equal to 0.77 times the estimated residual variance from that regression.

The distributions of the exogenous variables can play an important role in determining the better instrument variable. In a simple model, the less variance in the aggregate demand schedule the more likely is it that the interest rate is the better instrument variable. Inability to forecast exogenous variables like government expenditures and exports contributes directly to variance of aggregate demand. Thus, if there are schemes that forecast those variables with smaller error variance than do our autoregressive schemes, our failure to use them would seem, on the whole, to favor demand deposits as the optimum monetary instrument variable.

**RESULTS.** We derived opportunity loci for the first quarter of 1969 using 100 random drawings.<sup>6</sup> With  $r$  as the instrument variable

$$E(Y) = 884.9 - .819r$$

and

$$V(Y) = 361.0 - 2(.671)r + .088r^2$$

where  $r$  is measured as a per cent per annum and  $Y$  is measured in billions of dollars at an annual rate. Therefore, the  $r$ -locus is

$$V(Y) = 102,012 - 231.4 E(Y) + .13166 [E(Y)]^2$$

The highest value of  $E(Y)$  for which the locus has any meaning is  $E(Y) = 884.9$ , since there  $r = 0$ . At  $r = 10$ ,  $E(Y) = 876.7$ . The locus is drawn in Figure 1 for approximately that range of values. We would expect our estimated locus to most closely approximate the locus obtained from the original nonlinear model in the vicinity of  $r = r^0$ , the value around which we linearized, or in the vicinity of  $E(Y) = 880.3$ .

With  $m$  as instrument,

$$E(Y) = 805.8 + .495m$$

and

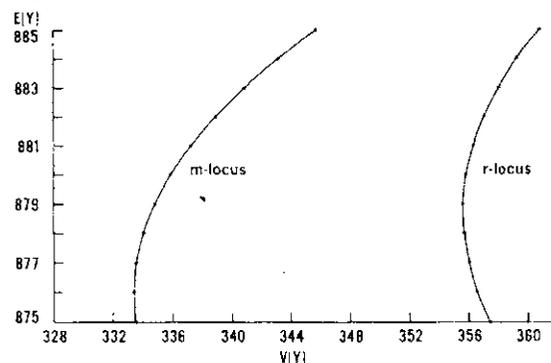
$$V(Y) = 1067.0 - 2(5.178)m + .0365m^2$$

where  $m$  is in billions of dollars. Therefore, the  $m$ -locus (also shown in Figure 1) is

$$V(Y) = 114,713.7 - 261.2E(Y) + .14909[E(Y)]^2$$

Note that in Figure 1  $m$  dominates  $r$  as an instrument variable. For any expected value of  $Y$ , the variance of  $Y$  is smaller with  $m$  as the instrument variable than with  $r$  as the instrument variable. But the difference between the

FIGURE 1  
MEAN-VARIANCE LOCI



<sup>6</sup> Deriving loci for 1969-Q1, we linearized the FR structure around values for 1968-Q4.

variances at, say,  $E(Y) = 880$  is 20, and 20 is not a significant difference. For a sample of 100 drawings, a 90 per cent confidence interval around the variance of the  $m$ -locus at  $E(Y) = 880$  ranges from 269 to 432, whereas the corresponding interval for the  $r$ -locus ranges from 285 to 457. There is, therefore, considerable overlap of the confidence intervals.

### III. THE REAL INCOME-VARIANCE OF PRICE UTILITY FUNCTION

We also derived the first-quarter 1969 opportunity loci relevant for maximization of expected utility, where

$$U = \log X - b[100(P - P^0)/P^0]^2$$

$X$  is real GNP in 1958 prices,  $P$  is the GNP deflator, and  $P^0$  is the deflator for the fourth quarter of 1968. Iso-expected utility contours for this function are straight lines with slope  $b$  in the  $[E \log X, 10^4 E(P - P^0/P^0)^2]$  plane. The log function implies risk aversion; at a

given value of the variance of the deflator, fair gambles on  $X$  are always rejected. The relevant opportunity loci consist of all attainable combinations of  $E \log X$  and  $10^4 E[(P - P^0)/P^0]^2$ . These were obtained for  $r$  and for  $m$  as follows.

Whether  $r$  or  $m$  is used as the instrument variable, there are reduced-form equations for both real income and the deflator. Let

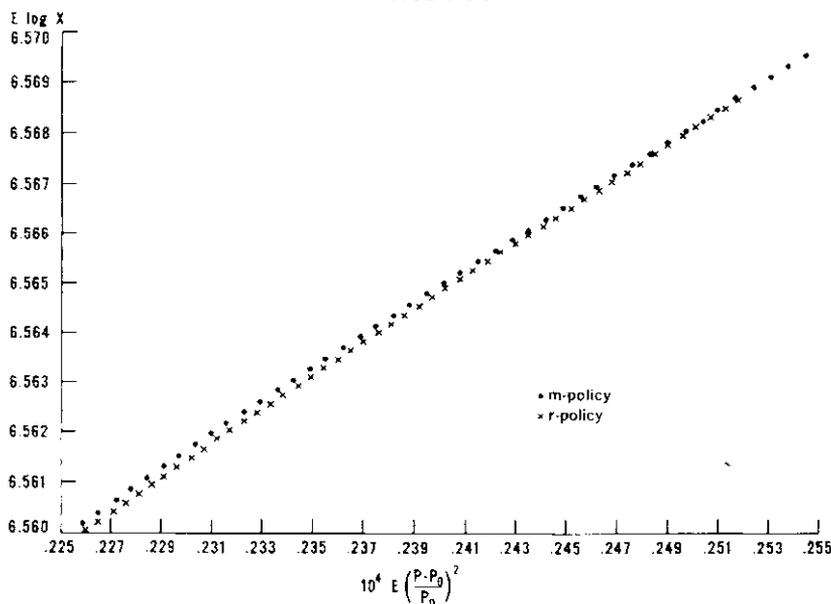
$$\begin{aligned} X &= b_1 r + b_2 \\ P &= b_3 r + b_4 \end{aligned}$$

be those for  $r$ . Thus,

$$E(\log X) = E \log (b_1 r + b_2)$$

so  $E \log X$  cannot be written as a function of  $r$  and of the moments of  $b_1$  and  $b_2$ . It is possible, however, to compute  $E \log X$  for each value of  $r$  in a reasonable range. We let  $r$  range from 1 per cent to 10 per cent. For each value of  $r$ , we computed and averaged  $\log (b_1 r + b_2)$  over the sample of values of  $b_1$  and  $b_2$  and took the resulting average as our estimate of  $E \log X$ . From the reduced form for  $P$ , we have

FIGURE 2  
THE MEAN INCOME-PRICE VARIANCE LOCI



$$E \left[ \frac{P - P^0}{P^0} \right]^2 = 1/(P^0)^2 [r^2 E(b_3^2) + E(b_4^2) + (P^0)^2 - 2P^0(rEb_3 + Eb_4) + 2rE(b_3b_4)]$$

Selected values of  $E \log X$  and  $10^4 E \left[ \frac{P - P^0}{P} \right]^2$  are given in Table 1. Some values for the  $m$ -locus, which were obtained in the same way using the reduced-form equations for  $m$ , are also given in Table 1. Both loci are shown in Figure 2.

TABLE 1: Selected Values For Real Income—Price Variance Loci

$r$	$r$ -locus		$m$	$m$ -locus	
	$E \log X$	$10^4 E \left( \frac{P - P^0}{P^0} \right)^2$		$E \log X$	$10^4 E \left( \frac{P - P^0}{P^0} \right)^2$
1	6.569	.252	140.1	6.559	.222
2	6.568	.250	142.1	6.560	.226
3	6.567	.247	144.1	6.561	.229
4	6.566	.244	146.1	6.562	.232
5	6.565	.241	148.1	6.563	.235
6	6.564	.239	150.1	6.564	.239
7	6.563	.236	152.1	6.566	.242
8	6.563	.233	154.1	6.567	.245
9	6.562	.231	156.1	6.568	.249
10	6.561	.228	158.1	6.569	.252

Once again  $m$  dominates  $r$ ; at each value of  $E \log X$  the variance of the deflator is smaller for the  $m$ -locus than it is for the  $r$ -locus. The difference, however, is miniscule. At  $E \log X = 6.5647$ , which corresponds to  $r = r^0$  for the  $r$ -locus, the percentage variance of the deflator for the  $m$ -locus is 0.2393, while that for the  $r$ -locus is 0.2397. For a sample of 100 draw-

ings, 90 per cent confidence intervals around those estimates are almost coincident.

#### IV. CONCLUSION

As indicated in the introduction, we think that little attention should be paid to our experimental findings. It is not only because our samples were too small, but also because, to calculate numerical loci, it is necessary to assume a utility function and, what is more, an economic structure. And to accept calculated loci, or a comparison thereof, is to accept the assumed utility function and the assumed structure. Even if our samples had been larger, we would not then have cared to press our findings. Before doing that, we would want to average over time<sup>7</sup> and several economic structures.

But more fundamentally, we feel that no monetary authority should decide once and for all, by statistical inference, which of its possible instrument variables to use. Unless faced with prohibitive costs, it should decide which variable to use at the beginning of every policy period or possibly every quarter. This ultimately is why we could in good conscience content ourselves only with offering a way of determining the optimum instrument variable (and with a sample of only 100 drawings).

<sup>7</sup> See the appendix, wherein we appraise the attempt of Holbrook and Shapiro to determine empirically the optimum monetary instrument variable.

## APPENDIX:

## Holbrook and Shapiro on the Optimum Monetary Instrument Variable

There has been one attempt that we know of, by Holbrook and Shapiro (H&S), to determine empirically the Federal Reserve's optimum instrument variable.<sup>1</sup> What H&S did was to calculate and then to compare certain variances of real GNP, variances associated with three possible monetary instrument variables: the narrowly defined stock of money, the monetary base, and, what would seem a rather surprising choice, the average rate on long-term Treasury bonds.<sup>2</sup> What they found was, for every calendar quarter in a long stretch of years, a smaller variance for the narrowly defined money stock than for both the monetary base and, by a much wider margin apparently, the average Treasury bond rate. Thus, their tentative conclusion was that, in setting its policy, the Federal Reserve ought to use the narrowly defined money stock.

But H&S calculated and so compared the wrong variances. They went astray, we suspect, because they forgot that there must be disturbance terms in their structural equations. Whatever the explanation, though, they cannot be regarded as having made a case, even a highly provisional case, for the narrowly defined money stock as the Federal Reserve's optimum monetary instrument variable.

H&S distinguished between actual GNP, denoted here by  $Y$ , and predicted GNP, denoted here by  $y_p$ . Suppose that

<sup>1</sup> See Robert Holbrook and Harold Shapiro, "The Choice of Optimal Intermediate Economic Targets," *American Economic Review*, May 1970, pp. 40-46.

<sup>2</sup> Holbrook and Shapiro referred to the narrowly defined stock of money, the money base, and the average rate on Treasury bonds as possible intermediate target variables. But they assumed that the Federal Reserve is able to determine exactly any one of these three variables, so it is quite proper for us to refer to them here as possible instrument variables.

$$\begin{aligned} (1) \quad C &= a_1 Y + U_1 \\ (2) \quad I &= a_2 + a_3 r + U_2 \\ (3) \quad r &= a_4 + a_5 Y + a_6 m + U_3 \\ (4) \quad Y &= C + I \end{aligned}$$

where  $C$  and  $I$  are, respectively, consumption and investment,  $r$  and  $m$  are the two possible monetary instrument variables, respectively, the rate of interest and the stock of money and  $U_1$ ,  $U_2$ , and  $U_3$  are random disturbances.<sup>3</sup> Then

$$(5) \quad Y(r) = \frac{1}{1 - a_1} (a_2 + a_3 r + U_1 + U_2)$$

and

$$(6) \quad Y(m) = \frac{1}{1 - a_1 - a_3 a_5} (a_2 + a_3 a_4 + a_3 a_6 m + U_1 + U_2 + a_3 U_3)$$

where  $Y(r)$  is actual GNP with  $r$  as the instrument variable and  $Y(m)$  is actual GNP with  $m$  as the instrument variable. Also

$$(5a) \quad y_p(r) = \frac{1}{1 - \hat{a}_1} (\hat{a}_2 + \hat{a}_3 r)$$

and

$$(6a) \quad y_p(m) = \frac{1}{1 - \hat{a}_1 - \hat{a}_3 \hat{a}_5} (\hat{a}_2 + \hat{a}_3 \hat{a}_4 + \hat{a}_3 \hat{a}_6 m)$$

where  $y_p(r)$  is predicted GNP with  $r$  as the instrument variable,  $y_p(m)$  is predicted GNP with  $m$  as the instrument variable, and  $\hat{a}_i$  is the estimator of  $a_i$ .

<sup>3</sup> This economic structure is far simpler than the one specified by H&S. But since we want only to illustrate wherein they went wrong, we do not need even a faintly realistic structure or more than two possible monetary instrument variables. H&S failed to include disturbances in describing their model, but they must surely belong there, for otherwise the model must be rejected unless the data fit it exactly.

The loss function explicitly assumed by H&S is

$$(7) \quad L(x) = [Y_p(x) - Y(x)]^2 \quad (x = r, m)$$

where  $Y_p(r)$  and  $Y_p(m)$  are the first-order Taylor expansions of, respectively,  $y_p(r)$  and  $y_p(m)$  around the point  $a = (a_1, a_2, \dots, a_6)$ .

Since

$$(8) \quad Y_p(r) = \frac{1}{1 - a_1} [\hat{a}_2 + \hat{a}_3 r + (\hat{a}_1 - a_1)Y]$$

and

$$(9) \quad Y_p(m) = \frac{1}{1 - a_1 - a_3 a_5} [\hat{a}_2 + a_3 \hat{a}_4 + a_3 m \hat{a}_6 + (\hat{a}_1 - a_1)Y + (\hat{a}_3 - a_3)r + a_3(\hat{a}_5 - a_5)Y]$$

it follows that<sup>4</sup>

$$(10) \quad EL(r) = E[Y_p(r) - Y(r)]^2 = VY_p(r) + V \left( \frac{U_1 + U_2}{1 - a_1} \right)$$

and

$$(11) \quad EL(m) = E[Y_p(m) - Y(m)]^2 = VY_p(m) + V \left( \frac{U_1 + U_2 + a_3 U_3}{1 - a_1 - a_3 a_5} \right)$$

$EL(r)$  is the expected loss with  $r$  as the instrument variable and  $EL(m)$  is the expected loss with  $m$  as the instrument variable.

The straightforward procedure would seem to be to minimize  $EL(r)$  by the choice of  $r$  and to minimize  $EL(m)$  by the choice of  $m$  and then to compare the respective minima. But doing so would amount to assuming that the monetary authority does not care about the expected value of  $Y$ . H&S therefore assumed that "the policy maker . . . select(s) the value of each intermediate target variable such that the expected value of income is equal to desired income, and then . . . choose(s) among (instrument) variables that one which minimizes the expected squared deviation of actual from desired income." So H&S would have the

<sup>4</sup> This formula for forecast error holds exactly only in the post-sample period, for in the sample period there is also a covariance term.

monetary authority minimize  $EL(x)$ , but subject to the constraint

$$(12) \quad EY_p(x) = \bar{Y}$$

where  $\bar{Y}$  is the target value of  $Y$ . But they themselves did not compute their constrained minima of  $EL(x)$ , that is,  $EL(\bar{x})$ .

They forgot to calculate the second terms on the right-hand sides of equations 10 and 11.<sup>5</sup> This is hardly a minor oversight. Those terms would remain even if the sample size were indefinitely large. And we suspect that for their estimated model the omitted terms are relatively large. A ranking of instruments by  $VY_p(x)$  in no way implies a ranking by  $EL(x)$ .

Even if H&S had not forgotten the second terms on the right-hand sides in equations 10 and 11, they would have ended up calculating the wrong variances. For in calculating variances, they used actual values of both  $r$  and  $m$  (that is,  $r_a$  and  $m_a$ ). And as is easily shown,  $EY_p(m_a)$  is not in general equal to  $EY_p(r_a)$ . The expectation of  $Y_p$  at  $r = r_a$  is, by equation 8,

$$(13) \quad EY_p(r_a) = \frac{1}{1 - a_1} [a_2 + a_3 r_a]$$

assuming unbiased estimators of the  $a_i$ 's. From equation 3, it follows that

$$(14) \quad m_a = \frac{1}{a_6} [r_a - a_4 - a_5 Y(r_a) - U_3]$$

and from (5) that

$$(15) \quad m_a = \frac{1}{a_6(1 - a_1)} [(1 - a_1 - a_3 a_5)r_a - a_4 + a_1 a_4 - (a_5 U_1 + a_5 U_2) + U_3 - a_1 U_3]$$

But the expectation of  $Y_p(m)$  at  $m = m_a$  is, by equations 9 and 15,

$$(16) \quad EY_p(m_a) = \frac{1}{(1 - a_1)} [a_2 + a_3 r_a - \frac{a_3(a_5 U_1 + a_5 U_2 + U_3 - a_1 U_3)}{(1 - a_1 - a_3 a_5)(1 - a_1)}]$$

<sup>5</sup> In footnote 7, p. 45, they recognize but do not deal with this omission.

So at any point in time  $EY_p(m_a) \neq EY_p(r_a)$  unless, by some chance, all  $U_i$ 's happen to be zero.

Thus, if actual or observed values of both (all) possible instrument variables are used in

calculating variances, the resulting variances will correspond to different mean values of  $Y_p$ , and a comparison of variances corresponding to the same value of  $EY_p$ , which is what H&S proposed, is not achieved.