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NOTES ON SEQUENTIAL OLIGOPOLY

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A Sequential Move Oligopoly Game

Two firms, labelled $i = 1, 2$, play a sequential move Cournot (quantity-setting) game. In each period t, $t = 0, 1, \ldots$, firm 1 moves first and picks an output level q_{1t}. Than having seen q_{1t}, firm 2 moves and picks an output level q_{2t}. We call q_{it} the action of firm i at time t. We assume q_{it} is a member of the action space A_i for all t. The period t payoff to firm i when actions q_{1t} and q_{2t} are taken is

$$(1.1) \qquad \pi_{it}(q_{1t},q_{2t}) = p(q_{1t}+q_{2t})q_{it}$$

where $p(\cdot)$ is the industry demand function. We assume $p(q)$ is differentiable, monotonically decreasing in q on a finite interval $[0,m]$ and that $p(q) \to 0$ as q increases to m and $p(q)$ equals zero for all $q \geq m$.

A strategy σ_i for firm i is a sequence of functions $\sigma_{i1}, \sigma_{i2}, \ldots,$ one for each period t. The function for period t determines player i's actions as a function of the actions taken by both players times before time t. Let the history faced by player 1 at time t be denoted h_{1t} where

$$h_{1t} = (q_{11},q_{21},\ldots,q_{1t-1},q_{2t-1}).$$

That is, h_{1t} records the output levels of both players in all periods before t. Let $H_{1t} = \{h_{1t} | q_{is} \in A_i \text{ for all } 1 \leq s < t\}$. In period t player 2 moves after having observed the current output q_{1t} produced by player 1. Thus the history fucing player 2 at time t is $h_{2t} = (h_{1t};q_{1t})$. Let

$$H_{2t} = \{h_{2t} | q_{is} \in A_1 \text{ for } 1 \leq s \leq t \text{ and } q_{2s} \in A_2 \text{ for } 1 \leq s < t\}.$$

A strategy σ_i for player i is a sequence of functions $\{\sigma_{it}\}_{t=1}^{\infty}$ where σ_{it}: $H_{it} \to A_i$. Let the strategy space of player i be

$$S_i = \{\sigma_i = (\sigma_{it})_{t=1}^{\infty} | \sigma_{it}: H_{it} \to A_i\}.$$

Let $\sigma = (\sigma_1, \sigma_2)$ and $S = S_1 \times S_2$. We need to define payoffs over strategies. We first define payoffs over outcomes. An outcome path q^0 is a collection of actions for both players, one each t. That is $q^0 = \{q_{1t}, q_{2t}\}_{t=1}^{\infty}$. The payoff to firm i under outcome path q^0 is

$$(1.2) \qquad V_i(q^0) = \sum_{t=1}^{\infty} \delta^t \pi_i(q_{1t}, q_{2t}).$$

Likewise, the payoff to firm i from t onwards under the outcome path q^t from t onwards is

$$(1.3) \qquad V_{it}(q^t) = \sum_{s=t}^{\infty} \delta^{s-t} \pi_i(q_{is}, q_{2s})$$

where $q^t = (q_{1t}, q_{2t}, q_{1t+1}, q_{2t+1}, \dots)$.

Given any history h_{1t} a strategy vector $\sigma_1(\cdot|h_{1t})$, $\sigma_2(\cdot|h_{2t})$ generates an outcome path from t onward, which is inductively defined as,

$$(1.4) \qquad q_{1t} = \sigma_1(h_{1t}|h_{1t})$$

$$q_{2t} = \sigma_2(h_{2t}|h_{1t}) \text{ where } h_{2t} = (h_{1t}; q_{1t})$$

$$q_{1t+1} = \sigma_1(h_{1t+1}|h_{1t}) \text{ where } h_{1t+1} = (h_{1t}; q_{1t}, q_{2t})$$

$$q_{2t+1} = \sigma_2(h_{2t+1}|h_{1t}) \text{ where } h_{2t+1} = (h_{1t}; q_{1t}, q_{2t}, q_{1t+1}), \text{ and soon.}$$

Payoffs over strategies $\sigma_1(\cdot|h_{1t})$, $\sigma_2(\cdot|h_{2t})$ are given by

$$V_{it}(\sigma_1(\cdot|h_{1t}), \sigma_2(\cdot|h_{2t})) = \sum_{s=t}^{\infty} \int^{s-t} \pi_i(q_{1s}, q_{2s})$$

where $q^t = (q_{1t}, q_{2t}, q_{1t+1}, q_{2t+1}, \dots)$ is defined by (1.4).

Let $S_1(h_{1t})$ denote the set of strategies for player i from t onward, given history h_{1t}. That is

$$(1.6) \qquad S_1(h_{1t}) = \{\sigma_1(\cdot|h_{1t}) = (\sigma_{1s}(\cdot|h_{1t}))_{s=t}^{\infty} \mid \sigma_{1s}(\cdot|h_{1t}): H_{1t}^{S} \to A_1\}$$

where $H_{1t}^{s} = \{h_{1t}^{s}=(q_{1t},q_{2t},\ldots,q_{1s-1},q_{2s-1})|q_{ir}\in A_i \text{ all } t\le r<s\}$. Let $S_2(h_{2t})$ be defined in an analogous fashion. We then have

<u>Definition</u>. $\sigma = (\sigma_1,\sigma_2)\in S$ is a subgame perfect Nash equilibrium if for each $t = 1, 2, \ldots$. The following conditions hold: for each $h_{1t} \in H_{1t}$.

$$(1.7)\qquad V_{1t}\big(\sigma_1(\cdot|h_{1t}),\sigma_2(\cdot|h_{2t})\big) \ge V_{1t}\big(\sigma_1'(\cdot|h_{1t}),\sigma_2(\cdot|h_{2t}')\big)$$

for all $\sigma_1'(\cdot|h_{1t}) \in S_1(h_{1t})$ where $h_{2t}' = \big(h_{1t},\sigma_1'(h_{1t}|h_{1t})\big)$ and for each $h_{2t} \in H_{2t}$

$$(1.8)\qquad V_{2t}\big(\sigma_1(\cdot|h_{1t}),\sigma_2(\cdot|h_{2t})\big) \ge V_{2t}\big(\sigma_1(\cdot|h_{1t}),\sigma_2'(\cdot|h_{2t})\big)$$

for all $\sigma_2'(\cdot|h_{2t}) \in S_2(h_{2t})$.

Notice that in each period t player 1 is a "Stackelberg leader" in the sense that player 1 when considering a deviation to some $\sigma_1'(\cdot|h_{1t})$ takes account of the fact that the action he adopts at t, say $\sigma_1'(h_{1t}|h_{1t})$, will affect the action taken by player 2 by affecting the <u>history</u> that player 2 confronts when it is his turn to move.